



SCHAUDER BASES AND SEQUENCE SPACES IN GENERAL TOPOLOGICAL VECTOR SPACES

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ABSTRACT. Minkowski functionals of balanced neighbourhoods of zero are used to define sequence spaces in which sequences are sequences in general non locally convex topological vector spaces. The classical result “Every basis in a complete metrizable topological vector space is a Schauder basis” is generalized for such sequence spaces.

KEYWORDS: Directed set, Net, Schauder basis, Topological vector space.

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1. INTRODUCTION

It is being easy to introduce many sequence spaces in which members are sequences in locally convex topological vector spaces. It is being difficult even to introduce classical sequence spaces in general non locally convex topological vector spaces. Similar difficulties appear while defining many sequence spaces even in Banach spaces when sequences are considered in terms of ideals of the set of natural numbers. However, the specific space of all bounded sequences modulo an ideal and the specific space of all sequences converging to zero modulo an ideal are studied in the article [5] for sequences in Banach spaces. A similar advantage is observed in the present article even for defining specific sequence spaces in general non locally convex topological vector spaces. Minkowski functionals of balanced neighbourhoods of zero in topological vector spaces are used for this purpose in the present article. The ultimate aim of this article is to find a generalization of the result: “Every basis in a complete metrizable topological vector space is a Schauder basis”. All vector spaces considered in this article are over the real field \mathbb{R} or over the complex field \mathbb{C} . All topological vector spaces [4] considered in this article are Hausdorff spaces.

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2. CLASSICAL SEQUENCE SPACES

Let us begin with a topological vector space X , and let \mathcal{U} denote the collection of all balanced open neighbourhoods of zero in this topological vector space. To each $U \in \mathcal{U}$, let us consider the Minkowski functional $\mu_U : X \rightarrow [0, \infty)$ defined by $\mu_U(x) = \inf\{t > 0 : x \in tU\}$. Here $\mu_U(\lambda x) = |\lambda|\mu_U(x), \forall x \in X$, and for every scalar λ . If $U, V \in \mathcal{U}$ such that $V + V \subseteq U$, then $\mu_U(x + y) \leq \max\{\mu_V(x), \mu_V(y)\} \leq \mu_V(x) + \mu_V(y), \forall x, y \in X$. If X is also a topological algebra in which the multiplication operation is jointly continuous and if $U, V \in \mathcal{U}$ such that $V \cdot V \subseteq U$, then $\mu_U(xy) \leq \mu_V(x)\mu_V(y), \forall x, y \in X$. These properties are used along with Proposition 2.1.17 and Theorem 2.2.14 given in [1], for defining the following sequence spaces. For each $U \in \mathcal{U}$, and for each sequence $(x_n)_{n=1}^\infty$ in X , let us define $p_U((x_n)_{n=1}^\infty) = \sup\{\mu_U(x_i) : i = 1, 2, 3, \dots\}$. Let $\ell^\infty(X)$ denote the vector space of all bounded sequences in X . For each $U \in \mathcal{U}$, let $U^{(\infty)} = \{(x_n)_{n=1}^\infty \in \ell^\infty(X) : p_U((x_n)_{n=1}^\infty) < 1\}$. Then there is a unique vector topology on $\ell^\infty(X)$ such that $\{U^{(\infty)} : U \in \mathcal{U}\}$ is a local base at zero. Let us consider only this vector topology on $\ell^\infty(X)$ that makes $\ell^\infty(X)$ into a topological vector space. If X is also a topological algebra, then $\ell^\infty(X)$ is also a topological algebra with respect to this vector topology and with respect to the natural multiplication. Within this $\ell^\infty(X)$, it is possible to consider the vector subspace $c_0(X)$ of all sequences converging to zero in X . It is observed that $c_0(X)$ is also a two sided ideal in $\ell^\infty(X)$, when X is also a topological algebra. Let us now fix $0 < p \leq 1$. For each $U \in \mathcal{U}$, and for each sequence $(x_n)_{n=1}^\infty$ in X , let us define $q_U((x_n)_{n=1}^\infty) = \sum_{n=1}^\infty (\mu_U(x_n))^p$. Define $\ell^p(X) = \{(x_n)_{n=1}^\infty \in \ell^\infty(X) : q_U((x_n)_{n=1}^\infty) < \infty, \forall U \in \mathcal{U}\}$. This is a vector space under a natural coordinatewise addition and a natural coordinatewise scalar multiplication. For each $U \in \mathcal{U}$, let $U^{(p)} = \{(x_n)_{n=1}^\infty \in \ell^\infty(X) : q_U((x_n)_{n=1}^\infty) < 1\}$. Then there is a unique vector topology on $\ell^p(X)$ such that $\{U^{(p)} : U \in \mathcal{U}\}$ is a local base at zero. Let us consider only this vector topology on $\ell^p(X)$ that makes $\ell^p(X)$ into a topological vector space. If X is also a topological algebra, then $\ell^p(X)$ is also a topological algebra with respect to this vector topology and with respect to a natural multiplication.

These classical sequence spaces have sequences defined on the directed set \mathbb{N} of all natural numbers. A directed set $\mathbb{N} \times \mathbb{N}$ with a specific order relation was considered in [2] for an extension of the result: “Every basis in a complete metrizable topological vector space is a Schauder basis”. A common extension for the directed sets \mathbb{N} and $\mathbb{N} \times \mathbb{N}$ is a directed set (D, \leq) satisfying the following conditions: (i) $\{\alpha \in D : \alpha \leq \beta\}$ is finite for every $\beta \in D$, and (ii) $\{\alpha \in D : \alpha \geq \beta\}$ is infinite for every $\beta \in D$. This type of extension of directed sets was considered in the article [3], again to generalize the result: “Every basis in a complete metrizable topological vector space is a Schauder basis”. It is assumed in the present article also that all directed sets with the notation (D, \leq) satisfy these two conditions (i) and (ii).

Let us begin with a directed set (D, \leq) , along with X and \mathcal{U} . Let $\ell^\infty(D, X)$ denote the vector space of all bounded nets $(x_\delta)_{\delta \in D}$ in X . Here, the net $(x_\delta)_{\delta \in D}$ is bounded means that the set $\{x_\delta : \delta \in D\}$ is a bounded set. Let $c_0(D, X)$ denote the vector space of all bounded nets $(x_\delta)_{\delta \in D}$ converging to zero in X . For each $U \in \mathcal{U}$, and for each net $(x_\delta)_{\delta \in D}$ in X , let us define $p_U((x_\delta)_{\delta \in D}) = \sup\{\mu_U(x_\delta) : \delta \in D\}$. With this notation, it is possible to make $\ell^\infty(D, X)$ into a topological vector space. For each $U \in \mathcal{U}$, and for each net $(x_\delta)_{\delta \in D}$ in X , and for each fixed $p \in (0, 1]$, let us define $q_U((x_\delta)_{\delta \in D}) = \sum_{\delta \in D} (\mu_U(x_\delta))^p$. Here, $\sum_{\delta \in D} (\mu_U(x_\delta))^p$

means $\sup \left\{ \sum_{\alpha \leq \beta} (\mu_U(x_\alpha))^p : \beta \in D \right\}$. With this notation, it is naturally possible to make the vector space $\ell^p(D, X) = \{(x_\delta)_{\delta \in D} \in \ell^\infty(D, X) : q_U((x_\delta)_{\delta \in D}) < \infty, \forall U \in \mathcal{U}\}$ into a topological vector space. All these spaces are complete, if X is complete, and all these spaces are sequentially complete, if X is sequentially complete. These spaces are examples for sequence spaces to be considered in the next section.

3. GENERAL SEQUENCE SPACES

Let X, \mathcal{U} and D be as in the previous section. To each $\delta \in D$, let us also fix a nonzero topological vector subspace X_δ of X . Let

$$Y = \left\{ (x_\delta)_{\delta \in D} : x_\delta \in X_\delta, \forall \delta \in D, \left(\sum_{\alpha \leq \beta} x_\alpha \right)_{\beta \in D} \right. \\ \left. \text{is a Cauchy net as well as a bounded net in } X \right\}.$$

Then Y is a vector space with respect to the natural operations. To each $U \in \mathcal{U}$, let us define $A_U = \{(x_\delta)_{\delta \in D} \in Y : \sum_{\alpha \leq \beta} x_\alpha \in U, \forall \beta \in D\}$. For each $U \in \mathcal{U}$, the set A_U is balanced and absorbing in Y . If $U, V, W \in \mathcal{U}$ and $W \subseteq U \cap V$, then $A_W \subseteq A_U \cap A_V$. If $U, V \in \mathcal{U}$ and $V + V \subseteq U$, then $A_V + A_V \subseteq A_U$. So, by Proposition 2.1.17 given in [1], the collection $\mathcal{V} = \{A_U : U \in \mathcal{U}\}$ induces a vector topology on Y such that Y becomes a topological vector space in which \mathcal{V} becomes a local base for the neighbourhood system of zero in Y . Let us consider Y as a topological vector space with that topology, in the rest of the article. To each $\alpha \in D$, let $P_\alpha : Y \rightarrow X_\alpha \subseteq X$ be the mapping defined by $P_\alpha((x_\delta)_{\delta \in D}) = x_\alpha, \forall (x_\delta)_{\delta \in D} \in Y$.

Theorem 3.1. *If each X_δ is sequentially complete, then Y is also sequentially complete. If D is such that $(\sum_{\alpha \leq \beta} x_\alpha)_{\beta \in D}$ is bounded in X whenever it is a Cauchy net in X , with $x_\alpha \in X_\alpha, \forall \alpha$, and if each X_δ is complete, then Y is also complete. Each P_α is continuous.*

Proof. Suppose that each X_δ is sequentially complete, for the first part. Consider a Cauchy sequence $\left((x_\delta^{(n)})_{\delta \in D} \right)_{n=1}^\infty$ in Y . For a given $U \in \mathcal{U}$, let us find $V \in \mathcal{U}$ such that $\bar{V} + \bar{V} + \bar{V} \subseteq U$. For this V , there is a $\lambda \in (0, 1]$ such that $\lambda (x_\delta^{(n)})_{\delta \in D} \in A_V, \forall n = 1, 2, 3, \dots$. For this V , there is an integer k such that $\sum_{\alpha \leq \beta} (x_\alpha^{(i)} - x_\alpha^{(j)}) \in V, \forall i, j \geq k, \forall \beta \in D$. This relation implies that the sequence $\left(\sum_{\alpha \leq \beta} x_\alpha^{(n)} \right)_{n=1}^\infty$ is Cauchy in $X, \forall \beta \in D$. This relation implies that $\left(x_\beta^{(n)} \right)_{n=1}^\infty$ is a Cauchy sequence in X_β , if the cardinality of the set $\{\alpha \in D : \alpha \leq \beta\}$ is one. If the set $\{\alpha \in D : \alpha \leq \beta\}$ has two elements, namely, α and β satisfying $\alpha < \beta$, then $\left(x_\alpha^{(n)} \right)_{n=1}^\infty$ is Cauchy in X_α , and $\left(x_\alpha^{(n)} + x_\beta^{(n)} \right)_{n=1}^\infty$ is Cauchy in X , and then in this case $\left(x_\beta^{(n)} \right)_{n=1}^\infty$ is a Cauchy sequence in X_β . By induction on the cardinality of the set $\{\alpha \in D : \alpha \leq \beta\}$, it can be verified that $\left(x_\alpha^{(n)} \right)_{n=1}^\infty$ is Cauchy in X_α , for every $\alpha \in D$. Since each X_α is sequentially complete, let $\left(x_\alpha^{(n)} \right)_{n=1}^\infty$ converge to some $x_\alpha \in X_\alpha$. The earlier relation $\sum_{\alpha \leq \beta} (x_\alpha^{(i)} - x_\alpha^{(j)}) \in V$,

$\forall i, j \geq k, \forall \beta \in D$ implies that $\sum_{\alpha \leq \beta} (x_\alpha^{(k)} - x_\alpha) \in \bar{V} \subseteq U, \forall \beta \in D$. So, it is possible to conclude that $\lambda \sum_{\alpha \leq \beta} x_\alpha \in U, \forall \beta \in D$. Moreover, it is possible to find a $\delta \in D$ such that $(\sum_{\alpha \leq \beta} x_\alpha^{(k)} - \sum_{\alpha \leq \gamma} x_\alpha^{(k)}) \in V, \forall \beta, \gamma \geq \delta$ in D . This implies that $(\sum_{\alpha \leq \beta} x_\alpha - \sum_{\alpha \leq \gamma} x_\alpha) \in \bar{V} + \bar{V} + \bar{V} \subseteq U, \forall \beta, \gamma \geq \delta$ in D . So, it is possible to conclude that $(x_\delta)_{\delta \in D} \in Y$, and that the sequence $\left((x_\delta^{(n)})_{\delta \in D} \right)_{n=1}^\infty$ converges to $(x_\delta)_{\delta \in D}$ in Y . Thus Y is also sequentially complete. The second part follows when suitable modifications are made in the previous arguments. For the third part, let us consider a net $\left((x_\delta^{(\gamma)})_{\delta \in D} \right)_{\gamma \in E}$ that converges to zero in Y . Here, (E, \leq) is a general directed set which need not satisfy the conditions (i) and (ii) mentioned in the previous section. Let us fix $U \in \mathcal{U}$ arbitrarily. Then there is an $\epsilon \in E$ such that $\sum_{\alpha \leq \beta} x_\alpha^{(\gamma)} \in U, \forall \beta \in D, \forall \gamma \geq \epsilon$ in E . If the cardinality of the set $\{\alpha \in D : \alpha \leq \beta\}$ is one, then the net $(x_\beta^{(\gamma)})_{\gamma \in E}$ converges to zero in X_β . By induction on the cardinality of the set $\{\alpha \in D : \alpha \leq \beta\}$, it is possible to conclude that the net $(x_\beta^{(\gamma)})_{\gamma \in E}$ converges to zero in X_β , for every $\beta \in D$. Thus, if a net $\left((x_\delta^{(\gamma)})_{\delta \in D} \right)_{\gamma \in E}$ converges to zero in Y , then the net $(x_\beta^{(\gamma)})_{\gamma \in E}$ converges to zero in X_β , for every $\beta \in D$. This means that each P_β is continuous. ■

Theorem 3.2. *Suppose $\{x_\alpha : \alpha \in D\}$ be a linearly independent set in X such that $x_\alpha \neq x_\beta$ whenever $\alpha \neq \beta$. Suppose that X is a complete metrizable topological vector space such that for every given $x \in X$, there is a unique set $\{f_\alpha(x) : \alpha \in D\}$ of scalars such that $x = \sum_{\alpha \in D} f_\alpha(x)x_\alpha$. Let us further assume that the net $(\sum_{\alpha \leq \beta} f_\alpha(x)x_\alpha)_{\beta \in D}$ is a bounded net for each $x \in X$. Then each f_α is a continuous linear functional on X .*

Proof. To each $\beta \in D$, let $X_\beta = \{\lambda x_\beta : \lambda \text{ is a scalar}\}$, and let us consider Y for these X_β . Then Y is also a complete metrizable topological vector space. Define a map $T : Y \rightarrow X$ by $T((\lambda_\delta x_\delta)_{\delta \in D}) = \sum_{\delta \in D} \lambda_\delta x_\delta, \forall (\lambda_\delta x_\delta)_{\delta \in D} \in Y$. Here $\sum_{\delta \in D} \lambda_\delta x_\delta$ means the limit of the net $(\sum_{\alpha \leq \beta} \lambda_\alpha x_\alpha)_{\beta \in D}$ so that it refers to an ordered convergence. This mapping T is a bijective continuous linear map. By the open mapping theorem, the inverse mapping should also be continuous. So, if a sequence $(\sum_{\delta \in D} f_\delta(x^{(n)})x_\delta)_{n=1}^\infty$ converges to zero in X , then the sequence $((f_\delta(x^{(n)})x_\delta)_{\delta \in D})_{n=1}^\infty$ also converges to zero in Y . By the previous theorem, the sequence $(f_\delta(x^{(n)})x_\delta)_{n=1}^\infty$ converges to zero in X_δ , for every fixed $\delta \in D$. So, the sequence $(f_\delta(x^{(n)}))_{n=1}^\infty$ of scalars converges to zero, for every fixed $\delta \in D$. This completes the proof. ■

Corollary 3.3. *Every basis in a complete metrizable topological vector space is a Schauder basis.*

Proof. Every coefficient functional is continuous, by Theorem 3.2. ■

The following Corollary 3.4 can be considered as a classical application to Theorem 3.2.

Corollary 3.4. *Suppose X , x_α and f_α be as in the statement of Theorem 3.2. Let $(z_\alpha)_{\alpha \in D}$ be a net in a topological vector space Z . Let $T : X \rightarrow Z$ be a linear map such that $T(\sum_{\alpha \in D} f_\alpha(x)x_\alpha) = \sum_{\alpha \in D} \lambda_\alpha f_\alpha(x)z_\alpha$, $\forall x \in X$, and for some fixed net $(\lambda_\alpha)_{\alpha \in D}$ of scalars. Let us further assume that the net $(\sum_{\alpha \leq \beta} \lambda_\alpha f_\alpha(x)x_\alpha)_{\beta \in D}$ is a bounded net, for each $x \in X$. Then T is continuous.*

Proof. For each $\beta \in D$, and for each $x \in X$, let $T_\beta(x) = \sum_{\alpha \leq \beta} \lambda_\alpha f_\alpha(x)x_\alpha$. Let $(x_n)_{n=1}^\infty$ be a sequence in X which converges to zero. Then $(f_\alpha(x_n))_{n=1}^\infty$ converges to zero, for every $\alpha \in D$. So, $(\sum_{\alpha \leq \beta} \lambda_\alpha f_\alpha(x_n)x_\alpha)_{n=1}^\infty$ converges to zero, for every $\beta \in D$. Thus each $T_\beta : X \rightarrow Z$ is a continuous linear map. Moreover, $\{T_\beta(x) : \beta \in D\}$ is bounded, for every $x \in X$. Then, by the uniform boundedness principle, $\{T_\beta : X \rightarrow Z : \beta \in D\}$ is an equicontinuous family of linear mappings such that $T(x)$ is the limit of the net $(T_\beta(x))_{\beta \in D}$, for every $x \in X$. Fix a neighbourhood W of zero in Z . Find a neighbourhood V of zero in Z such that \bar{V} , the closure of V , is contained in W . Then, there is a neighbourhood U of zero in X such that $T_\beta(U) \subseteq V$, $\forall \beta \in D$ so that $T(U) \subseteq \bar{V} \subseteq W$. This proves that T is continuous. ■

4. CONCLUSION

Section 2 provides examples for Theorem 3.2. Let $D = \{\alpha, \beta\} \cup \mathbb{N}$ with an assumption $\alpha < n, \beta < n, \forall n = 1, 2, 3, \dots$ and an assumption that α and β are not comparable. This is also an example for the directed sets used in Theorem 3.2. So, Theorem 3.2 can be considered as a generalization of Corollary 3.3. There is a need for further research to relax the condition “the net $(\sum_{\alpha \leq \beta} f_\alpha(x)x_\alpha)_{\beta \in D}$ is a bounded net for each $x \in X$ ” in the statement of Theorem 3.2.

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