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A COMPARATIVE STUDY OF LAPLACE DECOMPOSITION METHOD AND VARIATIONAL ITERATION METHOD FOR SOLVING NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this study, we compare the Laplace decomposition approach to the variational iteration method. This research focuses on comparing methodologies to resolve integro-differential equations that are nonlinear.The result show how practical and successful these methods are. We compare the results to four cases to assess the solution's correctness.

KEYWORDS:Nonlinear Integro-Differential Equations, Laplace decomposition method, Variational iteration method.
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1. Introduction

An Integro-Differential Equation is one that includes both the integral and derivative of unknown functions. Solving Integro-differential Equations is critical in science and engineering [15, 2]. In many scientific and technical domains, complicated physical processes are described by means of nonlinear problems. Nonlinear phenomena can be seen in a wide range of scientific domains, including chemical kinetics, solid state physics, fluid dynamics, mathematical biology and plasma physics. Numerous physical processes ,including the formation of glass, heat transmission, diffusion in general, diffusion of neutrons and coexistence of biological species with varying rates of generation involve the use of Integro-differential equations without linearity [15]. Integro-differential equations that are not linear fall into two categories: nonlinear Volterra equations and others nonlinear Fredholm equations. In this paper, we look at two successful approaches regarding the resolution of Volterra

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integro-differential equations that are not linear: LDM and VIM. The following is one kind of Volterra integro-differential equation that is not linear:

$$\frac{\mathbf{d^j v}}{\mathbf{dx^j}} = g(x) + \int_0^x K(x,t)G(v(t))dt, \tag{1.1}$$

where $G(v(t))$ function that is nonlinear of $v(t)$.

The present paper has the following structure.We define LDM and VIM in part 2, show the comparison results with four instances in section 3, and provide a conclusion in section 4.

2. Methods description

2.1. **Laplace Decomposition Method.** Combining the Adomian Decomposition and Laplace Transform techniques are also referred to as the Laplace Decomposition method (LDM).This methods main benefits is its ability to find a nonlinear equation's precise or approximate solution [9]. Differential equations can be successfully solved using the Laplace Decomposition method (LDM),which was initially presented by Suheil A. Khuri [11, 12]. When equation (1.1) is run through both sides using the Laplace transform, the result is

$$\begin{aligned} &s^j \mathbf{L}\{v(x)\} - s^{j-1} v(0) - s^{j-2} v'(0) - ... - v^{j-1}(0) \\ &= \mathbf{L}\{g(x)\} + \mathbf{L}\{K(x-t)\} + \mathbf{L}\{G(v(t))\} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \mathbf{L}\{v(x)\} = \frac{1}{s} v(0) + v(0) + \frac{1}{s^2} v'(0) + ... + \frac{1}{s^j} v^{(j-1)}(0) \\ + \frac{1}{s^j}\mathbf{L}\{g(x)\} + \frac{1}{s^j}\mathbf{L}\{K(x-t)\}\mathbf{L}\{G(v(t))\} \end{aligned} \tag{2.2}$$

In order to accomplish this, the linear expression v(x) on the left is first expressed using an endless succession of parts provided by,

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \tag{2.3}$$

recursively find the components $v_n(x), n \geq 0$.

For treating the non-linear component $G(v(x))$, the Adomian polynomial shall be embodied by an endless series, A_n we apply the Adomian polynomial get around its difficulties [15, 3, 16] in the format,

$$G(v(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{2.4}$$

where,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(\sum_{j=0}^{n} \lambda^j v_j\right)_{\lambda=0, n=0,1,2,...}$$

is obtained for all forms of nonlinearity types. (2.3) and (2.4) into (2.2) result in

$$\mathbf{L}\left(\sum_{j=0}^{n} v_n(x)\right) = \frac{1}{s}v(0) + \frac{1}{s^2}v'(0) + ... + \frac{1}{s^j}v^{(j-1)}(0) + \frac{1}{s^j}\mathbf{L}\{g(x)\} + \frac{1}{s^j}\mathbf{L}\{K(x-t)\}\mathbf{L}\left(\sum_{n=0}^{\infty} A_n(x)\right), \tag{2.5}$$

with the Adomian decomposition approach, the recursive connection listed below can be used

$$\mathbf{L}\{v_0(x)\} = \frac{1}{s}v(0) + \frac{1}{s^2}v'(0) + ... + \frac{1}{s^j}v^{(j-1)}(0) + \frac{1}{s^j}\mathbf{L}\{g(x)\}, \tag{2.6}$$

and

$$\mathbf{L}\{v(x)\} = \frac{1}{s^j}\mathbf{L}\{K(x-t)\}\mathbf{L}\{A_n(x)\}, n \geqslant 1. \tag{2.7}$$

When the first portion of (2.6) is subjected to the inverse Laplace transform $\boldsymbol{v_0(\mathbf{x})}$ is obtained which defined $\boldsymbol{A_0}$.Consequently, by using second portion of (2.7) the components of equation (2.3) will be fully determined.

2.2. **Variational Iteration Method.** Ji-Huan He developed the Variational iteration technique (VIM) [7, 8].If there is a closed form solution, VIM offers quickly converging successive approximations of the precise answer. Without requiring any special limitations, the VIM manages both linear and nonlinear issues are treated similarly [15]. It is necessary to specify the starting conditions in order to fully determine the precise solution. For the equation for integro-differential that is not linear (1.1) the correction functional is,

$$v_{n+1}(x) = v_n(x) + \int_0^x \lambda(\psi)\left[v_n^{(j)}(\psi) - f(\psi) - \int_0^{\psi}[K(\psi,r)G(\tilde{v}_n(r))dr]d\psi\right]. \tag{2.8}$$

There are two key phases involved in using the Variational iteration method. Prior to anything else, the Lagrange multiplier λ [18, 13, 14] must be found. This can be done best by utilizing a constrained variation and integration by parts. Either a function or constant can be the Lagrange multiplier λ. After λ has been established, the following approximations $v_{(n+1)}(x)$, for $n \geqslant 0$ of the answer $v(x)$, should be computed using an iteration formula that is not constrained in any way. Any selected function can serve as the zeroth approximation v_0. However ,for the selective zeroth approximation v_0, it is preferable to utilize the initial values $v(0), v'(0), \cdots$

$$\begin{aligned}
&v' + g\left(v(\psi), v'(\psi)\right) = 0,\ \lambda = -\ 1,\\
&v_0(x) = v(0),\ \ for\ first\ order\ v_n'\\
&v'' + g\left(v(\psi), v'(\psi), v''(\psi)\right) = 0, \lambda = \psi - x\\
&v_0(x) = v(0) + xv'(0), \text{ for second order } v_n'',\\
&v''' + g\left(v(\psi), v'(\psi), v''(\psi), v'''(\psi)\right) = 0,\ \lambda = -\frac{1}{2!}(\psi - x)^2,\\
&v_0(x) = v(0) + xv'(0) + \frac{1}{2!}x^2v''(0), \text{ for third order } v_n''',
\end{aligned} \tag{2.9}$$

So on. As a consequence, the answer is provided by

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x). \tag{2.10}$$

3. Main Result

Example 3.1. Take the integro-differential equation that is nonlinear,

$$\frac{dv}{dx} = \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x.x - \frac{3}{e^x} - \frac{1}{4e^{2x}} + \int_0^x (x-t)\, v^2(t)\, dt, v(0) = 2, \tag{3.1}$$

Using Laplace Decomposition Method. Using the provided initial condition and the Laplace transforms of equation (3.1) ,we have

$$sv(s) = 2 + \frac{9}{4s} - \frac{5}{2s^2} - \frac{1}{s^3} - \frac{3}{s+1} - \frac{1}{4(s+2)} + \frac{1}{s^2} L\left\{v^2(x)\right\},$$

$$v(s) = \frac{2}{s} + \frac{9}{s(4s)} - \frac{5}{s(2s^2)} - \frac{1}{s(s^3)} - \frac{3}{s(s+1)} - \frac{1}{4s(s+2)} + \frac{1}{s^3} L\left\{v^2(x)\right\} \tag{3.2}$$

Using the reverse Laplace transformation of the equation (3.2),we get

$$v(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \ldots + L^{-1}\left[\frac{1}{s^3} L\ \left\{v^2(x)\right\}\right] \tag{3.3}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \quad and \quad v^2(x) = \sum_{n=0}^{\infty} A_n \tag{3.4}$$

substitute equation (3.4) into equation (3.3) we get ,

$$\sum_{n=0}^{\infty} v_n(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \ldots + L^{-1}\left[\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.5}$$

When we compare the equation above's two sides, we obtain

$$v_0(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \ldots$$

$$v_1(x) = L^{-1}\left[\frac{1}{s^3} L[A_0]\right],$$

$$v_2(x) = L^{-1}\left[\frac{1}{s^3} L[A_1]\right],$$

$$\vdots$$

where, $A_0 = v_0^2$, $A_1 = 2v_0v_1$, $A_2 = 2v_0v_2 + v_1^2 \cdots$. and so on we get the following recursive relation

$$v_0(x) = 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \ldots,$$

$$v_1(x) = \frac{2}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{20}x^5 + \ldots.$$

$$\vdots$$

According to (3.4), the series solution is supplied by,

$$v(x) = 2 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots,$$

that arrives to the precise solution,

$$v(x) = 1 + e^{-x}$$

This is the precise solution to equation (3.1).

Using Variational Iteration Technique.

$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v'_n(t) - \frac{9}{4} + \frac{5}{2}t + \frac{1}{2}t^2 + 3e^{-t} + \frac{1}{4}e^{-2t} \; - \int_0^t \big((t-r)\, v_n^2(r) \big) dr \right] dt \tag{3.6}$$

In the case of the first-order integro-differential equation, we utilized λ= -1. Using the above initial condition let's choose $v_0(x) = v(0) = 2$.Following are the consecutive estimations obtained by including the correction functional with this selection.

$$\begin{aligned} v_0(x) &= 2, \\ v_1(x) &= 2 - x + \frac{x^2}{2!} - 5\frac{x^3}{3!} + 5\frac{x^4}{4!} - 7\frac{x^5}{5!} + \ldots, \\ v_2(x) &= 2 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + 5\frac{x^4}{4!} - \frac{x^5}{5!} + \ldots, \\ &\cdot \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

further approximations follow in this manner.

Admittedly, the VIM uses

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \,. \tag{3.7}$$

This provides a precise solution by,

$$v(x) = 1 + e^{-x}.$$

We validated through substitution.

Example 3.2. Take the integro-differential equation that is nonlinear

$$\frac{dv}{dx} = 1 - \frac{1}{3}e^x + \frac{1}{3}e^{-2x} + \int_0^x e^{x-t} v^2(t)\, dt,\; v(0) = 0. \tag{3.8}$$

Using Laplace Decomposition Method. Using the provided initial condition and the Laplace transforms of equation (3.8) ,we have

$$sv(s) = \frac{1}{s} - \frac{1}{3(s-1)} + \frac{1}{3(s+2)} + \frac{1}{s-1} L\left[v^2(x)\right]$$

$$v(s) = \frac{1}{s.s} - \frac{1}{3s(s-1)} + \frac{1}{3s(s+2)} + \frac{1}{s(s-1)} L\left[v^2(x)\right] \tag{3.9}$$

Using the reverse Laplace transformation of the equation (3.9),we get,

$$v(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{1}{8}x^4 - \ldots + L^{-1}\left[\frac{1}{S(S-1)} L\left[v^2(x)\right] \right] \tag{3.10}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x)=\sum_{n=0}^{\infty} v_n(x) \; and \qquad v^2(x)=\sum_{n=0}^{\infty} A_n \tag{3.11}$$

substitute equation (3.11) into equation (3.10) we get ,

$$\sum_{n=0}^{\infty} v_n(x)=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{1}{8}x^4-...+L^{-1}\left[\frac{1}{s(s-1)}L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.12}$$

When we compare the equation above's two sides, we obtain

$$\begin{aligned}
v_0(x)&=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{1}{8}x^4-\dots\\
v_1(x)&=L^{-1}\left[\frac{1}{s(s-1)}L[A_0]\right],\\
v_2(x)&=L^{-1}\left[\frac{1}{s(s-1)}L[A_1]\right],\\
&\cdot\\
&\cdot\\
&\cdot\\
&\cdot
\end{aligned}$$

where, $A_0=v_0^2$, $A_1=2v_0v_1$, $A_2=2v_0v_2+v_1^2$ .... and so on
We get the following recursive relation ,

$$\begin{aligned}
v_0(x)&=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{1}{8}x^4-\dots\\
v_1(x)&=\frac{x^4}{12}-\frac{x^5}{30}+\frac{x^6}{72}-\frac{x^7}{126}+\dots\\
&\cdot\\
&\cdot\\
&\cdot\\
&\cdot
\end{aligned}$$

According to (3.11), the series solution is supplied by,

$$v(x)=1-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots\right).$$

that arrives to the precise solution,

$$v(x)=1-e^{-x}.$$

Using Variational Iteration Technique. For (3.8), the correction functional is provided by

$$v_{n+1}(x)=v_n(x) \quad -\int_0^x\left[v'_n(t)-1+\frac{1}{3}e^t-\frac{1}{3}e^{-2t}-\int_0^t\left(e^{t-r}v_n^2(r)\right)dr\right]dt \tag{3.13}$$

In the case of the first-order integro-differential equation, we utilized λ= -1. Using the above initial condition let's choose $v_0(x)=v(0)=0$.Following are the consecutive estimations obtained by including the correction functional with this selection.

$$\begin{aligned}
v_0(x)&=0,\\
v_1(x)&=x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{8}+\dots,
\end{aligned}$$

$$v_2(x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \dots,$$
$$\vdots$$

further approximations follow in this manner.
Admittedly, the VIM uses,

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) , \tag{3.14}$$

This provides a precise solution by,

$$v(x) = 1 - e^{-x}.$$

We validated through substitution.

Example 3.3. Take the integro-differential equation that is nonlinear,

$$\frac{dv}{dx} = -1 + \int_0^x (x-t) v^2(t) \, dt, \qquad v(0) = 0 \tag{3.15}$$

Using Laplace Decomposition Method:
Using the provided initial condition and the Laplace transforms of equation (3.15) ,we have

$$sv(s) = -\frac{1}{s} + \frac{1}{s^2} L\left[v^2(x)\right] \tag{3.16}$$

$$v(s) = -\frac{1}{s.s} + \frac{1}{s.s^2} L\left[v^2(x)\right] \tag{3.17}$$

Using the reverse Laplace transformation of the equation (3.17),we get,

$$v(x) = -x + L^{-1}\left[\frac{1}{s^3} L\left[v^2(x)\right]\right]. \tag{3.18}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \; and \qquad v^2(x) = \sum_{n=0}^{\infty} A_n \tag{3.19}$$

substitute equation (3.19) into equation (3.18) we get ,

$$\sum_{n=0}^{\infty} v_n(x) = -x + L^{-1}\left[\frac{1}{s^3} L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.20}$$

When we compare the equation above's two sides, we obtain

$$v_0(x) = -x,$$
$$v_1(x) = L^{-1}\left[\frac{1}{s^3} L[A_0]\right],$$
$$v_2(x) = L^{-1}\left[\frac{1}{s^3} L[A_1]\right]$$
$$\vdots$$

Where, $A_0 = v_0^2$, $A_1 = 2v_0v_1$, $A_2 = 2v_0v_2 + v_1^2$ .... and so on
We get the following recursive relation ,

$$v_0 = -x,$$

$$v_1 = \frac{x^5}{60},$$

$$v_2 = \frac{-x^9}{15120},$$

$$\vdots$$

According to (3.19), the series solution is supplied by,

$$v(x) = -x + \frac{x^5}{60} - \frac{x^9}{15120} + \dots$$

Using Variational Iteration Technique:
For (3.15), the correction functional is provided by,

$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v'_n(t) + 1 - \int_0^t \big((t-r)\, v_n^2(r)\big) dr \right] dt \tag{3.21}$$

In the case of the first-order integro-differential equation, we utilized λ= -1.
Using the above initial condition let's choose $v_0(x) = v(0) = 0$.Following are the consecutive
estimations obtained by including the correction functional with this selection.

$$\begin{aligned} v_0(x) &= 0 \\ v_1(x) &= -x, \\ v_2(x) &= -x + \frac{x^5}{60}, \\ v_3(x) &= -x + \frac{x^5}{15} - \frac{x^9}{15120} + \dots \\ &\vdots \end{aligned} \tag{3.22}$$

further approximations follow in this manner.
Admittedly, the VIM uses

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \ , \tag{3.23}$$

This gives solution

$$v(x) = -x + \frac{x^5}{60} - \frac{x^9}{15120} + \dots$$

We validated through substitution.

Example 3.4. Take the integro-differential equation that is nonlinear,

$$\frac{dv}{dx} = x + \int_0^x v^2(t)\, dt, \qquad v(0) = 0 \tag{3.24}$$

Here kernel $K(x,t) = 1$

Using Laplace Decomposition Method. Using the provided initial condition and the Laplace transforms of equation (3.24),we have

$$sv(s) = \frac{1}{s^2} + \frac{1}{s}L\left[v^2(x)\right] \tag{3.25}$$

$$v(s) = \frac{1}{s.s^2} + \frac{1}{s.s}L\left[v^2(x)\right]. \tag{3.26}$$

Using the reverse Laplace transformation of the equation (3.26),we get,

$$v(x) = \frac{x^2}{2} + L^{-1}\left[\frac{1}{s^2}L\left[v^2(x)\right]\right]. \tag{3.27}$$

The solution is decomposed as an infinite sum and nonlinear term by Adomian polynomial as given below

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \quad and \quad v^2(x) = \sum_{n=0}^{\infty} A_n \tag{3.28}$$

substitute equation (3.28) into equation (3.27) we get ,

$$\sum_{n=0}^{\infty} v_n(x) = \frac{x^2}{2} + L^{-1}\left[\frac{1}{s^2}L\left[\sum_{n=0}^{\infty} A_n\right]\right]. \tag{3.29}$$

When we compare the equation above's two sides, we obtain

$$\begin{aligned} v_0(x) &= \frac{x^2}{2}, \\ v_1(x) &= L^{-1}\left[\frac{1}{s^2}L[A_0]\right], \\ v_2(x) &= L^{-1}\left[\frac{1}{s^2}L[A_1]\right], \\ &\vdots \end{aligned}$$

Where, $A_0 = v_0^2$, $A_1 = 2v_0v_1$, $A_2 = 2v_0v_2 + v_1^2 \ldots.$ and so on We get the following recursive relation

$$\begin{aligned} v_0 &= \frac{x^2}{2}, \\ v_1 &= \frac{x^6}{120}, \\ v_2 &= \frac{x^{10}}{10080}, \\ &\vdots \end{aligned}$$

According to (3.28), the series solution is supplied by,

$$v(x) = \frac{x^2}{2} + \frac{x^6}{120} + \frac{x^{10}}{10080} + \ldots$$

Using Variational Iteration Tecnique:

For (3.24), the correction functional is provided by,

$$v_{n+1}(x) = v_n(x) - \int_0^x \left[v'_n(t) - t - \int_0^t \left(v_n^2(r)\right)dr\right] dt \tag{3.30}$$

In the case of the first-order integro-differential equation, we utilized λ= -1.

Using the above initial condition let's choose $v_0(x) = v(0) = 0$.Following are the consecutive estimations obtained by including the correction functional with this selection.

$$\begin{aligned} v_0(x) &= 0, \\ v_1(x) &= \frac{x^2}{2}, \\ v_2(x) &= \frac{x^2}{2} + \frac{x^6}{120}, \\ v_3(x) &= \frac{x^2}{2} + \frac{x^6}{120} + \frac{x^{10}}{10800} + \dots, \\ &\;\;\vdots \end{aligned} \tag{3.31}$$

further approximations follow in this manner. Admittedly, the VIM uses

$$v(x) = \lim_{n \longrightarrow \infty} v_n(x) \;, \tag{3.32}$$

This gives solution

$$v(x) = \frac{x^2}{2} + \frac{x^6}{120} + \frac{x^{10}}{10800} + \dots.$$

We validated through substitution.

4. Conclusion

This work presents the successful application of Lagrangian multiplier (VIM) and Lagrangian differentiation (LDM) techniques for solving integro-differential nonlinear equations. Both methods yield approximations with greater accuracy or closed forms of solutions when available. The LDM is a powerful tool that can deal with both nonlinear and linear integro-differential equations, and for nonlinear operators, the VIM does not have any specific criteria, such as linearization or Adomian polynomials. While VIM requires the evaluation of the Lagrangian multiplier λ, both methods yield the same solution for the aforementioned examples. These two methods are strong and righteous. Based on the comparison of these two powerful methods, therefore, it may be said that VIM is simpler for finding the nonlinear integro-differential equations.

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