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CONVERGENCE BEHAVIOR OF MODIFIED-BERNSTEIN-KANTROVINCH-STANCU OPERATORS

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ABSTRACT. The study introduces a Kantrovinch-Stancu type modification of the modified-Bernstein operator, examining its convergence properties for Hölder's class of functions. It evaluates the rate of convergence through the modulus of continuity and Peetre's K-functional, providing insights into the efficiency of the proposed operators. Additionally, the research establishes a Vornovskaya type asymptotic result and investigates weighted approximation with polynomial growth, shedding light on the behavior of approximations under varying conditions. To illustrate the convergence behavior empirically, the study employs MATLAB software to present numerical examples, offering tangible evidence of the theoretical findings. Through this comprehensive analysis, the study contributes to understanding the performance and applicability of the Kantrovinch-Stancu modification in approximation theory, with implications for various fields relying on function approximation techniques.

KEYWORDS: Modulus of continuity, Kantrovinch operator, Bernstein operator, Moment estimates.
AMS Subject Classification: 41A10, 41A25.

1. Introduction and Preliminaries

Positive linear operators are widely used in various fields of science and engineering. This widely spread area provides us the key tools for exploring the Computer-aided geometric designs, signal processing, image compression, data analysis, numerical analysis, and solution to ordinary and partial differential equations that

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arises in mathematical modeling of real word phenomena. A very famous polynomial in this regards, was studied by Bernstein [1] and the Bernstein operator for every bounded function $\psi \in C[0,1]$, $n \geq 1$ and $t \in [0,1]$ is defined as

$$B_n(\psi;t) = \sum_{i=0}^{n} p_{n,i}(t)\psi\left(\frac{i}{n}\right),$$

and $p_{n,i}(t) = \binom{n}{i} t^{i-1}(1-t)^{n-i-1}$ is Bernstein basis function. Usta [2] presented a new modification for $\psi \in C[0,1],\ n \in \mathbb{N},\ t \in (0,1)$ as

$$\mathcal{B}_n(\psi;t) = \sum_{k=0}^{n} \binom{n}{k} (k-nt)^2 t^{k-1}(1-t)^{n-k-1}\psi\left(\frac{k}{n}\right). \tag{1.1}$$

Recently, Sofyahoğlu [3] introduced a parametric generalization of (1.1). Thereafter, different modification of the above operator have become interest to many researchers. For more details on parametric generalizations, we refer the readers to [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Kantrovinch [14] introduced a modification involving integral for the class of Lebesgue integrable functions on [0,1] given by

$$K_n(\psi;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \psi(u)du, \tag{1.2}$$

where $t \in (0,1)$. Recently, [15] introduced some approximation properties of Szász-Kantorovich type operators allied with d-symmetric d-orthogonal Brenke type polynomials. Also, [16] considered bivariate Summation-integral type hybrid operators and studied their approximation behavior. For applications point of view, refer [17, 18, 19, 20, 21].

The motivation behind the study stems from the need to enhance function approximation techniques, particularly for functions within Hölder's class. Traditional Bernstein operators, while effective, may not always offer optimal convergence rates for diverse functions. By introducing a Kantrovinch-Stancu type modification, this research aims to improve approximation efficiency. Investigating convergence properties through modulus of continuity and Peetre's K-functional provides a deeper understanding of how these new operators perform. The practical application of these theoretical insights, supported by MATLAB simulations, underscores the relevance of this work in advancing approximation theory and its applications across various fields that rely on accurate function representation.

We now introduce Kantrovinch-Stancu modification of the operator given by equation (1.1) based on Stancu parameters $0 \leq \alpha_1 \leq \alpha_2$, as follows:

$$\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t) = (n+\alpha_2)\sum_{k=0}^{n} \binom{n}{k} (k-nt)^2 t^{k-1}(1-t)^{n-k-1}\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} \nu(u)du,\ t \in (0,1). \tag{1.3}$$

2. Moment Estimation

Using the preliminaries, we can prove the following identities for Modified-Bernstein-Kantrovinch-Stancu operators :

Lemma 2.1. *(see [2]) The modified-Bernstein operators $\mathcal{B}_n(.;t)$, for $n \in \mathbb{N}$, satisfy the following identities:*

(i) $\mathcal{B}_n(1;t) = 1$;

(ii) $\mathcal{B}_n(y;t) = \left(\frac{n-2}{n}\right)t + \frac{1}{n}$;
(iii) $\mathcal{B}_n(y^2;t) = \left(\frac{n^2-7n+6}{n^2}\right)t^2 + \left(\frac{5n-6}{n^2}\right)t + \frac{1}{n^2}$;
(iv) $\mathcal{B}_n(y^3;t) = \left(\frac{n^3-15n^2+38n-24}{n^3}\right)t^3 + 12\left(\frac{n^2-4n+3}{n^3}\right)t^2 + \left(\frac{13n-14}{n^3}\right)t + \frac{1}{n^3}$.

Lemma 2.2. *For $n \in \mathbb{N}$ the operator $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu(y);t)$ satisfies the followings:*
(i) $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(1;t) = 1$;
(ii) $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t) = \left(\frac{n-2}{n+\alpha_2}\right)t + \frac{1}{n+\alpha_2}\left(\frac{3}{2}+\alpha_1\right)$;
(iii) $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^2;t) = \frac{1}{(n+\alpha_2)^2}\left\{(n^2-7n+6)t^2 + (6n-8+2\alpha_1(n-2))t + (\alpha_1+1)(\alpha_2+2) + \frac{1}{3}\right\}$.

Proof. Using the linear property of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)$, we've

$$\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t) = \frac{n}{n+\alpha_2}\mathcal{B}_{n,a}(y;t) + \frac{1}{n+\alpha_2}\left(\alpha_1+\frac{1}{2}\right)\mathcal{B}_{n,a}(1;t).$$

By using preliminaries, we can see part (2) is true. In a similar manner, we can prove other parts of above result. □

Let us denote the r^{th} order moment of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^r;t)$ by $\gamma_{n,r}^{(\alpha_1,\alpha_2)}(t)$.

Lemma 2.3. *For $n \in \mathbb{N}$, the r^{th} $(r = 1,2,4)$ ordered moments of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$ are given by*
(i) $\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t) = -\left(\frac{2+\alpha_1}{n+\alpha_2}\right)t + \frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)$;
(ii) $\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t) = \frac{1}{(n+\alpha_2)^2}\left\{(-3n+6+\alpha_2^2+4\alpha_2)t^2 + (3n-8-2\alpha_1\alpha_2-4\alpha_1-3\alpha_2)t + (\alpha_1+1)(\alpha_2) + \frac{1}{3}\right\}$;

Proof. Using the linear property of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$ and lemma (2.2), above lemma can be derived easily. □

Corollary 2.4. *For $n \in \mathbb{N}$, operator $\mathcal{K}_n(\alpha_1,\alpha_2)(.;t)$ satisfies the followings:*
(i) $\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t);t) = -(2+\alpha_2)t + \left(\alpha_1+\frac{3}{2}\right)\frac{3}{2}$;
(ii) $\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t) = 3t(1-t)$.

3. Approximation Properties of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$

3.1. **Local Approximation.**

Theorem 3.1. *Let $\nu \in C(0,1)$, then*

$$\lim_{n\longrightarrow\infty} \mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t) = \nu(t),$$

uniformly on $(0,1)$.

Proof. Using lemma (2.2), we have

$$\lim_{n\longrightarrow\infty} \mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^k;t) = t^k;\ (k=0,\ 1,\ 2),$$

uniformly on (0,1). The required result is immediately given by Korovkin type theorem [22]. □

3.2. **Rate of Convergence.** For $\nu \in C(0,1)$, the modulus of continuity of ν is defined as

$$\omega(\nu,\zeta) = \sup_{|y-t|\leq\zeta}\left\{\sup_{t\in(0,1)}|\nu(y)-\nu(t)|\right\}.$$

Also from [23], we can write

$$|\nu(y)-\nu(t)| \leq \left(1+\frac{(y-t)^2}{\zeta^2}\right)\omega(f,\zeta).$$

By [24], $\exists$ a constant $M>0$ such that

$$K(\nu;\zeta) \leq M\omega_2(\nu,\sqrt{\zeta}),\ \zeta>0, \tag{3.1}$$

where Peetre's functional $K(\nu;\zeta)$ is given by

$$K(\nu;\zeta) = \inf_{f\in C^2[0,1]}\{\|\nu-f\|+\zeta\,\|f''\|\},\ \zeta>0,$$

with $C^2[0,1]=\{\nu\in C[0,1]:\nu',\nu''\in C[0,1]\}$ and

$$\omega_2(\nu,\sqrt{\eta}) = \sup_{0<|h|<\sqrt{\eta}}\left\{\sup_{t,t+2h\in(0,1)}|\nu(t+2h)-2\nu(t+h)+\nu(t)|\right\}$$

is the second ordered modulus of continuity of ν on (0,1).

Theorem 3.2. *Let $t\in(0,1)$ and $\nu\in C[0,1]$. Then we have*

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \leq 2\omega\left(\nu,\sqrt{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)}\right),$$

where $\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)=\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)$, is the second ordered central moment of nth proposed operator.

Proof. For $\nu\in C[0,1]$, we obtain

$$\begin{aligned}\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| &= (n+\alpha_2)\sum_{k=0}^{n}p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}}|\nu(y)-\nu(t)|\,dy\\ &\leq (n+\alpha_2)\sum_{k=0}^{n}p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}}\left(1+\frac{(y-t)^2}{\zeta^2}\right)\omega(f,\zeta)dy\\ &= \left(1+\frac{1}{\zeta^2}\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)\right)\omega(\nu,\zeta).\end{aligned}$$

By taking $\zeta^2=\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)$, we reach the required result. □

Next, we define Hölder's class of functions for $\alpha\in(0,1]$ as follows:

$$\mathcal{H}_\alpha(0,1)=\{\nu\in C(0,1):|\nu(y)-\nu(t)|\leq M_\nu\,|y-t|^\alpha;\ y,t\in(0,1)\}.$$

The following theorem gives the rate of convergence for Hölder's class of functions:

Theorem 3.3. *Let $t\in(0,1)$ and $\nu\in\mathcal{H}_\alpha(0,1)$. Then we have*

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \leq M\left(\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)\right)^{\frac{\alpha}{2}},$$

where $\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)$ is the second ordered central moment of n^{th} proposed operator.

Proof. For $\nu \in \mathcal{H}_\alpha(0,1)$, consider

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| = n\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} |\nu(y)-\nu(t)|\,dy.$$

On applying Hölder's inequality with $p=\frac{2}{\alpha},\ q=\frac{2}{2-\alpha}$ twice, we are led to

$$\begin{aligned}\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| &\le \left\{n\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} |\nu(y)-\nu(t)|^{\frac{2}{\alpha}}\,dy\right\}^{\frac{\alpha}{2}}\\ &\le M\left\{n\sum_{k=0}^{n} p_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}} |y-t|^{2}\,dy\right\}^{\frac{\alpha}{2}}\\ &= M\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)^{\frac{\alpha}{2}},\end{aligned}$$

which completes the result. □

Theorem 3.4. *Let $\nu \in C[0,1]$ and $t \in (0,1)$. Then for all $n \in \mathbb{N}$, ∃ a positive constant M such that*

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| \le M\omega_2\left(\nu;\frac{1}{2}\sqrt{\frac{1}{2}\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}}\right)+2\omega\left(\nu,\left|\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right|\right).$$

Proof. Firstly, we define an auxiliary operator

$$A_n^{(\alpha_1,\alpha_2)}(\psi;t)=\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)+\psi(t). \quad (3.2)$$

Then, we have $A_n^{(\alpha_1,\alpha_2)}(1;t)=1$ and $A_n^{(\alpha_1,\alpha_2)}(y-t;t)=0$. Now Taylor's expansion for $\psi \in C^2[0,1]$ is given by

$$\psi(y)=\psi(t)+(y-t)\psi'(t)+\int_t^y (y-u)\psi''(u)du,\ t\in(0,1).$$

Applying auxiliary operator to both sides of above expansion, we obtain

$$\begin{aligned}A_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi(t) &= \mathcal{K}_n^{(\alpha_1,\alpha_2)}\left(\int_t^y (y-u)g''(u)du;t\right)\\ &\quad -\int_t^{\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)}\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)-u\right)\psi''(u)du.\end{aligned} \quad (3.3)$$

Now,

$$\left|\int_t^y (y-u)\psi''(u)du\right| \le \frac{1}{2}\|\psi''\|(y-t)^2$$

and

$$\begin{aligned}&\left|\int_t^{\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)}\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)-u\right)\psi''(u)du\right|\\ &\qquad \le \frac{1}{2}\|\psi''\|\left(\frac{-2-\alpha_2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)^2\\ &\qquad = \frac{1}{2}\|\psi''\|\left(\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right)^2.\end{aligned}$$

Rewriting equation (3.3), we obtain

$$\begin{aligned}\left|A_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi(t)\right| &\leq \frac{1}{2}\|\psi''\|\,\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)+\frac{1}{2}\|\psi''\|\left(\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right)^2\\ &=\frac{1}{2}\|\psi''\|\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}.\end{aligned} \tag{3.4}$$

Also,

$$\left|A_n^{(\alpha_1,\alpha_2)}(\psi;t)\right|\leq 3\|\psi\|. \tag{3.5}$$

In the view of equations (3.4) and (3.5), we get

$$\begin{aligned}\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right| &=\left| A_n^{(\alpha_1,\alpha_2)}(\nu;t)+\nu\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)-\nu(t)-\nu(t)+\psi(t)\right.\\ &\quad\left.-\psi(t)+A_n^{(\alpha_1,\alpha_2)}(\psi;t)-A_n^{(\alpha_1,\alpha_2)}(\psi;t)\right|\\ &\leq\left|A_n^{(\alpha_1,\alpha_2)}(\nu-\psi;t)-(\nu-\psi)(t)\right|\\ &\quad+\left|A_n^{(\alpha_1,\alpha_2)}(\psi;t)-\psi(t)\right|+\left|\nu\left(\frac{n-2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right)-\nu(t)\right|\\ &\leq 4\|\nu-\psi\|+\frac{1}{2}\|\psi''\|\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}\\ &\quad+\omega(\nu,\zeta)\left(1+\frac{1}{\zeta}\left|\frac{-2-\alpha_2}{n+\alpha_2}t+\frac{1}{n+\alpha_2}\left(\alpha_1+\frac{3}{2}\right)\right|\right).\end{aligned}$$

Taking infimum to RHS of above equation over $\psi\in C^2[0,1]$ and $\zeta=\left|\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right|$, we are led to

$$\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right|\leq 4K\left(\nu;\frac{1}{8}\left\{\gamma_{n,2}^{(\alpha_1,\alpha_2)}(t)+\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)^2\right\}\right)+2\omega\left(\nu,\left|\gamma_{n,1}^{(\alpha_1,\alpha_2)}(t)\right|\right).$$

We reach the required result immediately by using equation (3.1). □

3.3. **Voronovskaya-type Asymptotic Result.** In this subsection, we derive an asymptotic formula for the proposed operator as follows:

Theorem 3.5. *Let $\nu\in C^2[0,1]$. and $t\in(0,1)$. Then, we have*

$$\lim_{n\longrightarrow\infty} n(\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t))=\left\{(-2-\alpha_2)t+\left(\alpha_1+\frac{3}{2}\right)\right\}\nu'(t)+\frac{3}{2}t(1-t)\nu''(t).$$

Proof. From Peano form of remainder of Taylor's expansion, we can write

$$\nu(y)=\nu(t)+(y-t)\nu'(t)+\frac{1}{2}(y-t)^2\nu''(t)+(y-t)^2\epsilon(y,t), \tag{3.6}$$

where $\epsilon(y,t)=\frac{\nu''(z)-\nu''(t)}{2}$ for some z lying between t and y. Also, $\lim_{y\longrightarrow t}\epsilon(y,t)=0$. Now, operating the equation (3.6) by $\mathcal{K}_n(.;t)$, we get

$$\begin{aligned}\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t) &=\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t);t)\nu'(t)+\frac{1}{2}\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)\nu''(t)\\ &\quad+\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\epsilon(y,t)(y-t)^2;t).\end{aligned}$$

Using corollary (2.4) and Cauchy-Schwartz inequality, we can deduce

$$\begin{aligned}\lim_{n\longrightarrow\infty} n(\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)) &= \nu'(t)\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t);t)\\ &+\frac{1}{2}\nu''(t)\lim_{n\longrightarrow\infty} n\mathcal{K}_n^{(\alpha_1,\alpha_2)}((y-t)^2;t)\\ &+\lim_{n\longrightarrow\infty}\left(n\mathcal{K}_n^{(\alpha_1,\alpha_2)}\left((y-t)^2\epsilon(y,t);t\right)\right)\\ &\leq\left\{(-2-\alpha_2)t+\left(\alpha_1+\frac{3}{2}\right)\right\}\nu'(t)+\frac{3}{2}t(1-t)\nu''(t)\\ &+\lim_{n\longrightarrow\infty}\sqrt{n^2\mathcal{K}_n^{(\alpha_1,\alpha_2)}\left((y-t)^4;t\right)}\sqrt{\mathcal{K}_n^{(\alpha_1,\alpha_2)}\left(\epsilon^2(y,t);t\right)}.\end{aligned}\tag{3.7}$$

By theorem (3.1), we have

$$\lim_{n\longrightarrow\infty}\mathcal{K}_n^{(\alpha_1,\alpha_2)}\left(\epsilon^2(y,t);t\right)=\epsilon^2(t,t)=0.$$

Using above equation in (3.7), we are led to the required result. □

3.4. **Weighted Approximation.** Consider a weight function $\sigma(t) = 1+t^2$ on (0,1). Let $B_\sigma(0,1)$ denotes the space of all functions φ on (0,1) such that

$$|\varphi(t)|\leq M_\varphi\sigma(t)$$

and $C_\sigma(0,1)$ be the subspace of all continuous functions in $B_\sigma(0,1)$ endowed with norm $\|.\|_\sigma$ given by

$$\|\varphi\|_\sigma=\sup_{t\in(0,1)}\frac{\varphi(t)}{\sigma(t)}.$$

Next, we prove an inequality and convergence for the operator $\mathcal{K}_n(.;t)$ in weighted space as follows:

Lemma 3.1. *Let $\nu\in C_\sigma(0,1)$. Then following inequality holds for $\mathcal{K}_n(\nu;t)$*

$$\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)\right\|_\sigma\leq M\|\nu\|_\sigma.$$

Proof. By using definition of proposed operator, we may write

$$\begin{aligned}\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)\right\|_\sigma &= \sup_{t\in(0,1)}\frac{\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)\right|}{\sigma(t)}\\ &\leq\|\nu\|_\sigma\sup_{t\in(0,1)}\frac{n}{1+t^2}\sum_{k=0}^{n}b_{n,k}(t)\int_{\frac{(k+\alpha_1)}{(n+\alpha_2)}}^{\frac{(k+1+\alpha_1)}{(n+\alpha_2)}}(1+u^2)du\\ &=\|\nu\|_\sigma\sup_{t\in(0,1)}\frac{1}{1+t^2}\{1+\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^2;t)\}\leq M\|\nu\|_\sigma.\end{aligned}$$

□

Theorem 3.6. *For $\nu\in C_\sigma(0,1)$, the newly modified operator $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(.;t)$ satisfies*

$$\lim_{n\longrightarrow\infty}\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right\|_\sigma=0.$$

Proof. From lemma (2.2), we obtain

$$\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t)-t\right\|_\sigma=\sup_{t\in(0,1)}\frac{\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y;t)-t\right|}{1+t^2}\leq\frac{1}{n+\alpha_2}\left|\alpha_1-\alpha_2-\frac{1}{2}\right|.$$

Also,

$$\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^2;t)-t^2\right\|_\sigma = \sup_{t\in(0,1)} \frac{\left|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\nu;t)-\nu(t)\right|}{1+t^2}$$
$$\leq \frac{1}{(n+\alpha_2)^2}\left\{n(2\alpha_1-2\alpha_2-1)+\alpha_1^2-\alpha_2^2-\alpha_1+\frac{1}{3}\right\}.$$

Thus, in limiting condition, we can write

$$\lim_{n\longrightarrow\infty}\left\|\mathcal{K}_n^{(\alpha_1,\alpha_2)}(y^j;t)-t^j\right\| = 0;\ j=0,1,2.$$

Then, the weighted convergence holds for all $\nu \in C_\sigma(0,1)$ from the results given by Gadjiev [25]. □

4. Graphical Analysis

Now, we introduce some simulation results in order to substantiate the convergence behavior of $\mathcal{K}_n^{(\alpha_1,\alpha_2)}(\psi;t)$ for continuous function ψ by using MATLAB.

To test the approximation behavior of newly defined operators, let us consider a polynomial function $\psi(t)=t^3-t^2+\frac{t}{10}+0.1$. As the new sequence of operators is defined on $(0,1)$, so for that we will consider approximation over equally spaced grids in $[0.0005, 0.9995]$. Figure (1) and (2) shows the approximation and error in the approximation by proposed operator to $\psi(t)$ respectively for $n=20,50$ and 100 at $\alpha_1=\alpha_2=0$.

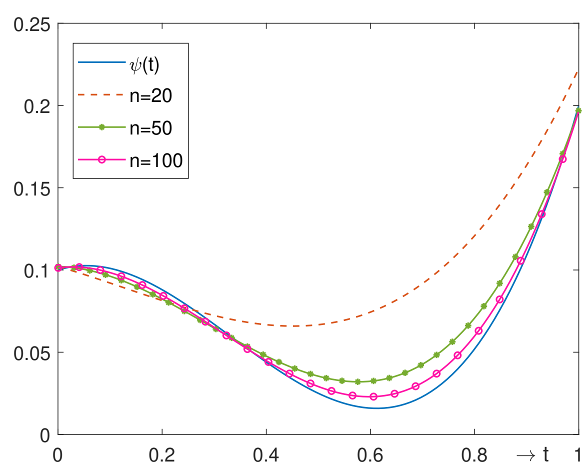

Figure 1. Approximation by proposed operator $\mathcal{K}_n^{(0,0)}(\psi;t)$ to ψ at different values of n.

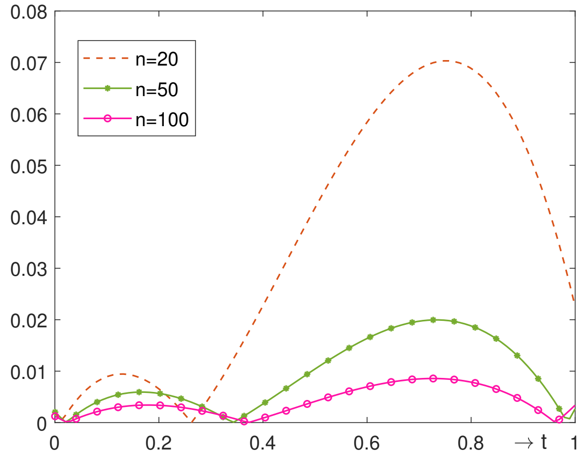

FIGURE 2. Error in the approximation by proposed operator $\mathcal{K}_n^{(0,0)}(\psi;t)$ to ψ at different values of n.

5. CONCLUSION

In this manuscript, we presented modified-Bernstein-Kantrovinch-Stancu operators. We discussed their rate of convergence, asymptotic formula, and weighted approximation of these operators with polynomial growth. Also, we included some numerical simulations in order to test the newly defined operators.

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REFERENCES

1. S. Bernstein, Démonstration du théoréme de weierstrass fondée surle calculde probabilités, Communications of the Kharkov Mathematical Society, 1913.
2. F. Usta, On new modification of bernstein operators: theory and applications, Iranian Journal of Science and Technology, Transactions A: Science, 2020, 44(4), 1119–1124.
3. M. Sofyalıoğlu, K. Kanat, B. Cekim, Parametric generalization of the modified Bernstein operators, Filomat, 2022, 36(5), 1699–1709.
4. Q.B. Cai, B.Y. Lian, G. Zhou, Approximation properties of λ-Bernstein operators, Journal of inequalities and applications, 2018: 61 (2018), 1–11.
5. N. L. Braha, T. Mansour, H.M. Srivastava, A parametric generalization of the Baskakov-Schurer-Szász-Stancu approximation operators, Symmetry, 2021, 13(6), 980.
6. P. N. Agrawal, B. Baxhaku, R. Shukla, On q-analogue of a parametric generalization of Baskakov operators, Mathematical Methods in the Applied Sciences, 2021, 44(7), 5989–6004.
7. A. Kajla, M. Mursaleen, T. Acar, Durrmeyer-type generalization of parametric Bernstein operators, Symmetry 2020, 12(7), Article ID: 1141.
8. S. A. Mohiuddine, Approximation by bivariate generalized Bernstein-Schurer operators and associated GBS operators, Advances in Difference Equations, (2020) 2020:676.
9. S. A. Mohiuddine, N. Ahmad, F. Özger, A. Alotaibi, B. Hazarika, Approximation by the parametric generalization of Baskakov-Kantorovich operators linking with Stancu operators, Iranian Journal of Science and Technology, Transactions A: Science 45 (2021) 593–605.
10. S. A. Mohiuddine, K. K. Singh, A. Alotaibi, On the order of approximation by modified summation-integral-typeoperators based on two parameters, Demonstratio Mathematica (2023), 56: 20220182.
11. J. Yadav, S. A. Mohiuddine, A. Kajla, A. Alotaibi, α-Bernstein-integral type operators, Bulletin of the Iranian Mathematical Society, 2023, 49, Article 59.

12. S. Mohiuddine, F. Özger, Approximation of functions by stancu variant of Bernstein–Kantorovich operators based on shape parameter α, Revista de la Real Academia de Ciencias Exactas, Fásicas y Naturales. Serie A. Matemáticas, 2020, 114(2), 1–17.
13. S. Sonker, P. Moond, Rate of convergence of parametrically generalized bivariate Baskakov-Stancu operators, Filomat, 37(27), 2023, 9197–9214.
14. L.V. Kantorovich, Sur certains développements suivant les polynˆomes de la forme de s. Bernstein, I, II, CR Acad. URSS 1930, 563(568), 595–600.
15. A. Kumar, A. Senapati, T. Som, Approximation by szász–kantorovich type operators associated with d-symmetric d-orthogonal polynomials of brenke type, The Journal of Analysis, 2023, 1–17.
16. M. Raiz, R.S. Rajawat, L.N. Mishra, V. N. Mishra, Approximation on bivariate of durrmeyer operators based on beta function, The Journal of Analysis, 2023, 1–23.
17. B. B. Jena, S. K. Paikray, S. A. Mohiuddine and V. N. Mishra, Relatively equi-stiaistical convergence via deferred Nörlund mean based on difference operator of fractional order and related approximation theorems, AIMS Mathematics, 5 (1), 2019, 650-672.
18. B.B. Jena, S. K. Paikray, H. Dutta, Statistically Riemann integrable and summable sequence of functions via deferred Cesàro mean, Bulletin of Iranian Mathematical Society, (2022), 48 (4), 1293-1309.
19. S. K. Paikray, P. Parida, S. A. Mohiuddine, A certain class of relatively equi-statistical fuzzy approximation theorems, European Journal of Pure and Applied Mathematics 2020, 13, 1212–1230.
20. H. M. Srivastava, B. B. Jena, S. K. Paikray, Some Korovkin-type approximation theorems associated with a certain deferred weighted statistical Riemann- integrable sequence of functions, Axioms, 11 (2022), 1–11.
21. H. M. Srivastava, B. B. Jena, S. K. Paikray, A certain class of equi-statistical convergence in the sense of the deferred power-series method, Axioms 12(2023), Article ID: 964.
22. P. Korovkin, On convergence of linear positive operators in the space of continuous functions,Doklady Akademii Nauk SSSR (DAN SSSR), 1953, 90, 961–964.
23. F. Altomare, M. Campiti, Korovkin-type Approximation Theory and Its Applications, Walter de Gruyter, New York, 2011.
24. R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Heidelberg, 1993.
25. A. Gadjiev, Theorems of the type of Korovkin's theorems. Matematicheskie Zametki, 1976, 20(5), 781–786.