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STRONG CONVERGENCE ALGORITHMS FOR EQUILIBRIUM PROBLEMS WITHOUT MONOTONICITY

BUI VAN DINH[1,*], NGUYEN THI THANH HA[1],
NGUYEN NGOC HAI[2], AND TRAN THI HUYEN THANH[1]

[1]Department of Mathematics, Faculty of Information Technology,
Le Quy Don Technical University, Hanoi, Vietnam
[2]Department of Scientific Fundamentals, Vietnam Trade Union University, Hanoi, Vietnam

ABSTRACT. In this paper, we introduce two new linesearch algorithms for solving a non-monotone equilibrium proplem in a real Hilbert space. Each method can be considered as a combination of the extragradient method with linesearch and shrinking projection methods. Then we show that the iterative sequence generated by each method converges strongly to a solution of the considered problem. A numerical example is also provided.

KEYWORDS: Non-monotonicity; equilibria; shrinking projection methods; strong convergence; Armijo linesearch; Hilbert space.
AMS Subject Classification: 90C25; 90C33; 65K10; 65K15

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. The strong convergence and the weak convergence in the Hilbert space $\mathbb{H}$ are denoted by '$\to$' and '$\rightharpoonup$', respectively.
Let Ω be an open convex subset in $\mathbb{H}$ containing a nonempty closed convex C, and $f : \Omega \times \Omega \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for every $x \in C$.

The equilibrium problem (shortly $\text{EP}(C, f)$), in the sense of Blum, Muu and Oettli [4, 21] (see also [15]), consists of finding $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C,$$

and its associated equilibrium problem

$$\text{Find } y^* \in C \text{ such that } f(x, y^*) \leq 0, \ \forall x \in C. \quad (1)$$

* *Corresponding author.*
Email address: vandinhb@gmail.com, nttha711@gmail.com, hainn@dhcd.edu.vn, thanh0712@gmail.com.

Note that problem (1) is called as the Minty equilibrium problem (MEP(C, f) for short) due to M. Castellani and M. Giuli [6]. We denote the solution set of EP(C, f) and MEP(C, f) by S_E and S_M, respectively.

Although problem EP(C, f) has a simple formulation, it encompasses, among its particular cases, many important problems in applied mathematics: convex optimization problem, variational inequality problem, fixed point problem, saddle point problem, Nash equilibrium problem in noncooperative game, and others; see, for example, [3, 4, 21], and the references quoted therein.

Recall that a bifunction f is said to be monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C,$$

and pseudo-monotone on C if

$$\forall x, y \in C, f(x, y) \geq 0 \implies f(y, x) \leq 0.$$

Solution methods for equilibrium problems with monotone or pseudo-monotone bifunctions [1, 9, 12, 13, 16, 17, 19, 22, 29] have been studied extensively by many researchers and they have been usually extended from those for variational inequality problems and other related problems [5, 14].

For obtaining a solution of a non Lipschitz type and pseudo-monotone equilibrium problem in Euclidean space, Tran *et al.* [27] proposed to combine extragradient algorithms [18] with Armijo linesearch rule [2] to get the following algorithm.

Algorithm 1.

Initialization. Pick $x^0 \in C$, $\eta, \mu \in (0, 1)$; $0 < \rho$;
$\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$.

Iteration k (k = 0, 1, 2, ...). Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min \left\{ f(x^k, y) + \frac{1}{2\rho} \|y - x^k\|^2 : \ y \in C \right\} \qquad CP(x^k)$$

to obtain its unique solutions y^k.
If $y^k = x^k$, then stop. Otherwise, go to Step 2.

Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\mu}{2\rho} \|x^k - y^k\|^2. \end{cases}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$.

Step 3. Select $w^k \in \partial_2 f(z^k, x^k)$, take $\sigma_k = \frac{f(z^k, x^k)}{\|w^k\|^2}$, and compute $x^{k+1} = P_C(x^k - \gamma_k.\sigma_k.w^k)$, and go to Step 1 with k is replaced by $k + 1$.

They showed that the sequence $\{x^k\}$ generated by the above algorithm converges to a solution of EP(C, f) provided that $S_E \neq \emptyset$.

In addition, to find a fixed point of a non-expansive self mapping T in real Hilbert spaces, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. Takahashi *et al.* [26] introduced the following iterative method, known as the shrinking projection method, which is the following:

Algorithm 2

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\alpha \in [0, 1)$, $\{\alpha_k\} \subset [0, \alpha]$ and set $C_0 = C$.

Iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:
Step 1. Compute

$$y^k = \alpha_k x^k + (1 - \alpha_k)Tx^k,$$
$$C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}.$$

Step 2. Compute $x^{k+1} = P_{C_{k+1}}(x^g)$, and go to Step 1 with k is replaced by $k+1$.

They proved that $\{x^k\}$ generated by Algorithm 2 converges strongly to $x^* = P_{Fix(T)}(x^g)$. In spired by above algorithms and recent works [7, 10, 25, 31], in this paper, we introduce algorithms for solving an equilibrium problem in a real Hilbert space without pseudo-monotonicity assumption of the bifunctions by combining Algorithm 1 with Algorithm 2. Then, we proved that the sequences generated by proposed algorithms strongly converges to a solution of S_E.

The rest of paper is organized as follows. The next section contains some preliminaries on the metric projection and equilibrium problems. The third section is devoted to introduce two algorithms for EP(C, f) and their strong convergence. In the last section, we present an application of the proposed algorithm for Nash-Cournot equilibrium models of electricity markets and its implementation.

2. Preliminaries

In this paper, we denote the metric projection operator on C by P_C, that is

$$P_C(x) \in C : \ \|x - P_C(x)\| \leq \|y - x\|, \ \forall y \in C.$$

It is well known that the projection operator onto a closed convex has the following properties.

Lemma 2.1. *Suppose that C is a nonempty closed convex subset in $\mathbb{H}$. Then*

(a) *$P_C(x)$ is singleton and well defined for every x;*
(b) *$z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0, \forall y \in C$;*
(c) *$\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C$.*

Definition 2.1. *A bifunction $\varphi : C \times C \to \mathbb{R}$ is said to be jointly weakly continuous on $C \times C$ if for all $x, y \in C$ and $\{x^k\}$, $\{y^k\}$ are two sequences in C converging weakly to x and y respectively, then $\varphi(x^k, y^k)$ converges to $\varphi(x, y)$.*

In the sequel, we need the following blanket assumptions

$(\mathcal{A}_1)$ $f(x, .)$ is convex on Ω for every $x \in C$;
$(\mathcal{A}_2)$ f is jointly weakly continuous on $\Omega \times \Omega$.

For each $z, \ x \in C$, by $\partial_2 f(z, x)$ we denote the subdifferential of the convex function $f(z, .)$ at x, i.e.,

$$\partial_2 f(z, x) := \{w \in \mathbb{H} : f(z, y) \geq f(z, x) + \langle w, y - x\rangle, \ \forall y \in C\}.$$

In particular,

$$\partial_2 f(z, z) = \{w \in \mathbb{H} : f(z, y) \geq \langle w, y - z\rangle, \ \forall y \in C\}.$$

The next lemma can be considered as an infinite-dimensional version of Theorem 24.5 in [24]

Lemma 2.2. [28, Proposition 4.3] *Let $f : \Omega \times \Omega \to \mathbb{R}$ be a function satisfying conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. Let $\bar{x}, \bar{y} \in \Omega$ and $\{x^k\}$, $\{y^k\}$ be two sequences in Ω converging weakly to $\bar{x}, \bar{y}$, respectively. Then, for any $\epsilon > 0$, there exist $\eta > 0$ and $k_\epsilon \in \mathbb{N}$ such that*

$$\partial_2 f(x^k, y^k) \subset \partial_2 f(\bar{x}, \bar{y}) + \frac{\epsilon}{\eta} B,$$

for every $k \geq k_\epsilon$, where B denotes the closed unit ball in $\mathbb{H}$.

Lemma 2.3. [20] *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$, *a point* $x^* \in C$ *is a solution of EP*(C, f) *if and only if it is a solution to the equilibrium problem:*

$$\text{Find } x^* \in C : f(x^*, y) + \frac{1}{2\rho}\|y - x^*\|^2 \geq 0, \ \forall y \in C. \qquad (AEP)$$

Lemma 2.4. [30] *Let* C *be a nonempty closed convex subset of* $\mathbb{H}$. *Let* $\{x^k\}$ *be a sequence in* $\mathbb{H}$ *and* $u \in \mathbb{H}$. *If any weak limit point of* $\{x^k\}$ *belongs to* C *and*

$$\|x^k - u\| \leq \|u - P_C(u)\|, \ \forall k.$$

Then $x^k \to P_C(u)$.

Lemma 2.5. [10] *Under assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$, *if* $\{z^k\} \subset C$ *is a sequence such that* $\{z^k\}$ *converges strongly to* $\bar{z}$ *and the sequence* $\{w^k\}$, *with* $w^k \in \partial_2 f(z^k, z^k)$, *converges weakly to* $\bar{w}$, *then* $\bar{w} \in \partial_2 f(\bar{z}, \bar{z})$.

Lemma 2.6. [11] *Let the equilibrium bifunction* f *satisfy the assumptions* $(\mathcal{A}_1)$ *on* Ω *and* $(\mathcal{A}_2)$ *on* C, *and* $\{x^k\} \subset C$, $0 < \underline{\rho} \leq \bar{\rho}$, $\{\rho_k\} \subset [\underline{\rho}, \ \bar{\rho}]$. *Consider the sequence* $\{y^k\}$ *defined as follows*

$$y^k = \arg\min \Big\{ \varphi(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : \ y \in C \Big\}.$$

Then, if $\{x^k\}$ *is bounded, then* $\{y^k\}$ *is also bounded.*

3. Main Results

Now we are in a position to present the first algorithm for solving a non-monotone equilibrium problem in a Hilbert space.

Algorithm 3.

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\eta, \mu \in (0, 1)$, $0 < \rho \leq \bar{\rho}$, $\{\rho_k\} \subset [\rho, \ \bar{\rho}]$, $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$. and set $C_0 = C$.

At each iteration k $(k = 0, 1, 2, ...)$. Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min \Big\{ f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 : \ y \in C \Big\} \qquad CP(x^k)$$

to obtain its unique solution y^k. If $y^k = x^k$, then stop. Otherwise, do Step 2.

Step 2. (The first Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \frac{\mu}{2\rho_k}\|x^k - y^k\|^2. \end{cases} \qquad (2)$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$.

Step 3. Select $w^k \in \partial_2 f(z^k, x^k)$, and compute $u^k = P_C(x^k - \gamma_k \sigma_k w^k)$, where $\sigma_k = \frac{f(z^k, x^k)}{\|w^k\|^2}$.

Step 4. Compute

$$x^{k+1} = P_{C_{k+1}}(x^g),$$

where $C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}$, and go to Step 1 with k is replaced by $k + 1$.

Remark 3.1. *If* $y^k = x^k$ *then* x^k *is a solution to EP*(C, f).

Before proving the convergence of Algorithm 1, let us recall the following lemma which was proved in [27].

Lemma 3.1. [27] *Suppose that the bifunction f satisfies assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then we have:*

(a) *The linesearch is well-defined;*
(b) $f(z^k, x^k) > 0$;
(c) $0 \notin \partial_2 f(z^k, x^k)$;
(d) *In addition, if $S_M \neq \emptyset$, then*

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|w^k\|)^2, \quad \textit{for all } x^* \in S_M. \tag{3}$$

Lemma 3.1 implies that the sequence $\{x^k\}$ generated by Algorithm 1 is well-defined. The following theorem establishes the strong convergence of $\{x^k\}$ to a solution of EP(C, f).

Theorem 3.2. *Suppose that bifunction f satisfies assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$. If the set S_M is nonempty, then the sequence $\{x^k\}$, $\{u^k\}$ generated by Algorithm 3 converge strongly to a solution x^* of EP(C, f).*

Proof. Take $\bar{x} \in S_M \subset C = C_0$. From Lemma 3.1, we have

$$\|u^k - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|w^k\|)^2. \tag{4}$$

Since $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$, we get

$$\|\bar{x} - u^k\| \leq \|\bar{x} - x^k\|. \tag{5}$$

By induction, we can conclude that $\bar{x} \in C_k$ for all k.

By Step 4, $x^k = P_{C_k}(x^g)$, we have

$$\|x^k - x^g\| \leq \|x - x^g\|, \ \forall x \in C_k, \tag{6}$$

so,

$$\|x^k - x^g\| \leq \|\bar{x} - x^g\|, \ \forall k. \tag{7}$$

Therefore, $\{x^k\}$ is bounded. Together with Lemma 2.2, $\{w^k\}$ is bounded. Combining with (5) we have $\{u^k\}$ is also bounded.
Since, $x^{k+1} \in C_k$ and (6), we have

$$\|x^k - x^g\| \leq \|x^{k+1} - x^g\|, \ \forall k. \tag{8}$$

Because $\{x^k\}$ is bounded, we get

$$\lim_{k\to\infty} \|x^k - x^g\| = \tau \geq 0. \tag{9}$$

In addition,

$$\begin{aligned}
\|x^{k+1} - x^k\|^2 &= \|x^{k+1} - x^g + x^g - x^k\|^2 \\
&= \|x^{k+1} - x^g\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^g, x^g - x^k\rangle \\
&= \|x^{k+1} - x^g\|^2 + \|x^g - x^k\|^2 + 2\langle x^{k+1} - x^k, x^g - x^k\rangle - 2\|x^g - x^k\|^2 \\
&\leq \|x^{k+1} - x^g\|^2 - \|x^k - x^g\|^2,
\end{aligned}$$

where the last inequality follows from the fact that $x^k = P_{C_k}(x^g)$ and $x^{k+1} \in C_k$, then $\langle x^{k+1} - x^k, x^g - x^k\rangle \leq 0$.
From (9), we obtain

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{10}$$

Because $x^{k+1} \in C_{k+1}$, one has

$$\begin{aligned}\|x^k - u^k\| &\leq \|x^k - x^{k+1}\| + \|x^{k+1} - u^k\| \\ &\leq 2\|x^k - x^{k+1}\|\end{aligned}$$

Take into account with (10) we get

$$\lim_{k\to\infty} \|u^k - x^k\| = 0. \tag{11}$$

Next, we show that $\{x^k\}$, $\{u^k\}$ converge strongly to $x^* = P_{\cap_{k=0}^{\infty} C_k}(x^g)$.

It is clear that C_k is nonempty, closed and convex set, it is also weakly closed. Since $C_{k+1} \subset C_k, \forall k$ and $x^k \in C_k$, $x^k \in C_{k_0}$ for all $k \geq k_0$. Let $\hat{x}$ be any weak accumulation point of the sequence $\{x^k\}$, i.e., there exists $\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_j} \rightharpoonup \hat{x}$ as $j \to \infty$. Since $\{x^{k_j}\} \subset C_{k_i}, \forall j \geq i$ and the weak closedness of C_{k_i}, it implies that $\hat{x} \in C_{k_i}, \forall i$. Hence $\hat{x} \in C_k, \forall k$, or $\hat{x} \in \cap_{k=0}^{\infty} C_k$.

Set $x^* = P_{\cap_{k=0}^{\infty} C_k}(x^g)$. From (7) we have,

$$\|x^k - x^g\| \leq \|x^* - x^g\|, \ \forall k. \tag{12}$$

We can conclude that x^k converges strongly to x^* by Lemma 2.4. Together with (11) we have u^k also converges strongly to x^*.

Next, we show that x^* solves EP(C, f).

In view of (4), it yields

$$\gamma_k(2 - \gamma_k)(\sigma_k\|w^k\|)^2 \leq \|x^k - u^k\| \big[\|x^k - \bar{x}\| + \|u^k - \bar{x}\|\big]. \tag{13}$$

Since $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0, 2)$, and (11), we get from (13) that

$$\lim_{k\to\infty} \sigma_k\|w^k\| = 0. \tag{14}$$

Since $\{x^k\}$ is bounded and Lemma 2.6, $\{y^k\}$ is bounded. Consequently, $\{z^k\}$ is also bounded. Using Lemma 2.5, $\{w^k\}$ is bounded, In view of (14) yields

$$\lim_{k\to\infty} f(z^k, x^k) = \lim_{k\to\infty} [\sigma_k\|w^k\|]\|w^k\| = 0. \tag{15}$$

We have

$$\begin{aligned}0 = f(z^k, z^k) &= f(z^k, (1 - \eta_k)x^k + \eta_k y^k) \\ &\leq (1 - \eta_k) f(z^k, x^k) + \eta_k f(z^k, y^k),\end{aligned}$$

so, we get from (2) that

$$\begin{aligned}f(z^k, x^k) &\geq \eta_k[f(z^k, x^k) - f(z^k, y^k)] \\ &\geq \frac{\mu}{2\rho_k}\eta_k\|x^k - y^k\|^2.\end{aligned}$$

Combining with (15) one has

$$\lim_{k\to\infty} \eta_k\|x^k - y^k\|^2 = 0. \tag{16}$$

We now consider two distinct cases:

Case 1. $\limsup_{k\to\infty} \eta_k > 0$.

Then there exists $\bar{\eta} > 0$ and a subsequence $\{\eta_{k_i}\} \subset \{\eta_k\}$ such that $\eta_{k_i} > \bar{\eta}$, $\forall i$, and from (16), one has

$$\lim_{i\to\infty} \|x^{k_i} - y^{k_i}\| = 0. \tag{17}$$

Remember that $x^k \to x^*$ and (17), it implies that $y^{k_i} \to x^*$ as $i \to \infty$.
By definition of y^{k_i} we have

$$f(x^{k_i}, y) + \frac{1}{2\rho_{k_i}}\|y - x^{k_i}\|^2 \geq f(x^{k_i}, y^{k_i}) + \frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2, \ \forall y \in C. \tag{18}$$

Without loss of generality, we assume that $\lim_{i\to\infty} \rho_{k_i} = \rho^*$. Letting $i \to \infty$, by jointly weak continuity of f and $x^{k_i} \to x^*$, $y^{k_i} \to x^*$, we obtain in the limit that

$$f(x^*, y) + \frac{1}{2\rho^*}\|y - x^*\|^2 \geq 0.$$

By Lemma 2.3, we conclude that

$$f(x^*, y) \geq 0, \ \forall y \in C.$$

Therefore, x^* is a solution of EP(C, f).

Case 2. $\lim_{k\to\infty} \eta_k = 0$.
Since $\{y^k\}$ is bounded, it implies that there exists $\{y^{k_i}\} \subset \{y^k\}$ such that $y^{k_i} \rightharpoonup \bar{y}$ as $i \to \infty$.

By the definition of y^{k_i}, we have

$$f(x^{k_i}, y^{k_i}) + \frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2 \leq 0. \tag{19}$$

In the other hand, by the Armijo linesearch rule (2), for $m_{k_i} - 1$, we have

$$f(z^{k_i, m_{k_i}-1}, x^{k_i}) - f(z^{k_i, m_{k_i}-1}, y^{k_i}) < \frac{\mu}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2. \tag{20}$$

Combining with (19) we get

$$f(x^{k_i}, y^{k_i}) \leq -\frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2 \leq \frac{1}{\mu}\big[f(z^{k_i, m_{k_i}-1}, y^{k_i}) - f(z^{k_i, m_{k_i}-1}, x^{k_i})\big]. \tag{21}$$

According to the linesearch rule, $z^{k_i, m_{k_i}-1} = (1-\eta^{m_{k_i}-1})x^{k_i} + \eta^{m_{k_i}-1}y^{k_i}$, $\eta^{m_{k_i}-1} \to 0$. Since x^{k_i} converges strongly to x^*, y^{k_i} converges weakly to $\bar{y}$, it implies that $z^{k_i, m_{k_i}-1}$ converges strongly to x^* as $i \to \infty$. In addition, $\{\frac{1}{\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2\}$ is bounded, without loss of generality, we may assume that $\lim_{i\to+\infty} \frac{1}{\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2$ exists. Hence, we get in the limit from (21) that

$$f(x^*, \bar{y}) \leq -\lim_{i\to+\infty} \frac{1}{2\rho_{k_i}}\|y^{k_i} - x^{k_i}\|^2 \leq \frac{1}{\mu} f(x^*, \bar{y}).$$

Therefore, $f(x^*, \bar{y}) = 0$ and $\lim_{i\to+\infty} \|y^{k_i} - x^{k_i}\|^2 = 0$. By the Case 1, we get that x^* is a solution of EP(C, f).

□

Replacing the linesearch rule 2 by the other one, we get the following algorithm.

Algorithm 4.

Initialization. Pick $x^0 = x^g \in C$, choose parameters $\eta, \mu \in (0,1)$, $0 < \rho \leq \bar{\rho}$, $\{\rho_k\} \subset [\rho,\ \bar{\rho}]$, $\gamma_k \in [\gamma, \bar{\gamma}] \subset (0,2)$. and set $C_0 = C$.

At each iteration k ($k = 0,1,2,...$). Having x^k do the following steps:

Step 1. Solve the strongly convex program

$$\min\left\{f(x^k, y) + \frac{1}{2\rho_k}\|y - x^k\|^2 :\ y \in C\right\} \qquad CP(x^k)$$

to obtain its unique solution y^k. If $y^k = x^k$, then stop. Otherwise, do Step 2.

Step 2. (The second Armijo linesearch rule) Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^k \\ f(z^{k,m}, y^k) + \frac{\mu}{2\rho_k}\|x^k - y^k\|^2 \leq 0. \end{cases} \tag{22}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m_k}$. If $0 \in \partial_2 f(z^k, z^k)$, then Stop. Otherwise, go to Step 3.

Step 3. Select $w^k \in \partial_2 f(z^k, z^k)$, and compute $u^k = P_C(x^k - \gamma_k \sigma_k w^k)$, where $\sigma_k = \frac{f(z^k, x^k)}{\|w^k\|^2}$.

Step 4. Compute

$$x^{k+1} = P_{C_{k+1}}(x^g),$$

where $C_{k+1} = \{x \in C_k : \|x - u^k\| \leq \|x - x^k\|\}$, and go to Step 1 with k is replaced by $k+1$.

Remark 3.2.
- *If $y^k = x^k$ then x^k is a solution to EP(C, f);*
- *If $0 \in \partial_2 f(z^k, z^k)$, then z^k is a solution to EP(C, f).*

Lemma 3.3. [27] *Suppose that the bifunction f satisfies assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, then we have:*

(a) *The linesearch is well-defined;*
(b) $f(z^k, y^k) < 0$*;*
(c) *If $S_M \neq \emptyset$, then*

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \gamma_k(2-\gamma_k)(\sigma_k\|w^k\|)^2, \quad \text{forall } x^* \in S_M. \tag{23}$$

Lemma 3.3 implies that the sequence $\{x^k\}$ generated by Algorithm 4 is well-defined.

The following theorem show us the convergence of Algorithm 4.

Theorem 3.4. *Suppose that bifunction f satisfies assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$. If the set S_M is nonempty, then the sequence $\{x^k\}$, $\{u^k\}$ generated by Algorithm 4 converge strongly to a solution x^* of EP(C, f).*

Proof. This theorem can be proved by the same arguments as in Theorem 3.2 so we obmit it.

4. Numerical examples

To illustrate the proposed algorithms, in this section, we consider an equilibrium problem arising in Nash-Cournot oligopolistic electricity market equilibrium model [8, 27]. In this model, there are n^c companies, each company i may possess I_i generating units. Let n^g be number of all generating units and x be the vector whose entry x_i stands for the power generating by unit i and $\sigma = \sum_{i=1}^{n^g} x_i$. We assume that the price p is a decreasing affine function of σ, that is

$$p(x) = 378.4 - 2\sum_{i=1}^{n^g} x_i = p(\sigma).$$

Then the profit made by company i is given by

$$f_i(x) = p(\sigma)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j),$$

where $c_j(x_j)$ is the cost for generating x_j given by

$$c_j(x_j) := \max\{c_j^0(x_j), c_j^1(x_j)\}$$

with

$$c_j^0(x_j) := \frac{\alpha_j^0}{2}x_j^2 + \beta_j^0 x_j + \gamma_j^0,\ c_j^1(x_j) := \alpha_j^1 x_j + \frac{\beta_j^1}{\beta_j^1+1}\gamma_j^{-1/\beta_j^1}(x_j)^{(\beta_j^1+1)/\beta_j^1},$$

where $\alpha_j^k, \beta_j^k, \gamma_j^k$ $(k = 0, 1)$ are given parameters.

Denote $x_j^{\min}$ and $x_j^{\max}$ is the lower and upper bounds for the power generating by the unit j. Then the strategy set of the model takes the form

$$C := \{x = (x_1, ..., x^{n^g})^T :\ x_j^{\min} \le x_j \le x_j^{\max},\ \forall j\}.$$

By setting $q^i := (q_1^i, ..., q_{n^g}^i)^T$ with

$$q_j^i = \begin{cases} 1 & \text{if } j \in I_i \\ 0 & \text{if } j \notin I_i \end{cases},$$

and define

$$A := 2\sum_{i=1}^{n^c}(1-q^i)(q^i)^T, \quad B := 2\sum_{i=1}^{n^c} q^i(q^i)^T, \tag{24}$$

$$a := -387.4\sum_{i=1}^{n^c} q^i, \text{and } c(x) := \sum_{j=1}^{n^g} c_j(x_j). \tag{25}$$

Then this oligopolistic equilibrium model can be written by the following equilibrium problem EP(C, f) (see [23, Page 155]):

Find $x^* \in C : f(x^*, y) = [(A+B)x^* + By + a]^T(y - x^*) + c(y) - c(x^*) \ge 0,\ \forall y \in C.$

It can be seen that, the matrix A is not positive semidefinite and $f(x, y) + f(y, x) = -(y - x)^T A(y - x)$, hence the bifunction f is nonmonotone and nonsmooth.

We test Algorithm 3 for this problem with corresponds to the first model in [8] where $n^c = 3$, and the parameters are given in the following tables:

We implement Algorithm 1 in Matlab R2014a running on a Laptop with Intel(R) Core(TM) i5-3230M CPU@2.60 GHz with 4 GB Ram. To terminate the Algorithm, we use the stopping criteria $\frac{\|x^{k+1}-x^k\|}{\max\{1,\|x^k\|\}} \le \epsilon$ with a tolerance $\epsilon = 10^{-3}$. The computation results are reported in Table 3 with some starting points and regularized parameters.

Com.	Gen.	$x^g_{\min}$	$x^g_{\max}$	$x^c_{\min}$	$x^c_{\max}$
1	1	0	80	0	80
2	2	0	80	0	130
2	3	0	50	0	130
3	4	0	55	0	125
3	5	0	30	0	125
3	6	0	40	0	125

TABLE 1. The lower and upper bounds of the power generation of the generating units and companies.

Gen.	α^0_j	β^0_j	γ^0_j	α^1_j	β^1_j	γ^1_j
1	0.0400	2.00	0.00	2.0000	1.0000	25.0000
2	0.0350	1.75	0.00	1.7500	1.0000	28.5714
3	0.1250	1.00	0.00	1.0000	1.0000	8.0000
4	0.0116	3.25	0.00	3.2500	1.0000	86.2069
5	0.0500	3.00	0.00	3.0000	1.0000	20.0000
6	0.0500	3.00	0.00	3.0000	1.0000	20.0000

TABLE 2. The parameters of the generating unit cost functions.

Iter(k)	ρ	x^k_1	x^k_2	x^k_3	x^k_4	x^k_5	x^k_6	Cpu(s)
0 691	0.1	0 46.6583	0 32.0728	0 15.0832	0 21.9862	0 12.3870	0 12.4071	136.0017
0 1166	0.5	0 46.6541	0 32.0750	0 15.0845	0 21.9224	0 12.4209	0 12.4389	151.3664
0 847	0.9	0 46.6440	0 31.9437	0 15.2014	0 21.6995	0 12.5953	0 12.4952	162.2410
0 629	0.1	30 46.6531	20 32.1041	10 15.0509	15 22.0089	10 12.4180	10 12.3606	122.1176
0 711 504	0.5	30 46.6416	20 31.9645	10 15.1811	15 21.6667	10 12.5630	10 12.5629	135.5798
0 711	0.9	30 46.6482	20 32.0263	10 15.1150	15 21.6827	10 12.5460	10 12.5657	147.0316

TABLE 3. Results computed with some starting points and regularized parameters.

5. CONCLUSION

. We have introduced two projection algorithms for finding a solution of a nonmonotone equilibrium problem in a real Hilbert space. The strong convergence of the proposed algorithms are obtained. We then have applied a proposed algorithm for a Nash-Cournot oligopolistic equilibrium model of electricity market. Some computation results are reported.

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