I. K. Argyros & S. George, J. Nonlinear Anal. Optim. Vol. 9(2) (2018), 129-138

Journal of Nonlinear Analysis and Optimization
Volume 9(2) (2018)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

BALL COMPARISON OF THREE METHODS OF CONVERGENCE ORDER SIX UNDER THE SAME SET OF CONDITIONS

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ABSTRACT. The aim of this paper is to compare the convergence radii of three methods of convergence order six under the same conditions. Moreover, we expand the applicability of these methods using only the first derivative in contrast to earlier works using hypotheses on derivatives up to order seven although these derivatives do not appear in the methods. Numerical examples complete this study.

KEYWORDS: High order methods; Banach space; local convergence; $\omega-$conditions.
AMS Subject Classification: Primary 65D10; Secondary 65D99

1. Introduction

In this paper we compare the convergence radii of following three sixth order iterative methods defined for $n = 0, 1, 2, \ldots,$ by [12]:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [-\frac{1}{2}I + \frac{9}{8}F'(y_n)^{-1}F'(x_n) \\ &\quad + \frac{3}{8}F'(x_n)^{-1}F'(y_n)]F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - \frac{9}{4}I + \frac{15}{8}F'(y_n)^{-1}F'(x_n)) \\ &\quad + \frac{11}{8}F'(x_n)^{-1}F'(y_n)]F'(y_n)^{-1}F(z_n), \end{aligned} \tag{1.1}$$

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[12]

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [\frac{5}{8}I + \frac{3}{8}(F'(y_n)^{-1}F'(x_n))^2] \\ &\quad \times F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - [-\frac{9}{4}I + \frac{15}{8}F'(y_n)^{-1}F'(x_n)) \\ &\quad + \frac{11}{8}F'(x_n)^{-1}F'(y_n)]F'(y_n)^{-1}F(z_n), \end{aligned} \tag{1.2}$$

and [14]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n - [\frac{23}{8}I - 3F'(x_n)^{-1}F'(y_n) + \frac{9}{8}(F'(x_n)^{-1}F'(y_n))^2] \\ &\quad \times F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= z_n - [\frac{5}{2}I - \frac{3}{2}F'(x_n)^{-1}F'(y_n)] \\ &\quad \times F'(x_n)^{-1}F(x_n) \end{aligned} \tag{1.3}$$

used for approximating a solution α of the equation

$$F(x) = 0. \tag{1.4}$$

Here: $F : \Omega \subset \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is a differentiable operator in the sense of Fréchet, $\mathcal{E}_1$ and $\mathcal{E}_2$ are Banach spaces and Ω is convex and open.

Earlier convergence analysis of these methods when $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}^k$ used, assumptions of the Fréchet derivatives of F of order up to seven [1, 2, 14] although these derivatives do not appear in these methods, limiting the applicability.

Example 1.1. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}$, $\Omega = [-\frac{5}{2}, \frac{3}{2}]$. Define F on Ω by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$\begin{aligned} F'(x) &= 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \log x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \log x^2 + 60x^2 = 24x + 22. \end{aligned}$$

Obviously $F'''(x)$ is not bounded on Ω. So, the convergence of methods (1.1), (1.2) and (1.3) is not guaranteed by the analysis in the earlier studies.

In this study, our analysis uses only the assumptions on the first Fréchet derivative of F. Thus, we extend the applicability of these methods and in the more general setting of Banach space valued operators. This technique can be used to extend the applicability of other iterative methods.

Notice that, solutions methods for equation (1.4) is an important area of research, since a plethora of problems from diverse disciplines such that Mathematics, Optimization, Mathematical Programming, Chemistry, Biology, Physics, Economics, Statistics, Engineering and other disciplines can be modeled into an equation of the form (1.4) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

The rest of the study is organized as follows. In Section 2 , the local convergence analysis is given and numerical examples are given in the last Section 4.

2. Local convergence

Let us introduce some real functions and parameters needed in the local convergence analysis. Consider a function $\omega_0 : S \longrightarrow S$ continuous and increasing with $w_0(0) = 0$, where $S = [0, \infty)$. Suppose that equation

$$\omega_0(t) = 1 \tag{2.1}$$

has at least one positive solution. We denote by ρ_0 the smallest such solution. Set $S_0 = [0, \rho_0)$. Let also $\omega : S_0 \longrightarrow S$ and $\omega_1 : S_0 \longrightarrow S$ be continuous and increasing functions with $\omega(0) = 0$. Define functions g_1 and $\bar{g}_1$ on the interval S_0 by

$$g_1(t) = \frac{\omega((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}$$

and

$$\bar{g}_1(t) = g_1(t) - 1.$$

Suppose that

$$\omega_1(0) < 3. \tag{2.2}$$

We obtain that $\bar{g}_1(0) = \frac{\omega_1(0)}{3} - 1 < 0$ and $\bar{g}_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. The intermediate value theorem guarantees the existence of at least one solution of the equation $\bar{g}_1(t) = 0$ in $(0, \rho_0)$. Denote by R_1 the smallest such solution. Suppose that equation

$$\omega_0(g_1(t)t) = 1 \tag{2.3}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Set $S_1 = [0, \rho_2)$, where $\rho_2 = \min\{\rho_0, \rho_1\}$. Define functions g_2 and $\bar{g}_2$ on S_1 by

$$\begin{aligned} g_2(t) \quad &= \quad \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{3}{8}\left[3\frac{\omega_0(g_1(t)t)+\omega_0(t)}{1-\omega_0(g_1(t)t)}\right. \\ &\quad + \left.\frac{w_0(g_1(t)t)+\omega_0(t)}{1-\omega_0(t)}\right]\frac{\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)} \end{aligned}$$

and

$$\bar{g}_2(t) = g_2(t) - 1.$$

We also get $\bar{g}_2(0) = -1$ and $\bar{g}_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Denote by R_2 the smallest solution of equation $\bar{g}_2(t) = 0$ in $(0, \rho_2)$. Suppose that

$$\omega_0(g_2(t)t) = 1 \tag{2.4}$$

has at least one positive solution. Denote by ρ_3 the smallest such solution. Set $S_2 = [0, \rho)$, where $\rho = \min\{\rho_2, \rho_3\}$. Define functions g_3 and $\bar{g}_3$ by

$$\begin{aligned} g_3(t) \quad &= \quad \left\{\frac{\int_0^1 \omega((1-\theta)g_2(t)t)d\theta}{1-\omega_0(g_2(t)t)}\right. \\ &\quad + \frac{(\omega_0(g_2(t)t)+\omega_0(g_1(t)t))\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{(1-\omega_0(g_2(t)t))(1-\omega_0(g_1(t)t))} \\ &\quad \frac{1}{8}\left[\frac{15(\omega_0(g_1(t)t)+\omega_0(t))}{1-\omega_0(g_1(t)t)}\right. \\ &\quad + \left.\left.\frac{11(w_0(g_1(t)t)+\omega_0(t))}{1-\omega_0(t)}\right]\frac{\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{1-\omega_0(g_1(t)t)}\right\} \end{aligned}$$

and

$$\bar{g}_3(t) = g_3(t) - 1.$$

We have again $\bar{g}_3(0) = -1$ and $\bar{g}_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Moreover, define a radius of convergence R by

$$R = \min\{R_i\}, \quad i = 1, 2, 3. \tag{2.5}$$

It follows that for each $t \in [0, R)$

$$0 \leq \omega_0(t) < 1,\ 0 \leq \omega_0(g_1(t)t) < 1,\ 0 \leq \omega_0(g_2(t)t) < 1, \tag{2.6}$$

and

$$0 \leq g_i(t) < 1,\ \ i = 1, 2, 3. \tag{2.7}$$

We base the local convergence analysis of method (1.1) on conditions (A):

(a1) $F : \Omega \longrightarrow \mathcal{E}_2$ is a continuously differentiable operator in the sense of Fréchet and there exists $\alpha \in \Omega$ such that $F(\alpha) = 0$ and $F'(\alpha)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$.

(a2) There exists function $\omega_0 : S \longrightarrow S$ continuous and increasing with $\omega_0(0) = 0$ and for each $x \in \Omega$

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq \omega_0(\|x - \alpha\|).$$

Set $\Omega_0 = \Omega \cap U(\alpha, \rho_0)$, where ρ_0 is given in (2.1).

(a3) There exist functions $\omega : S_0 \longrightarrow S, \omega_1 : S_0 \longrightarrow S$ such that for each $x, y \in \Omega_0$

$$\|F'(\alpha)^{-1}(F'(y) - F'(x))\| \leq \omega(\|y - x\|)$$

and

$$\|F'(\alpha)^{-1}F'(x)\| \leq \omega_1(\|x - \alpha\|)$$

where S_0 and S are defined previously.

(a4) $\bar{U}(\alpha, R) \subset \Omega, \rho_0, \rho_1, \rho_2$ exist and are given by (2.1), (2.3) and (2.4), respectively, (2.2) holds and R is given by (2.5).

(a5) There exists $R_1 \geq R$ such that

$$\int_0^1 \omega_0(\theta R_1) d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \bar{U}(\alpha, R_1)$.

Next, the local convergence analysis of method (1.1)is provided using the conditions (A) and the preceding notation.

Theorem 2.1. *Suppose that the conditions (A) hold. Then, sequence $\{x_n\}$ generated by (1.1), for $x_0 \in U(\alpha, R) - \{\alpha\}$ is well defined, remains in $U(\alpha, R)$ for each $n = 0, 1, 2, 3, \ldots$ and converges to α. Moreover, the following estimates hold*

$$\|y_n - \alpha\| \leq g_1(\|x - \alpha\|)\|x - \alpha\| \leq \|x - \alpha\| < R, \tag{2.8}$$

$$\|z_n - \alpha\| \leq g_2(\|x - \alpha\|)\|x - \alpha\| \leq \|x - \alpha\| \tag{2.9}$$

and

$$\|x_{n+1} - \alpha\| \leq g_3(\|x - \alpha\|)\|x - \alpha\| \leq \|x - \alpha\|, \tag{2.10}$$

where functions g_i are given previously and R is defined in (2.5). Furthermore, the limit point α is the only solution of equation $F(x) = 0$ in the set Ω_1.

Proof. We use mathematical induction to show (2.8) – (2.10). Let $x \in U(\alpha, R) - \{\alpha\}$. Using (2.5), (a1) and (a2), we get that

$$\|F'(\alpha)^{-1}(F'(x) - F'(\alpha))\| \leq \omega_0(\|x - \alpha\|) \leq \omega_0(R) < 1. \tag{2.11}$$

By the Banach perturbation lemma [6, 7, 10], $F'(x)^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$,

$$\|F'(x)^{-1}F'(\alpha)\| \leq \frac{1}{1 - \omega(\|x - \alpha\|)} \tag{2.12}$$

and the iterate y_0 is well defined by the first substep of method (1.1) for $n = 0$. We can write by (a1)that

$$y_0 - \alpha = x_0 - \alpha - F'(x_0)^{-1}F(x_0) + \frac{1}{3}F'(x_0)^{-1}F(x_0), \tag{2.13}$$

so by (2.5), (2.7) (for $i = 1$), (2.12) (for $x = x_0$) and (2.13), we have in turn that

$$\begin{aligned}
\|y_0 - \alpha\| &\leq \|F'(x_0)^{-1}F'(\alpha)\| \\
&\quad \|\int_0^1 F'(\alpha)^{-1}(F'(\alpha + \theta(x_0 - \alpha)) - F'(x_0))d\theta(x - \alpha)\| \\
&\quad + \frac{1}{3}\|F'(x_0)^{-1}F'(\alpha)\| \\
&\quad \|\int_0^1 F'(\alpha)^{-1}(F'(\alpha + \theta(x_0 - \alpha)) - F'(x_0))d\theta(x - \alpha)\| \\
&\leq \left[\frac{[\int_0^1 \omega((1-\theta)\|x_0 - \alpha\|)d\theta + \frac{1}{3}\int_0^1 \omega_1(\theta\|x_0 - \alpha\|)d\theta]}{1 - \omega_0(\|x_0 - \alpha\|)}\right] \\
&\quad \times\|x_0 - \alpha\| \\
&= g_1(\|x_0 - \alpha\|)\|x_0 - \alpha\| \leq \|x_0 - \alpha\| < R,
\end{aligned} \tag{2.14}$$

so (2.8) holds for $n = 0$ and $y_0 \in U(\alpha, R)$. Moreover, z_0 exists by (2.12) (for $x = y_0$). We can write

$$\begin{aligned}
z_0 - \alpha &= x_0 - \alpha - F'(x_0)^{-1}F(x_0) \\
&\quad -[-\frac{3}{2}I + \frac{9}{8}F'(y_0)^{-1}F'(x_0) + \frac{3}{8}F'(x_0)^{-1}F'(y_0)]F'(x_0)^{-1}F(x_0) \\
&= x_0 - \alpha - F'(x_0)^{-1}F(x_0) + \frac{3}{8}[3F'(y_0)^{-1}(F'(y_0) - F'(x_0)) \\
&\quad + F'(x_0)^{-1}(F'(x_0) - F'(y_0))]F'(x_0)^{-1}F(x_0),
\end{aligned} \tag{2.15}$$

where we used the estimations

$$\begin{aligned}
&-\frac{12}{8}I + \frac{9}{8}F'(y_0)^{-1}F'(x_0) + \frac{3}{8}F'(x_0)^{-1}F'(y_0) \\
= &-\frac{9}{8}(I - F'(y_0)^{-1}F'(x_0)) - \frac{3}{8}(I - F'(x_0)^{-1}F'(y_0)) \\
= &-\frac{3}{8}[3F'(y_0)^{-1}(F'(y_0) - F'(x_0)) + F'(x_0)^{-1}(F'(x_0) - F'(y_0))].
\end{aligned}$$

Then, by (2.5), (2.7) (for $i = 2$, (2.12) (for $x = y_0$), and (2.14), we have in turn that

$$\begin{aligned}
\|z_0 - \alpha\| &\leq \|x_0 - \alpha - F'(x_0)^{-1}F(x_0)\| + \frac{3}{8}[3\|F'(y_0)^{-1}F'(\alpha)\| \\
&\quad (\|F'(\alpha)^{-1}(F(y_0) - F'(\alpha))\| + \|F'(\alpha)^{-1}(F'(x_0) - F'(\alpha))\|) \\
&\quad +\|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}(\|F'(\alpha)^{-1}(F'(y_0) - F'(\alpha))\| \\
&\quad +\|F'(\alpha)^{-1}(F'(x_0) - F'(\alpha))\|)] \\
&\quad \|F'(x_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(x_0)\| \\
&\leq \left\{\frac{\int_0^1 \omega((1-\theta)\|x_0 - \alpha\|)d\theta}{1 - \omega_0(\|y_0 - \alpha\|)}\right. \\
&\quad + \frac{3}{8}\left[\frac{3(\omega_0(\|y_0 - \alpha\|) + \omega_0(\|x_0 - \alpha\|))}{1 - \omega_0(\|y_0 - \alpha\|)}\right.
\end{aligned}$$

$$
\begin{aligned}
& + \frac{\omega_0(\|x_0-\alpha\|)+\omega_0(\|y_0-\alpha\|)}{1-\omega_0(\|x_0-\alpha\|)}\Bigg] \\
& \left.\frac{\int_0^1 \omega_1(\theta\|x_0-\alpha\|)d\theta}{1-\omega_0(\|x_0-\alpha\|)}\right\} \|x_0-\alpha\| \\
\leq \quad & g_2(\|x_0-\alpha\|)\|x_0-\alpha\| \leq \|x_0-\alpha\|, \qquad (2.16)
\end{aligned}
$$

so (2.9) holds for $n=0$ and $z_0 \in U(\alpha, R)$. We also have by (2.12) (for $x = z_0$) that $F'(z_0)^{-1}$ exists. Then, we can write by the second substep of method (1.1) that

$$
\begin{aligned}
x_1-\alpha \quad = \quad & z_0-\alpha-F'(z_0)^{-1}F(z_0) \\
& +F'(z_0)^{-1}(F'(y_0)-F'(z_0))F'(y_0)^{-1}F(z_0) \\
& +\frac{1}{8}[15F'(y_0)^{-1}(F'(y_0)-F'(x_0))+11F'(x_0)^{-1}(F'(x_0)-F'(y_0))] \\
& F'(y_0)^{-1}F(z_0), \qquad (2.17)
\end{aligned}
$$

where we used estimations

$$
\begin{aligned}
& \frac{1}{8}[-26I+15F'(y_0)^{-1}F'(x_0)-11I+11F'(x_0)^{-1}F'(y_0)] \\
= \quad & -\frac{1}{8}[15(I-F'(y_0)^{-1}F'(x_0))+11(I-F'(x_0)^{-1}F'(y_0))] \\
= \quad & -\frac{1}{8}[15F'(y_0)^{-1}(F'(y_0)-F'(x_0))+11F'(x_0)^{-1}(F'(x_0)-F'(y_0))].
\end{aligned}
$$

Next, by (2.5), (2.7) (for $i=3$), (2.12) (for $x = x_0, z_0$), (2.16) and (2.17), we obtain in turn that

$$
\begin{aligned}
\|x_1-\alpha\| \quad \leq \quad & \|z_0-\alpha-F'(z_0)^{-1}F(z_0)\| \\
& +[\|F'(z_0)^{-1}F'(\alpha)\|(\|F'(\alpha)^{-1}(F'(y_0)-F'(x_0))\| \\
& +\|F'(\alpha)^{-1}(F'(z_0)-F'(\alpha))\|) \\
& \times\|F'(y_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(z_0)\| \\
& +\frac{1}{8}[15\|F'(y_0)^{-1}F'(\alpha)\|(\|F'(\alpha)^{-1}(F'(y_0)-F'(\alpha)\| \\
& +\|F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))\|)] \\
& +11\|F'(x_0)^{-1}F'(\alpha)\|(\|F'(\alpha)^{-1}(F'(x_0)-F'(\alpha))\| \\
& +\|F'(\alpha)^{-1}(F'(y_0-F'(\alpha))\|)] \\
& \times\|F'(y_0)^{-1}F'(\alpha)\|\|F'(\alpha)^{-1}F(z_0)\| \\
\leq \quad & \Bigg\{ \frac{\int_0^1 \omega((1-\theta)\|z_0-\alpha\|)d\theta}{1-\omega_0(\|z_0-\alpha\|)} \\
& + \frac{(\omega_0(\|z_0-\alpha\|)+\omega_0(\|y_0-\alpha\|))\int_0^1 \omega_1(\theta\|z_0-\alpha\|)d\theta}{(1-\omega_0(\|z_0-\alpha\|))(1-\omega_0(\|y_0-\alpha\|))} \\
& +\frac{1}{8}\Bigg[\frac{15(\omega_0(\|x_0-\alpha\|)+\omega_0(\|y_0-\alpha\|)}{1-\omega_0(\|y_0-\alpha\|)} \\
& + \frac{11(\omega_0(\|x_0-\alpha\|)+\omega_0(\|y_0-\alpha\|)}{1-\omega_0(\|x_0-\alpha\|)}\Bigg] \\
& \times \left.\frac{\int_0^1 \omega_1(\theta\|z_0-\alpha\|)d\theta}{1-\omega_0(\|y_0-\alpha\|)}\right\} \|z_0-\alpha\| \\
\leq \quad & g_3(\|x_0-\alpha\|)\|x_0-\alpha\| \leq \|x_0-\alpha\|, \qquad (2.18)
\end{aligned}
$$

so (2.10) holds for $n = 0$ and $x_1 \in U(\alpha, R)$. The induction for (2.8)– (2.10) is completed, if x_0, y_0, z_0, x_1 are replaced by x_j, y_j, z_j, x_{j+1} respectively, in the preceding estimations. It then follows from

$$\|x_{j+1} - \alpha\| \leq a\|x_j - \alpha\| < R,\ a = g_3(\|x_0 - \alpha\|) \in [0, 1) \tag{2.19}$$

that $\lim_{j\longrightarrow\infty} x_j =$ and $x_{j+1} \in U(\alpha, R)$. Finally, for the uniqueness part, let $\alpha_1 \in \Omega_1$ with $F(p_1) = 0$ with $F(\alpha_1) = 0$. Then, by (a2) and (a5), we get in turn that for $T = \int_0^1 F'(\alpha_1 + \theta(\alpha - \alpha_1))d\theta$ for

$$\begin{aligned} \|F'(\alpha)^{-1}(T - F'(\alpha))\| &\leq \int_0^1 \omega_0(\theta\|\alpha - \alpha_1\|)d\theta \\ &\leq \int_0^1 \omega_0(\theta R^*)d\theta < 1 \end{aligned} \tag{2.20}$$

leading to $T^{-1} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$. Then, by the identity

$$0 = F(\alpha) - F(\alpha_1) = T(\alpha - \alpha_1),$$

we deduce that $\alpha_1 = \alpha$.

□

Remark 2.1. The convergence order of method (1.1) can be determined using computing the computational order of convergence (COC) [7, 8, 11] given by

$$\xi = \frac{\ln(\frac{\|x_{n+2}-\alpha\|}{\|x_{n+1}-\alpha\|})}{\ln(\frac{\|x_{n+1}-\alpha\|}{\|x_n-\alpha\|})} \tag{2.21}$$

or the approximate computational order of convergence (ACOC) [7, 8, 11] given by

$$\xi^* = \frac{\ln(\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|})}{\ln(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|})}. \tag{2.22}$$

It turns out that the local convergence of method (1.2) (or method(1.3)) are given under the conditions (A) by modifying the definition of g_i functions to fit these methods as follows:

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{3}{8}\left[\left(\frac{(\omega_0(g_1(t)t)+\omega_0(t))}{1-\omega_0(g_1(t)t)}\right)^2 \right.\\ &\quad \left. + \frac{2(w_0(g_1(t)t)+\omega_0(t))}{(1-\omega_0(g_1(t)t))^2}\right]\frac{\int_0^1 \omega_1(\theta t)d\theta}{1-\omega_0(t)}, \\ \bar{g}_2(t) &= g_2(t) - 1, \end{aligned}$$

and g_3 and $\bar{g}_3$ as previously. The corresponding (2.15) Ostrowski-type representation in method (1.2) is:

$$\begin{aligned} z_n - \alpha &= x_n - \alpha - F'(x_n)^{-1}F(x_n) \\ &\quad + \frac{3}{8}[(F'(y_n)^{-1}(F'(y_n) - F'(x_n)))^2 \\ &\quad + 2F'(y_n)^{-1}(F'(y_n) - F'(x_n))F'(y_n)^{-1}F'(x_n)] \\ &\quad \times F'(x_n)^{-1}F(x_n), \end{aligned} \tag{2.23}$$

where the representations for functions g_1 and g_3 are the same. Moreover, the corresponding to (2.15) and (2.17) representations for method (1.3) are:

$$z_n - \alpha = x_n - \alpha - F'(x_n)^{-1}F(x_n)$$

$$+\frac{1}{8}[15F'(x_n)^{-1}(F'(x_n)-F'(y_n))$$
$$+9F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}(F'(y_n)-F'(x_n))]$$
$$\times F'(x_n)^{-1}F(z_n). \tag{2.24}$$

and

$$\begin{aligned} x_{n+1}-\alpha &= z_n-\alpha-F'(z_n)^{-1}F(z_n) \\ &\quad -\frac{3}{2}F'(x_n)^{-1}(F'(x_n)-F'(y_n)) \\ &\quad \times F'(x_n)^{-1}F(z_n). \end{aligned} \tag{2.25}$$

The g functions are:

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)} + \frac{1}{8}\left[\frac{15(\omega_0(g_1(t)t)+\omega_0(t))}{1-\omega_0(t)}\right. \\ &\quad \left. + \frac{9w_1(g_1(t)t)(\omega_0(g_1(t)t)+\omega_0(t))\int_0^1 \omega_1(\theta t)d\theta}{(1-\omega_0(t))^3}\right] \end{aligned}$$

and

$$\begin{aligned} g_3(t) &= \left\{\frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)}\right. \\ &\quad \left. + \frac{3}{2}\frac{(\omega_0(t)+\omega_0(g_1(t)t))\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{(1-\omega_0(t))^2}\right\} g_2(t). \end{aligned}$$

With the above changes and following the proof of Theorem 2.1, we arrive at the corresponding results for method (1.2) and method (1.3).

Theorem 2.2. *Suppose that the conditions (A) hold. Then, the conclusions of Theorem 2.1 hold for method (1.2) or method (1.3) with the above indicated changes.*

3. Numerical examples

Example 3.1. Let $\mathcal{B}_1=\mathcal{B}_2=\mathbb{R}^3, \Omega=\bar{U}(0,1), x^*=(0,0,0)^T$. Define function F on Ω for $u=(x,y,z)^T$ by

$$F(u)=(e^x-1,\frac{e-1}{2}y^2+y,z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v)=\begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.8)-(2.12), conditions, we get $\omega_0(t)=(e-1)t, \omega(t)=e^{\frac{1}{e-1}}t, \omega_1(t)=e^{\frac{1}{e-1}}$.

Then using the definition of r, we have that
$R_1=0.154406951357154070825217218043 69$
$R_2=0.0837447893717740837749019533475 8=R$
$R_3=0.11332932017032089355712543010668.$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$, $\omega_0(t) = 7.5t, \omega(t) = 15t, \omega_1(t) = 2$. This way, we have that
$R_1 = 0.022222222222222222222222222222222222$
$R_2 = 0.015951698098429258065866775950781 = R$
$R_3 = 0.021955106317595653175889225394712.$

Example 3.3. Let $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{R}$, $\Omega = [-\frac{5}{2}, \frac{1}{2}]$. Define F on Ω by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$F'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

Then, we get that $\varphi_0(t) = \varphi(t) = 147t, \psi(t) = 2$. So, we obtain
$R_1 = 0.0015117157974300831443688586545729$
$R_2 = 0.00088140170616351218649264787075026 = R$
$R_3 = 0.0012234803047134626755032549283442.$

Example 3.4. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 3, 5, 11] defined by

$$x(s) = \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.4), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2} + x(t))dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|).$$

Then, we get that $\omega_0(t) = \omega(t) = \frac{1}{8}(\frac{3}{2}t^{1/2} + t), \omega_1(t) = 1 + \omega_0(t)$. So, we obtain
$R_1 = 1.2$
$R_2 = 0.6078414862054067890895225900749 2 = R$
$R_3 = 0.77695598964350998105743428823189$.

4. Conclusion

A very important aspect in the study of iterative methods is the convergence region, since it determines the choices of the initial point. That is why we studied the convergence of three popular sixth order methods for solving nonlinear equations under the same set of conditions. The radii of convergence were evaluated on three test examples showing that in each example a different method has the largest radius of convergence.

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