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BEST PROXIMITY POINTS INVOLVING SIMULATION FUNCTIONS WITH τ-DISTANCE

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ABSTRACT. In this paper, we illustrate the best proximity point theorems in complete metric spaces for $\mathscr{Z}$-p-proximal contractions of the first kind and of the second kind involving the simulation functions using τ-distance with lower semicontinuity in its first variable. Ous results extend generalize the results in literature.

KEYWORDS: Best proximity point, Simulation functions, τ-Distance.
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1. Introduction

In 1922, Banach proved that if (X, d) is a complete metric space and the mapping T satisfies Banach contraction mapping principle, then T has a unique fixed point, that is $T(u) = u$ for some $u \in X$. Since the results of Banach, many authors have been studying fixed point and best proximity points of mappings in metric spaces. Their reseach are still being studied in many directions. In 1999, Suzuki [15] introduced the concept of τ-distance on a metric space, which is a generalized concept of w-distance. They also improve the generalizations of the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, including they discuss the relation between w-distance. In 2015, Khojasteh *et al.* [10] introduced the simulation function. Recently, simulation function have been used to study the best proximity points in metric spaces (see [12, 14, 2]).

In 2018, Kostić *et al.*[2] introduced a special type of w-distance, the w_0-distance, to extend best proximity results of Tchier *et al.* [14] involving simulation functions. In this paper, we generalize some best proximity points results in metric spaces involving simulation functions with τ-distance.

In this paper we prove the best proximity point results involving simulation

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functions with τ-distance, given by τ-distance is lower semicontinuous in its first variable.

2. Preliminaries

Here we recall some definition and some example of the simulation function ([1, 9, 10, 11, 12, 14]).

Definition 2.1. [1] Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping. Then ζ is called a *simulation function* if it satisfies the following conditions:
(ζ_1) $\zeta(t,s) < s - t$ for all $t, s > 0$
(ζ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

In 2015, Khojasteh *et al.*[10] introduced the simulation function as a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying $\zeta(0,0) = 0$ alongside the conditions (ζ_1) and (ζ_2) of Definition 2.1. On the other hand, Argoubi *et al.* [1] slightly modified the definition of Khojasteh *et al.* [10] by removing the condition $\zeta(0,0) = 0$. In this paper, we use a modified definition of Argoubi *et al.* [1].

The set of all simulation functions will be denoted by $\mathscr{Z}$.
The following, we recall some examples of simulation functions.

Example 1.1 [9] Let $\zeta_i : [0,\infty) \times [0,\infty) \to \mathbb{R}, i = 1, ..., 6$ be defined by

1. $\zeta_1(t,x) = \psi(s) - \phi(t)$ for all $t, s \in [0,\infty)$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\phi(t) = \psi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \le \phi(t)$ for all $t > 0$.
2. $\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty)^2 \to (0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$
3. $\zeta_3(t,s) = s - \varphi(s) - t$ for all $t, s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
4. If $\varphi : [0,\infty) \to [0,1)$ is a function such that $\limsup_{t\to r+} \varphi(t) < 1$ for all $r > 0$ we define
$$\zeta_4(t,s) = s\varphi(s) - t \text{ for all } t, s \in [0,\infty).$$
5. If $\eta : [0,\infty) \to [0,\infty)$ is an upper semi-continuous mapping such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, we define
$$\zeta_5(t,s) = \eta(s) - t \text{ for all } t, s \in [0,\infty).$$
6. If $\phi : [0,\infty) \to [0,\infty)$ is a function such that $\int_0^\varepsilon \phi(u)du$ exists and $\int_0^\varepsilon \phi(u)du > \varepsilon$ for each $\varepsilon > 0$, we define
$$\zeta_6(t,s) = s - \int_0^t \phi(u)du \text{ for all } t, s \in [0,\infty).$$

It is easily verified that each function $\zeta_i (i = 1, ..., 6)$ is the simulation function.

Definition 2.2. [2] A nonself mapping $T : A \to B$ is said to be a $\mathscr{Z}$-p-proximal contraction of the first kind if there exists $\zeta \in \mathscr{Z}$ such that

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(u,Ty) = d(A,B) \end{aligned}\right\} \Rightarrow \zeta(\mu(u,v), \mu(x,y)) \ge 0$$

for all $u, v, x, y \in A$.

Definition 2.3. [2] A non-self-mapping $T : A \to B$ is said to be a $\mathscr{Z}$-p-proximal contraction of the second kind if there exists $\zeta \in \mathscr{Z}$ such that

$$\left.\begin{array}{r} d(u, Tx) = d(A, B) \\ d(u, Ty) = d(A, B) \end{array}\right\} \Rightarrow \zeta(\mu(Tu, Tv), \mu(Tx, Ty)) \geq 0$$

for all $u, v, x, y \in A$.

In the case $p = d$, the notaion of $\mathscr{Z}$-p-proximal contraction are reduced to $\mathscr{Z}$-proximal contraction of Tchier *et al.* [14].

In Definition 2.2, if the simulation function ζ is given by $\zeta(t, s) = \alpha s - t$ for some $\alpha \in [0, 1)$, then the mapping T is called a *p-proximal contraction of the first kind.* Moreover if $p = d$, then T is a proximal contraction of the first kind.

We recall the following notation:

$\mathscr{G}_{A,p} = \{g : g$ is a continuous functions from (A, d) to (A, d) and $p(x, y) \leq p(gx, gy)$ for all $x, y \in A\}$

$\mathscr{T}_{g,p} = \{T : T$ is a function from A to B and $p(Tx, Ty) \leq p(Tgx, Tgy)$ for all $x, y \in A\}$.

In the case $p = d, \mathscr{G}_{A,p}$ is denoted by $\mathscr{G}_A$ and $\mathscr{T}_{g,p}$ by $\mathscr{T}_g$ (see [14]).

In 1999, Suzuki [15] introduced the concept of τ-distance on a metric space, which is a generalized concept of w-distance. They gave example of the τ-distance. Further They discuss the relation between w-distance. Kostić *et al.*[2] introduced the concept of w_0-distance, which is slightly different to the original w-distance of [8], in regard that the lower semicontinuity with respect to both variables is supposed.

Definition 2.4. [15] Let X be a metric space with metric d. Then a function $p : X \times X \to [0, \infty)$ is called the τ-distance on X if there exists a function η from $X \times [0, \infty) \to [0, \infty)$ and the following are satisfied:

- (τ_1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
- (τ_2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable;
- (τ_3) $\lim_n x_n = x$ and $\limsup_n \{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for all $w \in X$;
- (τ_4) $\limsup_n \{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;
- (τ_5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$

We may replace (τ_2) by the following $(\tau_2)^{'}$

- $(\tau_2)^{'}$ $\inf\{\eta(x, t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

We recall some properties of τ-distance. Let X be a metric space with metric d and let p be a τ-distance on X. Then a sequence $\{x_n\}$ of X is called p-Cauchy if there exists a function η from $X \times [0, \infty) \to [0, \infty)$ satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of X such that $\limsup_n \{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$.

We recall the following lemma, which can be found in [15].

Lemma 2.5. [15] *Let X be a metric space with metric d and let p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence. Moreover,* *if* $\{y_n\}$ *is a sequence satisfying* $\limsup_n \{p(x_n, y_m) : m \geq n\} = 0$*, then* $\{y_n\}$ *is also a p-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$.

Lemma 2.6. [15] *Let* X *be a metric space with metric* d *and* p *be a* τ*-distance on* X*. If a sequence* $\{x_n\}$ *of* X *satisfies* $\lim_n p(z, x_n) = 0$ *for some* $z \in X$*, then* $\{x_n\}$ *is a p-Cauchy sequence. Moreover, if a sequence* $\{y_n\}$ *of* X *also satisfies* $\lim_n p(z, y_n) = 0$*, then* $\lim_n d(x_n, y_n)$*. In particular for* $x, y, z \in X$*,* $p(z, x) = 0$ *and* $p(z, y) = 0$ *imply* $x = y$*.*

Lemma 2.7. [15] *Let* X *be metric space with metric* d *and let* p *be a* τ*-distance on* X*. If a sequence* $\{x_n\}$ *of* X *satisfies* $\limsup_n \{p(x_n, x_m) : m > n\} = 0$*, then* $\{x_n\}$ *is a p-Cauchy sequence. Moreover if a sequence* $\{y_n\}$ *of* X *satisfies* $\lim_n p(x_n, y_n) = 0$*, then* $\{y_n\}$ *is also a p-Cauchy sequence and* $\lim_n d(x_n, y_n) = 0$*.*

Let (X, d) be a metric space, A and B two nonempty subsets of X and $T : A \to B$ a non-self-mapping. The following notations will be used throughout the paper.

$$\begin{aligned} d(A, B) &= \inf\{d(x, y) : x \in A, y \in B\}; \\ d(y, A) &= \inf\{d(x, y) : x \in A\} = d(\{y\}, A); \\ A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}; \\ B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

Throughout this paper, the set of all best proximity points of a non-self mapping $T : A \to B$ will be denoted by

$$B_{est}(T) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

If $g : A \to A$, then we have

$$B^g_{est}(T) = \{x \in A : d(gx, Tx) = d(A, B)\}.$$

3. Main Results

Let (X, d) be a metric space, $p : X \times X \to [0, \infty)$ a τ-distance on X, and let A and B be two nonempty subsets of X (which need not be equal). For every $x, y \in X$,

$$\mu(x, y) := \max\{p(x, y), p(y, x)\}.$$

It is easily checked that the function $\mu : X \times X \to [0, \infty)$ has the following properties, for all $x, y, z \in X$;

(1) $\mu(x, y) = 0 \Rightarrow x = y$;
(2) $\mu(x, y) = \mu(y, x)$, i.e. μ is symmetic;
(3) $\mu(x, y) \leq \mu(x, z) + \mu(z, y)$, i.e. μ satisfies the triangle inequality.

Lemma 3.1. *Suppose that* $\{x_n\}$ *is sequence such that* $\lim_n \mu(x_n, x_{n+1}) = 0$*. If* $\lim_{n,m} \mu(x_n, x_m) \neq 0$*, then there are* $\epsilon > 0$ *and two subsequence* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *of* $\{x_n\}$ *such that* $\lim_k \mu(x_{n_k}, x_{m_k}) = \lim_k \mu(x_{n_k+1}, x_{m_k+1}) = \epsilon$*.*

Next, we prove our main results.

Theorem 3.2. *Let* A *and* B *be two nonempty subsets of a complete metric space (X,d) with a* τ*-distance* p*, such that* A_0 *is nonempty and closed. Let* $p(\cdot, x) : X \to [0, \infty)$ *be lower semicontinuous for any* $x \in X$*. Suppose that the mappings* $g : A \to A$ *and* $T : A \to B$ *satisfy the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the first kind;*
(b) $g \in \mathscr{G}_{A,p}$*;*

(c) $A_0 \subseteq g(A_0)$;
(d) $T(A_0) \subseteq B_0$.

Then there exists a unique element $x \in A_0$ such that $d(gx, Tx) = d(A, B)$ and $p(x, x) = 0$. Moreover, for any initial $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B).$$

Again, since Tx_1 is an element of $T(A_0)$ which is contained in B_0, and A_0 is contained in $g(A_0)$, it follows that there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B).$$

This process can be continued, for any $x_n \in A_0$ it is possible to find $x_{n+1} \in A_0$ such that

$$d(gx_{n+1}, Tx_n) = d(A, B).$$

If there exists $n_0 \in \mathbb{N}$ such that $\mu(x_{n_0}, x_{n_0-1}) = 0$, then $x_{n_0-1} = x_{n_0}$, which implies that $d(gx_{n_0-1}, Tx_{n_0-1}) = d(A, B)$. That is, x_{n_0-1} is a best proximity point of T under mapping g.

Assume that $\mu(x_n, x_{n-1}) > 0$ for all $n \in \mathbb{N}$. Since $g \in \mathscr{G}_{A,p}$, we have $\mu(gx_n, gx_{n-1}) > 0$ for all $n \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the first kind and $g \in \mathscr{G}_{A,p}$, we obtain

$$\begin{aligned} 0 \le \zeta(\mu(gx_{n+1}, gx_n), \mu(x_n, x_{n-1})) &< \mu(x_n, x_{n-1}) - \mu(gx_{n+1}, gx_n) \\ &\le \mu(x_n, x_{n-1}) - \mu(x_{n+1}, x_n). \end{aligned} \tag{3.1}$$

Thus

$$\mu(x_{n+1}, x_n) < \mu(x_n, x_{n-1}), \forall n \in \mathbb{N}. \tag{3.2}$$

This implies that the sequence $\{\mu(x_n, x_{n-1})\}$ is decreasing and so there exists

$$\lim_{n\to\infty} \mu(x_n, x_{n-1}) = r \ge 0. \tag{3.3}$$

Suppose that $r > 0$. From (3.1),

$$\mu(gx_{n+1}, gx_n) \le \mu(x_n, x_{n-1})$$

for every $n \in \mathbb{N}$. On the other hand, $g \in \mathscr{G}_{A,p}$ and hence

$$\mu(x_{n+1}, x_n) \le \mu(gx_{n+1}, gx_n) \le \mu(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. Let $n \to \infty$,

$$\lim_{n\to\infty} \mu(gx_{n+1}, gx_n) = r. \tag{3.4}$$

Now, using the simulation function property (ζ_2) we obtain

$$0 \le \limsup_{n\to\infty} \zeta(\mu(gx_{n+1}, gx_n), \mu(x_n, x_{n-1})) < 0$$

which is a contradiction. Hence we have $r = 0$ which imples that

$$\lim_{n\to\infty} \mu(x_n, x_{n-1}) = 0. \tag{3.5}$$

Now, let us prove that

$$\lim_{m,n\to\infty} \mu(x_n, x_m) = 0. \tag{3.6}$$

If (3.6) is not true, then

$$\lim_{m,n\to\infty} \mu(x_n, x_m) \ne 0. \tag{3.7}$$

From Lemma 3.1, then there exists $\epsilon > 0$ and two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} \mu(x_{n_k}, x_{m_k}) = \varepsilon. \tag{3.8}$$

and

$$\lim_{k\to\infty} \mu(x_{n_k+1}, x_{m_k+1}) = \varepsilon. \tag{3.9}$$

We can assume that $\mu(x_{n_k+1}, x_{m_k+1}) > 0$ for all $k \in \mathbb{N}$. Again, T is a $\mathscr{Z}$-p-proximal contraction of the first kind and $d(gx_{n_k+1}, Tx_{n_k}) = d(A, B) = d(gx_{m_k+1}, Tx_{m_k})$. By the property (ζ_1), we obtain

$$\begin{aligned}
0 &\leq \zeta(\mu(gx_{n_k+1}, gx_{m_k+1}), \mu(x_{n_k}, x_{m_k})) \\
&< \mu(x_{n_k}, x_{m_k}) - \mu(gx_{n_k+1}, gx_{m_k+1}) \\
&\leq \mu(x_{n_k}, x_{m_k}) - \mu(x_{n_k+1}, x_{m_k+1})
\end{aligned}$$

for all $k \in \mathbb{N}$. Thus the previous inequality with (3.8) and (3.9) imply that

$$\lim_{k\to\infty} \mu(gx_{n_k+1}, gx_{m_k+1}) = \varepsilon. \tag{3.10}$$

From (3.8) and (3.10) we see that the sequance $t_k := \mu(gx_{n_k+1}, gx_{m_k+1})$ and $s_k := \mu(x_{n_k}, x_{m_k})$ have the same positive limit. By the property (ζ_2), we conclude that

$$0 \leq \limsup_{k\to\infty} \zeta(t_k, s_k) < 0$$

which is a contradiction and hence (3.6) holds.
Since

$$\lim_{m,n\to\infty} \mu(x_n, x_m) = 0,$$

we have

$$\limsup_{m,n\to\infty} \{p(x_n, x_m) : m > n\} = 0.$$

By Lemma 2.7, we have $\{x_n\}$ is a p-Cauchy sequance in A_0. And by Lemma 2.5 we have $\{x_n\}$ is a Cauchy sequance in A_0. Since (X, d) is complete metric space and A_0 is a closed subset of X, there exists $\lim_{n\to\infty} x_n = x \in A_0$. Moreover, by the continuity of g we have $\lim_{n\to\infty} gx_n = gx$. Since $gx_n \in A_0$ for all $n \in \mathbb{N}$ and A_0 is closed, we also have $gx \in A_0$. On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$, there exists $z \in A_0$ such that $d(z, Tx) = d(A, B)$.

Let us prove that $z = gx$. If $z = gx_n$ for infinetely many $n \in \mathbb{N}$, then $z = gx$. Assume that $z \neq gx$, in which case there exists $n_0 \in \mathbb{N}$ such that $z \neq gx_n$ for all $n \geq n_0$. If $\mu(gx_n, z) = 0$ for some $n \geq n_0$, then $gx_n = z$. That is $\mu(gx_n, z) > 0$ for all $n \geq n_0$. Also there exists a subsequance $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \neq x$ for every $k \in \mathbb{N}$ (if that is not true, then $x_n = x$ for all $n \in \mathbb{N}$ and so $\mu(x_n, x_{n-1}) = 0$ for all $n \in \mathbb{N}$, which is contrary to (3.2)). Similary, we have $\mu(x_{n_k}, x) > 0$ for every $k \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the first kind and $g \in \mathscr{G}_{A,p}$, we obtain

$$\begin{aligned}
0 &\leq \zeta(\mu(gx_{n_k+1}, z), \mu(x_{n_k}, x)) \\
&< \mu(x_{n_k}, x) - \mu(gx_{n_k+1}, z) \\
&\leq \mu(gx_{n_k}, gx) - \mu(gx_{n_k+1}, z).
\end{aligned}$$

This imples that

$$\mu(gx_{n_k+1}, z) < \mu(gx_{n_k}, gx) \tag{3.11}$$

for every $k \in \mathbb{N}$ such that $n_k \geq n_0$.

Similarly argument as before we can show that

$$\lim_{m,n\to\infty} \mu(gx_n, gx_m) = 0.$$

This means that for any $\epsilon > 0$ there exists a $N_\epsilon \in \mathbb{N}$ such that $\mu(gx_n, gx_m) < \epsilon$ for all $m > n \geq N_\epsilon$. For a fixed $n \in \mathbb{N}$ with $n \geq \max\{n_0, N_\epsilon\}$ and

$$\lim_{m,n\to\infty} \mu(gx_n, gx_m) = 0,$$

we have

$$\limsup_{m,n\to\infty}\{p(gx_n, gx_m) : m > n\} = 0.$$

By Lemma 2.7, we have $\{gx_n\}$ is a p-Cauchy sequance in A_0.
Since gx_n is a p-Cauchy sequance in A_0, there exists a function η from $A_0 \times [0,\infty) \to [0,\infty)$ satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of A_0 such that

$$\limsup_{n\to\infty}\{\eta(z_n, p(z_n, gx_m)) : m \geq n\} = 0.$$

By (τ_3) and $p(\cdot, x) : X \to [0,\infty)$ is lower semicontinuous imply that

$$p(gx_n, gx) \leq \liminf_{m} p(gx_n, gx_m) < \epsilon.$$

and

$$p(gx, gx_n) \leq \liminf_{m} p(gx_m, gx_n) < \epsilon.$$

Therefore

$$\lim_{k\to\infty} p(gx_{n_k}, gx) = 0. \tag{3.12}$$

Similarly, $\lim_{k\to\infty} p(gx, gx_{n_k}) = 0$ which combind with (3.12) yields

$$\lim_{k\to\infty} \mu(gx_{n_k}, gx) = 0.$$

Then from (3.11) we have

$$\lim_{k\to\infty} \mu(gx_{n_k+1}, z) = 0. \tag{3.13}$$

Letting $k \to \infty$ in the following inequality and by (3.5), (3.13)

$$\mu(gx_{n_k}, z) \leq \mu(gx_{n_k}, gx_{n_k+1}) + \mu(gx_{n_k+1}, z),$$

we get $\lim_{k\to\infty} \mu(gx_{n_k}, z) = 0$. This implies

$$\lim_{k\to\infty} p(gx_{n_k}, z) = 0. \tag{3.14}$$

Since $\lim_{k\to\infty} gx_{n_k} = gx$, we obtain

$$p(gx, gx) = 0 \text{ and } p(gx, z) = 0.$$

By Lemma 2.6, imply that $z = gx$. Finally, from $d(z, Tx) = d(A, B)$, we get $d(gx, Tx) = d(A, B)$.

To prove the uniqueness, let y be in A_0 such that

$$d(gy, Ty) = d(A, B).$$

Assume that $\mu(gx, gy) \geq \mu(x, y) > 0$. Since $g \in \mathscr{G}_{A,p}$ and T is a $\mathscr{Z}$-p-proximal contraction of the first kind, we obtain

$$\begin{aligned} 0 &\leq \zeta(\mu(gx, gy), \mu(x, y)) \\ &< \mu(x, y) - \mu(gx, gy) \\ &\leq \mu(x, y) - \mu(x, y) = 0 \end{aligned}$$

which leads to a contradiction. Hence $\mu(x,y)=0$, which implies $x=y$.

By a similar argument we prove $p(x,x)=0$. Suppose to the contrary, that $\mu(x,x)=p(x,x)>0$. Then $\mu(gx,gx)>0$. Again, we have

$$\begin{aligned} 0 &\le \zeta(\mu(gx,gx),\mu(x,x)) \\ &< \mu(x,x)-\mu(gx,gx) \\ &\le \mu(x,x)-\mu(x,x)=0 \end{aligned}$$

which is a contradiction. □

If g is the identity mapping on A, then the preceding theorem yields the following corollary.

Corollary 3.3. *Let A and B be two nonempty subset of a complete metric space (X,d) with a τ-distance p, such that A_0 is nonempty and closed. Let $p(\cdot,x):X\to[0,\infty)$ is lower semicontinuous for any $x\in X$. Suppose that the mappings $T:A\to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the first kind;*
(b) *$T(A_0)\subseteq B_0$.*

Then there exists a unique best proximity point $x\in A_0$ of the mapping T, such that $p(x,x)=0$. Moreover, for every $x_0\in A_0$ there exists a sequence $\{x_n\}\subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$.

From Theorem 3.2 we can also obtain an interesting g-best proximity point result for a p-proximal contraction of the first kind.

Corollary 3.4. *Let A and B be two nonempty subset of a complete metric space (X,d) with a τ-distance p, such that A_0 is nonempty and closed. Let $p(\cdot,x):X\to[0,\infty)$ is lower semicontinuous for any $x\in X$. Suppose that the mappings $g:A\to A$ and $T:A\to B$ satisfies the folowing conditions:*

(a) *T is a p-proximal contraction of the first kind with respect $\alpha\in[0,1)$;*
(b) *$g\in\mathscr{G}_{A,p}$;*
(c) *$A_0\subseteq g(A_0)$;*
(d) *$T(A_0)\subseteq B_0$.*

Then there exists a unique element $x\in A_0$ such that $d(gx,Tx)=d(A,B)$ and $p(x,x)=0$. Moreover, for every $x_0\in A_0$ there exists a sequence $\{x_n\}\subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$.

Proof. Note that a p-proximal contraction of the first kind with respect to $\alpha\in[0,1)$ is a $\mathscr{Z}$-p-proximal contraction of the first kind with respect to the simulation function $\zeta:[0,\infty)\times[0,\infty)\Rightarrow\mathbb{R}$ defined by $\zeta(t,s)=\alpha s-t$ for all $t,s\ge 0$. □

By taking $p=d$ in Theorem 3.2 the main result of [14] is obtained.

Corollary 3.5. *Let A and B be two nonempty subset of a complete metric space (X,d), such that A_0 is nonempty and closed. Suppose that the mappings $g:A\to A$ and $T:A\to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-proximal contraction of the first kind;*
(b) *$g\in\mathscr{G}_A$;*
(c) *$A_0\subseteq g(A_0)$;*
(d) *$T(A_0)\subseteq B_0$.*

Then there exists a unique element $x\in A_0$ such that $d(gx,Tx)=d(A,B)$. Moreover, for every $x_0\in A_0$ there exists a sequence $\{x_n\}\subseteq A_0$ such that $d(gx_{n+1},Tx_n)=d(A,B)$ for all $n\in\mathbb{N}\cup\{0\}$, and $\{x_n\}$ converging to x.

Theorem 3.6. *Let A and B be two nonempty subsets of a complete metric space (X,d) with a τ-distance p, such that $T(A_0)$ is nonempty and closed. Let $p(\cdot,x): X \to [0,\infty)$ is lower semicontinuous for any $x \in X$. Suppose that the mappings $g: A \to A$ and $T: A \to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the second kind;*
(b) *T is injective on A_0*
(c) *$T \in \mathscr{T}_{g,p}$;*
(d) *$A_0 \subseteq g(A_0)$;*
(e) *$T(A_0) \subseteq B_0$.*

Then there exists a unique element $x \in A_0$ such that $d(gx,Tx) = d(A,B)$ and $p(Tx,Tx) = 0$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Proceeding as in Theorem 3.2 we can construct a sequance $\{x_n\}$ such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$. In the constructive process of $\{x_n\}$, if we have $Tx_n = Tx_m$ for some $m > n$, then we choose $x_{m+1} = x_{n+1}$.
Since T is a $\mathscr{Z}$-p-proximal constraction of the second kind, we have

$$\zeta(\mu(Tgx_n,Tgx_{n+1}),\mu(Tx_{n-1},Tx_n)) \geq 0$$

for every $n \in \mathbb{N}$. Since T is injective on A_0 and $T \in \mathscr{T}_{g,p}$, using the property (ζ_1) of a simulation function, we obtain that

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx_n,Tgx_{n+1}),\mu(Tx_{n-1},Tx_n)) \\ &< \mu(Tx_{n-1},Tx_n) - \mu(Tgx_n,Tgx_{n+1}) \\ &\leq \mu(Tx_{n-1},Tx_n) - \mu(Tx_n,Tx_{n+1}) \end{aligned} \tag{3.15}$$

for every $n \in \mathbb{N}$. Then we have

$$\mu(Tx_n,Tx_{n+1}) < \mu(Tx_{n-1},Tx_n),\ \forall n \in \mathbb{N} \tag{3.16}$$

which implies that the sequence $\{\mu(Tx_{n-1},Tx_n)\}$ is decreasing.

If there exists $n_0 \in \mathbb{N}$ such that $\mu(Tx_{n_0-1},Tx_{n_0}) = 0$, then $Tx_{n_0-1} = Tx_{n_0}$. By the injective of T on A_0 follows $x_{n_0-1} = x_{n_0}$. Then $d(gx_{n_0-1},Tx_{n_0}) = d(gx_{n_0},Tx_{n_0}) = d(A,B)$ and x_{n_0} is the best proximity point of T under mapping g. That is, $x_{n_0} \in B^g_{est}(T)$.
Now, let $\mu(Tx_{n-1},Tx_n) > 0$ for all $n \in \mathbb{N}$. Then there exists

$$\lim_{n\to\infty} \mu(Tx_{n-1},Tx_n) = r \geq 0.$$

Suppose $r > 0$. From (16) we can also deduce that

$$\mu(Tgx_n,Tgx_{n+1}) < \mu(Tx_{n-1},Tx_n).$$

On the other hand, $T \in \mathscr{T}_{g,p}$ and hence

$$\mu(Tx_n,Tx_{n+1}) \leq \mu(Tgx_n,Tgx_{n+1}) < \mu(Tx_{n-1},Tx_n)$$

for all $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mu(Tgx_n,Tgx_{n+1}) = r.$$

Using the property (ζ_2) of a simulation function, we get

$$0 \leq \limsup_{n\to\infty} \zeta(\mu(Tgx_{n+1},Tgx_n),\mu(Tx_{n-1},Tx_n)) < 0$$

which is a contradiction, and hence $r = 0$.
Therefore

$$\lim_{n\to\infty} \mu(Tx_{n-1}, Tx_n) = 0. \tag{3.17}$$

Next, we claim that

$$\lim_{m,n\to\infty} \mu(Tx_n, Tx_m) = 0. \tag{3.18}$$

Assume that (3.18) is not true, that is

$$\lim_{m,n\to\infty} \mu(Tx_n, Tx_m) \neq 0.$$

From Lemma 3.1, then there exists $\epsilon > 0$ and two subsequence $\{Tx_{n_k}\}$ and $\{Tx_{m_k}\}$ of $\{Tx_n\}$ such that

$$\lim_{k\to\infty} \mu(Tx_{n_k}, Tx_{m_k}) = \lim_{k\to\infty} \mu(Tx_{n_{k+1}}, Tx_{m_{k+1}}) = \epsilon. \tag{3.19}$$

Then there exists a subsequence of $\{x_{n_k}\}$, which we assume it is the whole sequence $\{x_{n_k}\}$, such that $\mu(Tx_{n_k}, Tx_{m_k}) > 0$ for all $k \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the second kind and $d(gx_{n_k+1}, Tx_{n_k}) = d(A, B) = d(gx_{m_k+1}, Tx_{m_k})$, we have

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx_{n_k+1}, Tgx_{m_k+1}), \mu(Tx_{n_k}, Tx_{m_k})) \\ &< \mu(Tx_{n_k}, Tx_{m_k}) - \mu(Tgx_{n_k+1}, Tgx_{m_k+1}) \\ &\leq \mu(Tx_{n_k}, Tx_{m_k}) - \mu(Tx_{n_k+1}, Tx_{m_k+1}) \end{aligned}$$

for all $k \in \mathbb{N}$. From the above ineqauality and (3.19),

$$\lim_{k\to\infty} \mu(Tgx_{n_k+1}, Tgx_{m_k+1}) = \epsilon.$$

Using the property (ζ_2) of a simulation function with $t_k := \mu(Tgx_{n_k+1}, Tgx_{m_k+1})$ and $s_k := \mu(Tx_{n_k}, Tx_{m_k})$, we get

$$0 \leq \limsup_{k\to\infty} \zeta(t_k, s_k) < 0$$

which is a contradiction and hence (3.18) holds.
Since

$$\lim_{m,n\to\infty} \mu(Tx_n, Tx_m) = 0$$

we have

$$\limsup_{m,n\to\infty} \{p(Tx_n, Tx_m) : m > n\} = 0.$$

It follows from Lemma 2.7 that $\{Tx_n\}$ is a p-Cauchy sequance in B_0, And from Lemma 2.5, we have $\{Tx_n\}$ is a Cauchy sequance in B_0.

Since (X, d) is a complete metric space and $T(A_0)$ is a closed subset of X, there exists $\lim_{n\to\infty} Tx_n = Tu \in T(A_0) \subseteq B_0$. Moreover, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A, B).$$

Since $A_0 \subseteq g(A_0)$, we obtain that $z = gx$ for some $x \in A_0$. Hence

$$d(gx, Tu) = d(A, B). \tag{3.20}$$

If $x_n = x$ holds for infinite values of $n \in \mathbb{N}$, then $Tx = Tu$. Assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \neq x$ for all $n \geq n_0$. Then there exists a sunsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Tx_{n_k} \neq Tu$ for all $k \in \mathbb{N}$. Since T is a $\mathscr{Z}$-p-proximal contraction of the second kind, we get

$$0 \leq \zeta(\mu(Tgx_{n_k+1}Tgx), \mu(Tx_{n_k}, Tu)) < \mu(Tx_{n_k}, Tu) - \mu(Tgx_{n_k+1}, Tgx).$$

Hence

$$\mu(Tx_{n_k+1}, Tx) \leq \mu(Tgx_{n_k+1}, Tgx) < \mu(Tx_{n_k}, Tu) \tag{3.21}$$

for all $k \in \mathbb{N}$ such that $n_k \geq n_0$, because $T \in \mathscr{T}_{g,p}$.

It follows from (3.18) we obtain that for any $\epsilon > 0$ there exists a $N_\epsilon \in \mathbb{N}$ such that $\mu(Tx_n, Tx_m) < \epsilon$ for every $m > n \geq N_\epsilon$.

Since $\{Tx_n\}$ is a p-Cauchy sequance in B_0, there exists a function η from $B_0 \times [0, \infty) \to [0, \infty)$ satisfying $(\tau_2) - (\tau_5)$ and a sequence $\{z_n\}$ of B_0 such that

$$\limsup_{n\to\infty}\{\eta(z_n, p(z_n, Tx_m)) : m \geq n\} = 0.$$

By (τ_3) and $p(\cdot, x) : X \to [0, \infty)$ is lower semicontinuous, imply that

$$p(Tx_n, Tu) \leq \liminf_m p(Tx_n, Tx_m) < \epsilon$$

and

$$p(Tu, Tx_n) \leq \liminf_m p(Tx_m, Tx_n) < \epsilon$$

for any fixed $n \geq \max\{n_0, N_\epsilon\}$, which implies that

$$\lim_{k\to\infty} p(Tx_{n_k}, Tu) = 0. \tag{3.22}$$

Similarly $\lim_{k\to\infty} p(Tu, Tx_{n_k}) = 0$, and hence $\lim_{k\to\infty} \mu(Tx_{n_k}, Tu) = 0$. Combine this and (3.21) to get $\lim_{k\to\infty} \mu(Tx_{n_k+1}, Tx) = 0$. By triangle inequality of μ,

$$\mu(Tx_{n_k}, Tx) \leq \mu(Tx_{n_k}, Tx_{n_k+1}) + \mu(Tx_{n_k+1}, Tx).$$

From (3.17) and passing to limit as $k \to \infty$, we obtain $\lim_{k\to\infty} \mu(Tx_{n_k}, Tx) = 0$. This implies that

$$\lim_{k\to\infty} p(Tx_{n_k}, Tx) = 0 \tag{3.23}$$

Since $\lim_{k\to\infty} Tx_{n_k} = Tu$, we obtain

$$p(Tu, Tu) = 0 \text{ and } p(Tu, Tx) = 0.$$

Using (3.22), (3.23) and Lemma 2.6 imply that $Tx = Tu$. By substituting this in (3.20), we get $d(gx, Tx) = d(A, B)$.

We will show the uniqueness, let y be in A_0 such that

$$d(gy, Ty) = d(A, B),$$

i.e, $y \in B^g_{est}(T)$. Suppose that $\mu(Tgx, Tgy) \geq \mu(Tx, Ty) > 0$. Since $T \in \mathscr{T}_{g,p}$ is a $\mathscr{Z}$-p-proximal contraction of the second kind, we have

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx, Tgy), \mu(Tx, Ty)) \\ &< \mu(Tx, Ty) - \mu(Tgx, Tgy) \\ &\leq \mu(Tx, Ty) - \mu(Tx, Ty) = 0 \end{aligned}$$

which is a contraction. Hence $\mu(Tx, Ty) = 0$, which means that $Tx = Ty$. From T is injective on A_0, imply that $x = y$.

Finally, suppose that $\mu(Tx, Tx) = p(Tx, Tx) > 0$. Then $\mu(Tgx, Tgx) > 0$. Using a similar argument as above, we have

$$\begin{aligned} 0 &\leq \zeta(\mu(Tgx, Tgx), \mu(Tx, Tx)) \\ &< \mu(Tx, Tx) - \mu(Tgx, Tgx) \\ &\leq \mu(Tx, Tx) - \mu(Tx, Tx) = 0 \end{aligned}$$

which is a contraction. Therefore $p(Tx, Tx) = 0$. □

The following best proximity point result is a special case of Theorem 3.6 when g is an identity map on A.

Corollary 3.7. *Let A and B be two nonempty subsets of a complete metric space (X, d) with a τ-distance p, such that $T(A_0)$ is nonempty and closed. Let $p(\cdot, x) : X \to [0, \infty)$ is lower semi continuous for any $x \in X$, Suppose that the mappings $T : A \to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-p-proximal contraction of the second kind;*
(b) *T is injective on A_0;*
(c) *$T(A_0) \subseteq B_0$.*

Then there exists a unique best proximity point $x \in A_0$ of T with $p(Tx, Tx) = 0$, and for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

By setting $p = d$ in Theorem 3.6 the main result of [14] is obtained.

Corollary 3.8. *Let A and B be two nonempty subsets of a complete metric space (X, d), such that $T(A_0)$ is nonempty and closed. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfies the folowing conditions:*

(a) *T is a $\mathscr{Z}$-proximal contraction of the second kind;*
(b) *T is injective on A_0*
(c) *$T \in \mathscr{T}_g$;*
(d) *$A_0 \subseteq g(A_0)$;*
(e) *$T(A_0) \subseteq B_0$.*

Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_n = x$.

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