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NEW CONVERGENCE THEOREMS FOR COMMON FIXED POINTS OF A WIDE RANGE OF NONLINEAR MAPPINGS

TAKANORI IBARAKI*[1] AND YUKIO TAKEUCHI[2]

[1] Department of Mathematics Education, Yokohama National University,
79-2 Tokiwadai, Hodogaya, Yokohama 240-8501, Japan
[2] Takahashi Institute for Nonlinear Analysis
1-11-11 Nakazato, Minami, Yokohama 232-0063, Japan

ABSTRACT. In this article, we present new convergence theorems for common fixed points of a wide range of nonlinear mappings in the Hilbert space setting.

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1. Introduction

In 1963, DeMarr [11] proved a common fixed point theorem for families of commuting nonexpansive mappings. After DeMarr, many researchers studied this subject and many results for families of nonexpansive mappings appeared; refer to Linhart [27], Ishikawa [15], Kuhfittig [25], Kitahara and Takahashi [16], Takahashi and Tamura [39], Suzuki [36, 35] and so on. For example, in the strictly convex Banach space setting, Linhart [27] presented an iteration scheme for common fixed points of infinite families of commuting nonexpansive self–mappings on a compact convex set. Motivated by Linhart's result, Suzuki [36] presented the following.

Theorem S. Let C be a compact convex subset of a strictly convex Banach space E. Let $\{T_n\}$ be a sequence of nonexpansive mappings on C with $\cap_{n=1}^{\infty} F(T_n) \neq$. Let $\{a_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n < 1$ and let $\{I_n\}$ be a sequence of subsets of N satisfying $I_n \subset I_{n+1}$ for $n \in N$ and $\cup_{n=1}^{\infty} I_n = N$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = (1 - \textstyle\sum_{i \in I_n} a_i) x_n + \sum_{i \in I_n} a_i T_i x_n \qquad \text{for } \; n \in N.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

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* *Corresponding author.*
Email address : ibaraki@ynu.ac.jp, aho314159@yahoo.co.jp.

On the other hand, in 1975, Baillon [6] proved the first nonlinear ergodic theorem for a nonexpansive mapping in a Hilbert space. After Baillon, many mean convergence theorems appeared. Furthermore, Takahashi and Takeuchi [40] proved a mean convergence theorem for attractive points of generalized hybrid mappings with neither closeness nor convexity of the domain. Also, Aoyama [1] and Kohsaka [19] proved convergence theorems for quasi–nonexpansive type mappings.

In 1997, Shimizu and Takahashi [32] studied a common fixed point problem for finite families of commutative nonexpansive mappings. They introduced an iteration scheme combined Halpern type and Baillon type, and proved a strong convergence theorem in Hilbert spaces. In 1998, Atsushiba and Takahashi [4] introduced an iteration scheme combined Mann type and Baillon type, and proved a weak convergence theorem for commutative two nonexpansive mappings, in uniformly convex Banach spaces. Suzuki [34] and Takeuchi [42] studied this problem in general Banach spaces.

Very recently, in the Hilbert space setting, Kohsaka [20] replaced nonexpansive mappings by (λ)–hybrid mappings in the main theorems of [32, 4]. Kohsaka [20] also presented the following theorem; also see Ibaraki and Takeuchi [13].

Theorem K. Let C be a bounded closed and convex subset of a Hilbert space H. Let S and T be (λ)–hybrid self–mappings on C with λ and μ. Assume $ST = TS$. Set $F = F(S) \cap F(T)$. Define a sequence $\{x_n\}$ in C by $x_1 \in C$ and

$$x_{n+1} = \tfrac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} S^i T^j x_1 \qquad \text{for} \quad n \in N.$$

Then the following hold.

(1) $\{P_F S^i T^j x_1\}_{(i,j)\in N_0^2}$ converges strongly to $u \in F$ in the sense of net.
(2) $\{x_n\}$ converges weakly to $u \in F$.

Remark. Of course, we can replace the boundedness of C by $F = F(S) \cap F(T) \neq$.

Motivated by the works as above, we hope to add something new. Then, specifically, we prove some convergence theorems for common fixed points of a wide range of nonlinear self–mappings on a closed convex subset of a Hilbert space.

2. Preliminaries

In this article, N and N_0 denote the sets of positive integers and non–negative integers, respectively. $N(i, j)$ denotes the set $\{k \in N_0 : i \leq k \leq j\}$ for $i, j \in N_0$ with $i \leq j$. In the case of $j < i$, we define $N(i, j) =$ and $\sum_{k=i}^{j}(\cdot) = 0$.

H denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ derived from $\langle \cdot, \cdot \rangle$. C always denotes a non–empty subset of H unless otherwise noted. Then, normally, "non–empty" is omitted. The following are basic:

(1) A closed convex subset C of H is weakly closed. A bounded sequence in H has a weakly convergent subsequence.

(2) Let $\{u_n\}$ be a sequence in H. Then $\{u_n\}$ converges weakly to $z \in H$ if every weak cluster point of $\{u_n\}$ and z are the same.

(3) H has the Opial property [30], that is, if $\{u_n\}$ is a sequence in H which converges weakly to $u \in H$, then, for $v \in H$ with $v \neq u$,

$$\liminf_n \|u_n - u\| < \liminf_n \|u_n - v\|.$$

(4) Let C be a closed convex subset of H. For $x \in H$, there is the unique point z_x of C satisfying $\|x - z_x\| = \inf\{\|x - z\| : z \in C\}$. z_x is called the unique nearest

point of C to x. Define a mapping P_C by $P_C x = z_x$ for $x \in H$. P_C is called the metric projection from H onto C. P_C satisfies the following: For $x \in H$ and $y \in C$,

$$0 \leq \langle x - P_C x, P_C x - y \rangle \quad \text{and} \quad \|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2.$$

Let C be a subset of H and T be a mapping from C into H. I denotes the identity mapping on C. Sometimes we denote I by T^0. $F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : x = Tx\}$. $A(T)$ denotes the set of attractive points of T, that is, $A(T) = \{x \in H : \|Ty - x\| \leq \|x - y\| \text{ for all } y \in C\}$; for the notion of attractive points, see Takahashi and Takeuchi [40]. $I - T$ is said to be demiclosed at 0 if $u \in F(T)$ holds whenever there is a sequence $\{x_n\}$ in C which converges weakly to $u \in C$ and satisfies $\lim_n \|Tx_n - x_n\| = 0$. In the case that C is compact and convex, $I - T$ is demiclosed at 0 if T is continuous on C.

T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$. Also T is called quasi–nonexpansive if $\neq F(T) \subset A(T)$. A nonexpansive mapping T with $F(T) \neq$ is quasi–nonexpansive.

T is said to satisfy condition (N_2) if there is $s \in [0, \infty)$ such that

$$\|x - Ty\| \leq \|x - y\| + s\|x - Tx\| \quad \text{for} \quad x, y \in C. \qquad (N_2)$$

A nonexpansive mapping satisfies (N_2) as $s = 1$. T satisfies $F(T) \subset A(T)$ if T satisfies (N_2). Then T is quasi–nonexpansive if T satisfies (N_2) and $F(T) \neq$.

Recently, some researchers study (N_2); see Suzuki [37], Falset and co–authors [12], Takahashi and Takeuchi [40], Kubota and Takeuchi [22], and Kubota and co–authors [21]. Also, some researchers study generalized hybrid mappings introduced by Kocourek and co–authors [18] or (λ)–hybrid mappings introduced by Aoyama and co–authors [2]. The class of generalized hybrid mappings is wider than the class of (λ)–hybrid mappings. Even so, the class of (λ)–hybrid mappings contains some important classes of nonlinear mappings.

In [2], they say as below: Let $\lambda \in R$. T is called λ–hybrid if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty \rangle \quad \text{for} \quad x, y \in C. \qquad (\lambda_h)$$

For example, the following expression appeared in Kohsaka [20]: Let S be a λ–hybrid self–mapping on C and T be a μ–hybrid self–mapping on C. To avoid confusion, we call T (λ)–hybrid if there is $\lambda \in R$ satisfying (λ_h). Then the expression becomes as below: Let S and T be (λ)–hybrid self–mappings on C with λ and μ.

A nonexpansive mapping is (λ)–hybrid as $\lambda = 1$. T satisfies $F(T) \subset A(T)$ if T is (λ)–hybrid. So a (λ)–hybrid mapping T is quasi–nonexpansive if $F(T) \neq$. Since the last term in (λ_h) is written by inner product, it is easy to deal with.

We had better give remarks for our way of thinking in this article.

Our way of thinking. Let C be a closed convex subset of a Hilbert space H. In later sections, we deal with a sequence $\{T_j\}$ of nonlinear self–mappings on C. To have a convergence theorem for common fixed points of $\{T_j\}$, maybe it is difficult to ignore the condition that $I - T_j$ is demiclosed at 0 for $j \in N$. Then we will assume the condition to express our assertions. Also we consider the following conditions:

$$\neq A = \cap_{j \in N} A(T_j), \qquad F = \cap_{j \in N} F(T_j) \subset A.$$

We give some notes for the conditions. For simplicity, we consider $\{S, T\}$ as $\{T_j\}$.

Let S and T be self–mappings on a closed convex subset C of a Hilbert space H. We denote by F the common fixed point set $F(S) \cap F(T)$ and by A the common attractive point set $A(S) \cap A(T)$. To have a convergence theorem finding a common fixed point of $\{S, T\}$, usually, we assume $F \neq$. In the case that both S and T are nonexpansive, $F \neq$ asserts $\neq F \subset A$ in cooperation with properties of nonexpansive mappings. However, we should be more careful about the fact that we do not make some beneficial results from $F \neq$ itself. In proofs of many such theorems, it seems that conditions corresponding to $A \neq$ and $F \subset A$ are essential.

We note the following:

(a) $F \subset A$ implies neither $F(S) \subset A(S)$ nor $F(T) \subset A(T)$.
(b) $F \neq$ does not imply $A \neq$ without the assumption $F \subset A$.
(c) $A \neq$ implies $F \neq$; see Lemma 3.7.
However, $A \neq$ does not imply $F \subset A$.
(d) Usually, it follows from the assumption $F \subset A$ that F is closed and convex.

In the case that $S = T$, $F \subset A$ and $F(S) \subset A(S)$ are equivalent.

Suppose both S and T are quasi–nonexpansive. In this case, $\{S, T\}$ has so good properties, that is, we know the following:

(e) $F = F(S) \cap F(T) \subset A(S) \cap A(T) = A$.
(f) $F(S)$, $F(T)$ and F are closed and convex.

It is important that, even if $\neq F \subset A$, neither S nor T need be quasi–nonexpansive. Furthermore, we easily find pairs of C and $\{S, T\}$ such that neither S nor T is quasi–nonexpansive, $\neq F \subset A$ and $ST \neq TS$. However, in general, we may need strict constraints on properties of $\{S, T\}$ to guaranty $A \neq$ in theory. Even so, to find a point of A is easier than to find directly a point of $F \cap A$.

Due to the reasons as above, to express our assertions connected with common fixed points of $\{T_j\}$, we assume the following:

(i) $I - T_j$ is demiclosed at 0 for $j \in N$.
(ii) $\neq A = \cap_{j \in N} A(T_j)$ and $F = \cap_{j \in N} F(T_j) \subset A$.

Here we present an example. For simplicity, we consider R^2 with the Euclidean norm. Maybe T_1 and T_2 in the example are closed to us and just ordinary mappings.

Example 2.1. Let $D = \{x = (s, t) \in R^2 : s \in [0, 1], t \in [\frac{1}{2}s, 2s]\}$. Then D is compact and convex. For $x = (s, t) \in D$, set $u_x = (\frac{1}{2}t, t)$ and $z_x = (s, \frac{1}{2}s)$. Let T_1 and T_2 be self–mappings on D defined by

$$T_1 x = \tfrac{1}{2}(x + u_x) = \tfrac{1}{2}((s, t) + (\tfrac{1}{2}t, t)) = (\tfrac{1}{2}s + \tfrac{1}{4}t, t),$$
$$T_2 x = \tfrac{1}{2}(x + z_x) = \tfrac{1}{2}((s, t) + (s, \tfrac{1}{2}s)) = (s, \tfrac{1}{4}s + \tfrac{1}{2}t) \qquad \text{for } x = (s, t) \in D.$$

Then we can easily observe the following:

- (i) holds, that is, $I - T_j$ is demiclosed at 0 for $j = 1, 2$.
- (ii) holds, that is, $\neq \cap_{j=1}^{2} F(T_j) \subset \cap_{j=1}^{2} A(T_j)$.

Also, we can easily confirm the following:

- Neither T_1 nor T_2 is quasi–nonexpansive (hemi–contractive).
- T_1 and T_2 are not commutative.
- $B = \frac{1}{2}T_1 + \frac{1}{2}T_2$ is nonexpansive and $F(B) = \{(0, 0)\} = \cap_{j=1}^{2} F(T_j)$.

We had better note the following: A real linear space L may have more than one norms. Then it may depend on norm whether $\neq F \subset A$ holds or not. In some cases, nonexpansiveness of T and $A(T)$ depend on norm. Quasi–nonexpansiveness of T depends on norm and the domain of T. However, $F(T)$ has no connection with norms if the formula of T does not contain any norm on L. Especially, in finite dimensional linear spaces, we may choose a convenient norm to find a point of F.

3. Lemmas

Many researchers take the following assertion or a similar assertion in their articles; for example, see Weng [43], Xu [45], and Aoyama and co–authors [2].

Lemma 3.1. *Let $\{\alpha_n\}$ be a sequence in $[0,1]$. Let $\{a_n\}$ and $\{c_n\}$ be sequences of non–negative real numbers and let $\{b_n\}$ be a sequence of real numbers. Suppose $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\limsup_n b_n \le 0$, $\sum_{n=1}^{\infty} c_n < \infty$, and $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n b_n + c_n$ for $n \in N$. Then $\lim_n a_n = 0$.*

In the Hilbert space setting, we present some lemmas needed in the sequel; some of them are obtained in suitable Banach spaces. The following is well–known.

Lemma 3.2. *Let $c \in [0,1]$ and $x, y \in H$. Then, the following holds:*

$$\|cx + (1-c)y\|^2 = c\|x\|^2 + (1-c)\|y\|^2 - c(1-c)\|x-y\|^2.$$

In the Hilbert space setting, the following lemma is an extension of Browder's demiclosed principle. This lemma was essentially proved in Suzuki [37].

Lemma 3.3. *Let C be a subset of H and let S be a mapping from C into H which satisfies (N_2). Suppose $\{x_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. We know that $\{x_n\}$ converges weakly to u, S satisfies condition (N_2) for some $s \in [0,\infty)$, and $\lim_n \|Sx_n - x_n\| = 0$. Then the following holds:

$$\begin{aligned}\liminf_n \|x_n - Su\| &\le \liminf_n(\|x_n - u\| + s\|x_n - Sx_n\|) \\ &= \liminf_n \|x_n - u\|. \end{aligned} \tag{3.1}$$

Arguing by contradiction, assume $u \ne Su$. Then, by the Opial property, we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. This contradicts to (3.1). □

The following is another extension due to Aoyama and co–authors [2].

Lemma 3.4. *Let C be a subset of H and let S be a (λ)–hybrid mapping with λ from C into H. Suppose $\{x_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_n \|Sx_n - x_n\| = 0$. Then $u \in F(S)$.*

Proof. We know that $\{x_n\}$ converges weakly to u and S is (λ)–hybrid with λ. By $\lim_n \|Sx_n - x_n\| = 0$, the following hold:

$$\liminf_n \|x_n - Su\| \le \liminf_n(\|x_n - Sx_n\| + \|Sx_n - Su\|) = \liminf_n \|Sx_n - Su\|,$$

$$\begin{aligned}\liminf_n \|x_n - Su\|^2 &\le \liminf_n \|Sx_n - Su\|^2 \\ &\le \liminf_n(\|x_n - u\|^2 + 2|1-\lambda|\|Sx_n - x_n\|\|Su - u\|) = \liminf_n \|x_n - u\|^2.\end{aligned}$$

Then we have

$$\liminf_n \|x_n - Su\| \le \liminf_n \|x_n - u\|. \tag{3.2}$$

Arguing by contradiction, assume $u \ne Su$. Then, by the Opial property, we have $\liminf_n \|x_n - u\| < \liminf_n \|x_n - Su\|$. This contradicts to (3.2). □

The following lemma is useful when we consider weak convergence theorems in the Hilbert space setting; for example, see Atsushiba and co–authors [3].

Lemma 3.5. *Let D be a subset of H. Let $\{u_n\}$ be a sequence in H such that $\{\|u_n - w\|\}$ converges for each $w \in D$. Suppose $\{u_{n_i}\}$ and $\{u_{n_j}\}$ are subsequences of $\{u_n\}$ which converge weakly to $u, v \in D$, respectively. Then $u = v$.*

Proof. Let $w \in D$. Then, since $\{\|u_n - w\|\}$ converges, any subsequence of $\{\|u_n - w\|\}$ converges to the same real number. Arguing by contradiction, assume $u \ne v$. Then, by $u, v \in D$ and the Opial property, we have the following:

$$\liminf_i \|u_{n_i} - u\| < \liminf_i \|u_{n_i} - v\| = \liminf_j \|u_{n_j} - v\|,$$

$$\liminf_j \|u_{n_j} - v\| < \liminf_j \|u_{n_j} - u\| = \liminf_i \|u_{n_i} - u\|.$$

Thus we have $\liminf_i \|u_{n_i} - u\| < \liminf_i \|u_{n_i} - u\|$. This is a contradiction. □

The following two lemmas are due to Takahashi and Takeuchi [40].

Lemma 3.6. *Let C be a subset of H and let T be a mapping from C into H. Then, $A(T)$ is a closed convex subset of H.*

Lemma 3.7. *Let C be a subset of H and let T be a self–mapping on C. Suppose $x \in A(T)$ and z_x is the unique nearest point of C to x. Then $z_x \in F(T)$. In particular, $A(T) \cap C \subset F(T)$. Furthermore, $A(T) \cap C = F(T)$ holds if $F(T) \subset A(T)$.*

We need the following lemma in the sequel.

Lemma 3.8. *Let C be a subset of H and let T be a mapping from C into H. Let $a \in [0,1]$, $x \in C$ and $w = ax + (1-a)Tx$. Suppose $v \in A(T)$. Then,*

$$a(1-a)\|Tx - x\|^2 \le \|x - v\|^2 - \|w - v\|^2. \tag{1}$$

Suppose further that C is bounded. Let $r > \sup_{x \in C} \|x - v\|$. Then,

$$\tfrac{a(1-a)}{2r}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|. \tag{2}$$

Proof. We show (1). By $v \in A(T)$ and Lemma 3.2, we have

$$\begin{aligned}\|w - v\|^2 &= \|a(x - v) + (1-a)(Tx - v)\|^2 \\ &= a\|x - v\|^2 + (1-a)\|Tx - v\|^2 - a(1-a)\|Tx - x\|^2 \\ &\le \|x - v\|^2 - a(1-a)\|Tx - x\|^2.\end{aligned}$$

Then we see $\|w - v\| \le \|x - v\|$ and $a(1-a)\|Tx - x\|^2 \le \|x - v\|^2 - \|w - v\|^2$.

There is $r \in (0, \infty)$ satisfying $r > \sup_{x \in C} \|x - v\|$ if C is bounded. We show (2). Set $s = \|x - v\|$ and $t = \|w - v\| \le \|x - v\|$. Then we know $0 \le s + t < 2r$ and

$$a(1-a)\|Tx - x\|^2 \le s^2 - t^2 = (s-t)(s+t).$$

In the case of $0 < s + t < 2r$, we immediately have

$$\tfrac{a(1-a)}{2r}\|Tx - x\|^2 \le \tfrac{a(1-a)}{s+t}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|.$$

In the case of $s + t = 0$, it is trivial that $\frac{a(1-a)}{2r}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|$. □

4. Convergence theorems

In this section, we present our main results. We begin our argument with considering the following sequences. Let $\{c_j\}$ be a sequence satisfying the following:

$$c_j \in (0,1) \text{ for } j \in N, \qquad \textstyle\sum_{j=1}^{\infty} c_j = 1. \tag{s}$$

Let $\{c_{n,j}\}$ be the double sequence such that, for each $n \in N$,

$$c_{n,j} = c_j \text{ for } j \in N(1, n-1), \qquad c_{n,n} = \textstyle\sum_{j=n}^{\infty} c_j = 1 - \sum_{j=1}^{n-1} c_j. \tag{ds}$$

Note $N(1,0) =$, $\sum_{j=1}^{0}(\cdot) = 0$ and $c_{1,1} = 1$. For $j \in N$, $c_{n,j} = c_j$ holds for $n > j$. Then the double sequence $\{c_{n,j}\}$ has the following properties:

$$\lim_n c_{n,j} = c_j \quad \text{for } j \in N, \qquad \textstyle\sum_{j=1}^{n} c_{n,j} = 1 \quad \text{for } n \in N.$$

For reference, we present a typical example of $\{c_j\}$ satisfying (s). Set $c_j = 1/2^j$ for $j \in N$. Then $\{c_j\}$ satisfies (s). For example, $\{c_{5,j}\}_{j \in N(1,5)} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}\}$.

The following lemma is important to have our weak convergence theorems.

Lemma 4.1. *Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C. Assume $A = \cap_{j\in N} A(T_j) \neq$. Let $S_n = \sum_{j=1}^n c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1-a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *$\{\|x_n - u\|\}$ converges for $u \in A$.*
(2) *$\lim_n \|T_j x_n - x_n\| = 0$ for $j \in N$.*

Proof. Fix any $u \in A = \cap_{j\in N} A(T_j)$. We know that, for $n \in N$ and $x \in C$,

$$\|S_n x - u\| \le \textstyle\sum_{j=1}^n c_{n,j} \|T_j x - u\| \le \|x - u\|.$$

So $u \in \cap_{n\in N} A(S_n)$. Then, $A \subset \cap_{n\in N} A(S_n)$. Set $D = \{x \in C : \|x-u\| \le \|x_1 - u\|\}$. Then D is bounded and convex. By the inequality as above, we easily see that each S_n is a self–mapping on D. Then $\{x_n\}$ is a sequence in D.

We show (1). By Lemma 3.8 (1), we see that, for $n \in N$,

$$0 \le a_n(1-a_n)\|S_n x_n - x_n\|^2 \le \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Then $\{\|x_n - u\|\}$ is non–increasing and converges.

We show (2). Since D is bounded, let $r \in (0,\infty)$ satisfy $r > \sup_{x\in D} \|x - u\|$. Recall properties of $\{c_{n,j}\}$. By using Lemma 3.8 (2), we easily see that, for $n \in N$,

$$\begin{aligned}
\|x_{n+1} - u\| &= \|a_n x_n + (1-a_n) S_n x_n - u\| \\
&\le \textstyle\sum_{j=1}^n c_{n,j} \|a_n x_n + (1-a_n) T_j x_n - u\| \\
&\le \textstyle\sum_{j=1}^n c_{n,j} (\|x_n - u\| - \frac{a_n(1-a_n)}{2r} \|T_j x_n - x_n\|^2) \\
&\le \|x_n - u\| - \tfrac{a(1-b)}{2r} \textstyle\sum_{j=1}^n c_{n,j} \|T_j x_n - x_n\|^2.
\end{aligned}$$

From this inequality, the following follows:

$$\tfrac{a(1-b)}{2r} \textstyle\sum_{j=1}^n c_{n,j} \|T_j x_n - x_n\|^2 \le \|x_n - u\| - \|x_{n+1} - u\|.$$

Since $\{\|x_n - u\|\}$ converges and $\frac{a(1-b)}{2r} > 0$, we see that, for $j \in N$,

$$\limsup_n c_{n,j} \|T_j x_n - x_n\|^2 \le \limsup_n (\textstyle\sum_{j=1}^n c_{n,j} \|T_j x_n - x_n\|^2) \le 0.$$

Then we have the following:

$$\lim_n \|T_j x_n - x_n\| = 0 \quad \text{for} \;\; j \in N.$$

□

4.1. **Weak convergence theorems.**

We present a weak convergence theorem which is one of our main results.

Theorem 4.2. *Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Set $F = \cap_{j\in N} F(T_j)$ and $A = \cap_{j\in N} A(T_j)$. Assume $\neq A$. Let $S_n = \sum_{j=1}^n c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1-a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *Every weak cluster point of* $\{x_n\}$ *is a point of* F.
(2) *In the case of* $F \subset A$, $\{x_n\}$ *converges weakly to some* $z \in F$.

Proof. We know that C is weakly closed and $\{x_n\}$ is a sequence in C. By Lemma 4.1, we also know that $\{\|x_n - u\|\}$ converges for $u \in A$ and

$$\lim_n \|T_j x_n - x_n\| = 0 \quad \text{for} \quad j \in N. \tag{4.1}$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence.

We show (1). Let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $z \in C$. Since $I - T_j$ is demiclosed at 0 for $j \in N$, by (4.1), $z \in F = \cap_{j\in N} F(T_j)$. Thus every weak cluster point of $\{x_n\}$ is a point of F. We show (2). Suppose $F = \cap_j F(T_j) \subset A$. Then, $\{\|x_n - u\|\}$ converges for $u \in F \subset A$. Let z be a weak cluster point of $\{x_n\}$. Then, by Lemma 3.5 and (1), every weak cluster point of $\{x_n\}$ and $z \in F$ are the same. Thus $\{x_n\}$ converges weakly to $z \in F$. □

Remark 4.3. Let $m \in N_0$. By observing proofs of Lemma 4.1 and Theorems 4.2, it is obvious that we can replace S_n by S_{n+m} in the iteration scheme in Theorems 4.2.

Here we present some results derived from Theorems 4.2.

Theorem 4.4. *Let* $a, b \in (0,1)$ *satisfy* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a,b]$. *Let* $k \in N$. *Let* C *be a closed convex subset of* H *and let* $\{T_j\}_{j\in N(1,k)}$ *be a finite sequence of self–mappings on* C *such that* $I - T_j$ *is demiclosed at* 0 *for* $j \in N(1,k)$. *Set* $F = \cap_{j\in N(1,k)} F(T_j)$ *and* $A = \cap_{j\in N(1,k)} A(T_j)$. *Assume* $\ne A$. *Let* $S = \frac{1}{k}\sum_{j=1}^{k} T_j$. *Define a sequence* $\{x_n\}$ *by* $x_1 \in C$ *and*

$$x_{n+1} = a_n x_n + (1 - a_n) S x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *Every weak cluster point of* $\{x_n\}$ *is a point of* F.
(2) *In the case of* $F \subset A$, $\{x_n\}$ *converges weakly to some* $z \in F$.

Proof. Define $\{c_j\}$ and $\{U_j\}$ by

$$c_j = \tfrac{1}{k},\ U_j = T_j \quad \text{for} \quad j \in N(1,k-1), \quad c_j = \tfrac{1}{2^{j-k+1}} \times \tfrac{1}{k},\ U_j = T_k \quad \text{for} \quad j \ge k.$$

Then we know $\sum_{j=k}^{\infty} c_j = \frac{1}{k}$ and $\sum_{j=1}^{\infty} c_j = 1$. Also, we easily see that each $I - U_j$ is demiclosed at 0, $\cap_{j\in N} A(U_j) = \cap_{j\in N(1,k)} A(T_j)$ and $\cap_{j\in N} F(U_j) = \cap_{j\in N(1,k)} F(T_j)$. Let $\{c_{n,j}\}$ be the sequence satisfying (ds). Then $\{c_{n,j}\}$ and $\{U_j\}$ satisfy all assumptions in Theorem 4.2. Fix any $n \ge k$. We confirm that $S_n = \sum_{j=1}^{n} c_{n,j} U_j$ becomes S. By the definitions of $\{c_{n,j}\}$ and $\{U_j\}$, we have

$$\begin{aligned}\sum_{j=1}^{n} c_{n,j} U_j &= \sum_{j=1}^{k-1} \tfrac{1}{k} T_j + \sum_{j=k}^{n-1} \left(\tfrac{1}{2^{j-k+1}} \times \tfrac{1}{k}\right) T_k + \sum_{j=n}^{\infty} \left(\tfrac{1}{2^{j-k+1}} \times \tfrac{1}{k}\right) T_k \\ &= \tfrac{1}{k} \sum_{j=1}^{k} T_j.\end{aligned}$$

From these, by Theorems 4.2 and Remark 4.3, we have the results. □

Theorem 4.5. *Let* $a, b \in (0,1)$ *satisfy* $a \le b$ *and let* $\{a_n\}$ *be a sequence in* $[a,b]$. *Let* $\{c_j\}$ *be a sequence satisfying* (s) *and let* $\{c_{n,j}\}$ *be the double sequence satisfying* (ds). *Let* C *be a closed convex subset of* H *and let* $\{T_j\}$ *be a sequence of self–mappings on* C *such that* T_j *satisfies* (N_2) *for* $j \in N$. *Assume* $F = \cap_{j\in N} F(T_j) \ne$. *Let* $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ *for* $n \in N$. *Define a sequence* $\{x_n\}$ *by* $x_1 \in C$ *and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ *converges weakly to some* $z \in F$.

Proof. Since each T_j satisfies (N_2), $\neq F \subset A = \cap_{j\in N} A(T_j)$ holds. By Lemma 3.3, $I - T_j$ is demiclosed at 0 for $j \in N$. By Theorem 4.2, we have the result. □

Theorem 4.6. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of (λ)–hybrid self–mappings on C. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since each T_j is (λ)–hybrid, $\neq F \subset A = \cap_{j\in N} A(T_j)$ holds. By Lemma 3.4, $I - T_j$ is demiclosed at 0 for $j \in N$. By Theorem 4.2, we have the result. □

Theorem 4.7. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of quasi–nonexpansive self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since each T_j is quasi–nonexpansive, $\neq F \subset A = \cap_{j\in N} A(T_j)$ holds. By Theorem 4.2, we have the result. □

Theorem 4.8. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let C be a closed convex subset of H and let T be a quasi–nonexpansive self–mapping on C such that $I - T$ is demiclosed at 0. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) T x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F(T)$.

The following is corresponding to Theorem S due to Suzuki.

Theorem 4.9. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of nonexpansive self–mappings on C. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F$.

We present some convergence theorems for sequences of non–self mappings which are derived from Theorem 4.2. In advance, we prepare a lemma.

Lemma 4.10. *Let C be a closed convex subset of H and let T be a quasi–nonexpansive mapping from C into H. Then $F(T) = F(P_C T)$.*

Proof. Note $F(T) \neq$. In general, $F(T) \subset F(P_C T)$ holds. We show the reverse.

Let $z \in F(P_C T)$ and $u \in F(T)$. Since T is quasi–nonexpansive, we have

$$\|Tz - z\|^2 + \|z - u\|^2 = \|Tz - P_C Tz\|^2 + \|P_C Tz - u\|^2 \leq \|Tz - u\|^2 \leq \|z - u\|^2.$$

This implies $Tz = z$. Thus we have $F(P_C T) \subset F(T)$. □

The following is a direct consequence of Theorem 4.2.

Theorem 4.11. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H such that $I - P_C T_j$ is demiclosed at 0 for $j \in N$. Set $F' = \cap_{j \in N} F(P_C T_j)$ and $A' = \cap_{j \in N} A(P_C T_j)$. Assume $A' \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} P_C T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the followings hold.

(1) *Every weak cluster point of $\{x_n\}$ is a point of F'.*
(2) *In the case of $F' \subset A'$, $\{x_n\}$ converges weakly to some $z \in F'$.*

Remark 4.12. Here we give an additional explanation for Theorem 4.11.

Set $F = \cap_{j \in N} F(T_j)$ and $A = \cap_{j \in N} A(T_j)$. We consider the case of $F' = F$. Suppose $\{T_j\}$ is a sequence of quasi–nonexpansive mappings with $\neq F$. By Lemma 4.10, $F' = F$. Then, $\neq F' = F \subset A \cap A'$ holds because $F \subset A$ and

$$\|P_C T_j y - u\| = \|P_C T_j y - P_C u\| \leq \|T_j y - u\| \leq \|y - u\| \qquad \text{for } y \in C,\ u \in F.$$

We note the following: For $j \in N$, $P_C T_j$ $(T_j P_C)$ is nonexpansive if T_j is nonexpansive; $I - P_C T_j$ is demiclosed at 0. Furthermore, for example, we know the following: Let T be a k–strictly pseudo–contractive mapping from C into H, where $k \in [0,1)$. Then, we can easily find a nonexpansive mapping S satisfying $A(T) = A(S)$ and $F(T) = F(S)$; see Zhou [46], and Atsushiba and co–authors [3].

Suppose further that C is compact and every T_j is continuous. Then $P_C T_j$ is continuous; $I - P_C T_j$ is demiclosed at 0. Such pairs of C and $\{T_j\}$ are typical examples. So, $\neq F' = F \subset A \cap A'$ and assumptions in Theorem 4.11 are satisfied.

Theorem 4.13. *Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H satisfying the following:*

(a) *T_1 is a self–mappings on C, and $T_j T_1$ are self–mappings on C for $j \geq 2$.*
(b) *$I - T_j$ is demiclosed at 0 for $j \in N$.*

Let $V_1 = T_1$ and $V_j = T_j T_1$ for $j \geq 2$. Set $F = \cap_{j \in N} F(T_j)$ and $A' = \cap_{j \in N} A(V_j)$. Assume $A' \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} V_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then the following hold:

(1) *Every weak cluster point of $\{x_n\}$ is a point of F.*
(2) *In the case of $F \subset A'$, $\{x_n\}$ converges weakly to some $z \in F$.*

Proof. By Lemma 4.1, we know that $\{\|x_n - u\|\}$ converges for $u \in A'$. Also we know $\lim_n \|V_j x_n - x_n\| = 0$ for $j \in N$. Then we have

$$\text{(i) } \lim_n \|T_1 x_n - x_n\| = 0, \qquad \text{(ii) } \lim_n \|T_j T_1 x_n - x_n\| = 0 \ \text{ for } \ j \geq 2.$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence.

We show (1). Let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $z \in C$. Since $I - T_1$ is demiclosed at 0, by (i), we see $z \in F(T_1)$. Also, by (i), $\{T_1 x_{n_l}\}$ converges weakly to z. Furthermore, by (i) and (ii), we see

$$\lim_l \|T_j T_1 x_{n_l} - T_1 x_{n_l}\| = 0 \ \text{ for } \ j \geq 2. \tag{4.2}$$

From these, since $I - T_j$ is demiclosed at 0 for $j \in N$, we have $z \in F = \cap_{j \in N} F(T_j)$. Thus every weak cluster point of $\{x_n\}$ is a point of F.

We show (2). Suppose $F \subset A'$. Then $\{\|x_n - u\|\}$ converges for $u \in F \subset A'$. Let z be a weak cluster point of $\{x_n\}$. By Lemma 3.5 and (1), every weak cluster point of $\{x_n\}$ and $z \in F$ are coincide. Thus $\{x_n\}$ converges weakly to $z \in F$. □

Remark 4.14. In Theorem 4.13, set $A = \cap_{j \in N} A(T_j)$. For reference, we show $A \subset A'$. Let $u \in A$. Then, since T_1 is a self–mapping on C, we see

$$\|V_j x - u\| = \|T_j T_1 x - u\| \leq \|T_1 x - u\| \leq \|x - u\| \quad \text{for} \quad x \in C.$$

Note that we do not claim $A \neq$. For the theorem, we only present the following typical example: Let $C = [-1, 1] \subset R$ and let T_1 and T_2 be mappings defined by $T_1 x = x/2$ and $T_2 x = 2x$ for $x \in [-1, 1]$. Then it is obvious that $A(T_1) = \{0\}$, $A(T_2) =$, $F(T_2) = F(T_1) = \{0\}$, $T_2 T_1 = I$, $A(T_2 T_1) = R$ and $F(T_2 T_1) = C$. Furthermore, T_2 is not a self–mapping on C, $I - T_1$ and $I - T_2$ are demiclosed at 0, $A = A(T_1) \cap A(T_2) =$ and

$$\{0\} = F = F(T_1) \cap F(T_2) = F(T_1) \cap F(T_2 T_1) = A(T_1) \cap A(T_2 T_1) = A' = \{0\}.$$

4.2. **Strong convergence theorems.**

We present a strong convergence theorem which is our another main result. This theorem is connected with works of Aoyama [1], and Atsushiba and co–authors [3]; also see Maingé and Măruşter [28].

Theorem 4.15. *Let $b \in (0, 1)$ and let $\{a_n\}$ be a sequence in $(0, 1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Set $F = \cap_{j \in N} F(T_j)$ and $A = \cap_{j \in N} A(T_j)$. Assume $\neq A$ and $F \subset A$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1 - b) S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$.

Proof. Since C is convex, each S_n, each U_n and each $a_n q + (1 - a_n) U_n$ are self–mappings on C. Then $\{u_n\}$ is a sequence in C. By Lemma 3.6, A is closed and convex. Then we can consider the metric projection P_A. Set $v = P_A q \in A$ and $D = \{x \in C : \|x - v\| \leq \|u_1 - v\| + \|q - v\|\}$. Then D is bounded closed and convex. We know $q, u_1 \in D$ and $v \in A \subset \cap_{n \in N} A(S_n)$. Then, for $x \in D$ and $n \in N$, we have

$$\|U_n x - v\| \leq b\|x - v\| + (1 - b)\|S_n x - v\| \leq \|x - v\| \leq \|u_1 - v\| + \|q - v\|,$$

$$\begin{aligned} &\|a_n q + (1 - a_n) U_n x - v\| \\ &\qquad \leq a_n \|q - v\| + (1 - a_n)\|U_n x - v\| \leq \|u_1 - v\| + \|q - v\|. \end{aligned}$$

We confirmed that each U_n and each $a_n q + (1 - a_n) U_n$ are self–mappings on D, that is, we confirmed that $\{u_n\}$ and $\{U_n u_n\}$ are sequences in D.

We show that $\{u_n\}$ converges strongly to $v = P_A q$. We easily see that, for $n \in N$,

$$\|U_n u_n - u_n\| = \|(b u_n + (1 - b) S_n u_n) - u_n\| = (1 - b)\|S_n u_n - u_n\|. \tag{4.3}$$

By Lemma 3.8, we also see that, for $n \in N$,

$$\|U_n u_n - v\|^2 \leq \|u_n - v\|^2 - b(1 - b)\|S_n u_n - u_n\|^2. \tag{4.4}$$

Furthermore, it follows from (4.4) that, for $n \in N$,

$$\begin{aligned}\|u_{n+1} - v\|^2 &= \|a_n q + (1-a_n)U_n u_n - v\|^2 \\ &= \|(1-a_n)(U_n u_n - v) + a_n(q-v)\|^2 \\ &\le (1-a_n)\|U_n u_n - v\|^2 + a_n^2\|q-v\|^2 + 2a_n(1-a_n)\langle U_n u_n - v, q - v\rangle \\ &\le (1-a_n)\left(\|u_n - v\|^2 - b(1-b)\|S_n u_n - u_n\|^2\right) \\ &\qquad + a_n^2\|q-v\|^2 + 2a_n(1-a_n)\langle U_n u_n - v, q-v\rangle \\ &= (1-a_n)\|u_n - v\|^2 + a_n K_n, \end{aligned} \tag{4.5}$$

$$\text{where} \quad K_n = a_n\|q-v\|^2 + 2(1-a_n)\langle U_n u_n - v, q-v\rangle - \tfrac{(1-a_n)}{a_n} b(1-b)\|S_n u_n - u_n\|^2. \tag{4.6}$$

By $a_n, b \in (0,1)$ and (4.6), we easily see

$$\begin{aligned} K_n &\le a_n\|q-v\|^2 + 2(1-a_n)\langle U_n u_n - v, q-v\rangle \\ &\le \|q-v\|^2 + 2\|U_n u_n - v\|\|q-v\|. \end{aligned} \tag{4.7}$$

Then, since D is bounded, we know $\limsup_n K_n < \infty$. We show $\limsup_n K_n \le 0$.

Since D is weakly compact, there is a subsequence $\{n_l\}$ of $\{n\}$ such that $\{u_{n_l}\}$ converges weakly to some $u \in D$ and $\limsup_n K_n = \lim_l K_{n_l}$.

Consider the case of $\liminf_l \|S_{n_l} u_{n_l} - u_{n_l}\|^2 > 0$. Then there is $M > 0$ and a subsequence $\{n_{l_i}\}$ of $\{n_l\}$ satisfying $\|S_{n_{l_i}} u_{n_{l_i}} - u_{n_{l_i}}\|^2 > M > 0$. By $a_n, b \in (0,1)$, $\lim_n a_n = 0$ and (4.6), we know $K_{n_{l_i}} < 0$ for sufficiently large $i \in N$. Thus we have

$$\limsup_n K_n = \lim_l K_{n_l} = \lim_i K_{n_{l_i}} \le 0.$$

In the case of $\liminf_l \|S_{n_l} u_{n_l} - u_{n_l}\|^2 = 0$, by passing to subsequences, we may consider that $\{u_{n_l}\}$ converges weakly to $u \in D$ and satisfies the following:

$$\limsup_n K_n = \lim_l K_{n_l}, \qquad \lim_l \|S_{n_l} u_{n_l} - u_{n_l}\|^2 = 0.$$

By (4.3), $\lim_l \|U_{n_l} u_{n_l} - u_{n_l}\|^2 = 0$, that is, $\{U_{n_l} u_{n_l}\}$ also converges weakly to u.

Since D is bounded, there is $r \in (0,\infty)$ satisfying $r > \sup_{x\in D}\|x - v\|$. Recall properties of $\{c_{n,j}\}$. Then, by Lemma 3.8 (2), we see that, for $l \in N$,

$$\begin{aligned}\|U_{n_l} u_{n_l} - v\| &= \|b u_{n_l} + (1-b)S_{n_l} u_{n_l} - v\| \\ &\le \textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|b u_{n_l} + (1-b)T_j u_{n_l} - v\| \\ &\le \textstyle\sum_{j=1}^{n_l} c_{n_l,j}(\|u_{n_l} - v\| - \frac{b(1-b)}{2r}\|T_j u_{n_l} - u_{n_l}\|^2) \\ &= \|u_{n_l} - v\| - \tfrac{b(1-b)}{2r}\textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2. \end{aligned}$$

From this inequality, the following follows:

$$\tfrac{b(1-b)}{2r}\textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2 \le \|u_{n_l} - v\| - \|U_{n_l} u_{n_l} - v\| \le \|U_{n_l} u_{n_l} - u_{n_l}\|.$$

By $\lim_l \|U_{n_l} u_{n_l} - u_{n_l}\| = 0$ and $\frac{b(1-b)}{2r} > 0$, we see that, for $j \in N$,

$$\limsup_l c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2 \le \limsup_l (\textstyle\sum_{j=1}^{n_l} c_{n_l,j}\|T_j u_{n_l} - u_{n_l}\|^2) \le 0.$$

Then we have

$$\lim_l \|T_j u_{n_l} - u_{n_l}\| = 0 \quad \text{for} \quad j \in N.$$

Since $I - T_j$ is demiclosed at 0 for $j \in N$, we have $u \in F \subset A$.

Reconfirm that $\lim_l a_{n_l} = 0$, $v = P_A q$, and $\{U_{n_l} u_{n_l}\}$ converges weakly to $u \in A$. Then, by (4.7), we have the following:

$$\begin{aligned}\lim_l K_{n_l} &\le \lim_l \left(a_{n_l}\|v-q\|^2 + 2(1-a_{n_l})\langle U_{n_l}u_{n_l} - v, q - v\rangle\right)\\ &= 2\langle u - P_A q, q - P_A q\rangle \le 0.\end{aligned}$$

Thus we have $\limsup_n K_n = \lim_l K_{n_l} \le 0$.

We know that (4.5) holds. Then, by properties of $\{a_n\}$ and $\limsup_n K_n \le 0$, Lemma 3.1 asserts $\lim_n \|u_n - v\|^2 = 0$, that is, $\{u_n\}$ converges strongly to $v = P_A q$.

Finally, we show $v = P_A q = P_F q \in F$. Since D is closed and $\{u_n\} \subset D$, we know that $v = P_A q \in A \cap D$. By Lemma 3.7 and $F \subset A \cap C$, we can easily see that

$$A \cap C = (\cap_{j\in N} A(T_j)) \cap C = \cap_{j\in N}(A(T_j) \cap C) \subset \cap_{j\in N} F(T_j) = F \subset A \cap C.$$

Then $v = P_A q \in A \cap D \subset A \cap C = F$. We know that $F = A \cap C$ is closed and convex. Then we can consider the metric projection P_F. By $v \in F \subset A$, we know

$$\|q - v\| = \min_{y\in A}\|q-y\| \le \inf_{y\in F}\|q-y\| \le \|q-v\|.$$

This implies $\|q-v\| = \min_{y\in F}\|q-y\|$ and $v = P_F q$. Thus $v = P_A q = P_F q \in F$. □

We present some results follow from Theorem 4.15; refer to previous subsection.

Theorem 4.16. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^\infty a_n = \infty.$$

Let $k \in N$. Let C be a closed convex subset of H and let $\{T_j\}_{j\in N(1,k)}$ be a finite sequence of self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N(1,k)$. Set $F = \cap_{j\in N(1,k)} F(T_j)$ and $A = \cap_{j\in N(1,k)} A(T_j)$. Assume $\neq F \subset A$. Let $S = \frac{1}{k}\sum_{j=1}^k T_j$ and $U = bI + (1-b)S$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n) U u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$.

Theorem 4.17. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^\infty a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of self–mappings on C such that T_j satisfies (N_2) for $j \in N$. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^n c_{n,j} T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Theorem 4.18. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^\infty a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of (λ)–hybrid self–mappings on C. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^n c_{n,j} T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Theorem 4.19. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of quasi–nonexpansive self–mappings on C such that $I - T_j$ is demiclosed at 0 for $j \in N$. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Theorem 4.20. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let C be a closed convex subset of H and let T be a quasi–nonexpansive self–mapping on C such that $I - T$ is demiclosed at 0. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n)(b u_n + (1-b) T u_n) \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_{F(T)} q \in F(T)$.

Theorem 4.21. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of nonexpansive self–mappings on C. Assume $F = \cap_{j\in N} F(T_j) \neq$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Here we present strong convergence theorems for sequences of non–self mappings which are corresponding to Theorems 4.11 and 4.13; also see Remarks 4.12 and 4.14.

The following is a direct consequence of Theorem 4.15.

Theorem 4.22. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H such that $I - P_C T_j$ is demiclosed at 0 for $j \in N$. Set $F' = \cap_{j\in N} F(P_C T_j)$ and $A' = \cap_{j\in N} A(P_C T_j)$. Assume $\neq F' \subset A'$. Let $S_n = \sum_{j=1}^{n} c_{n,j} P_C T_j$ and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_{F'} q \in F'$.

Theorem 4.23. *Let $b \in (0,1)$ and let $\{a_n\}$ be a sequence in $(0,1)$ satisfying*

$$\lim_n a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a closed convex subset of H and let $\{T_j\}$ be a sequence of mappings from C into H satisfying the following:

(a) *T_1 is a self–mappings on C and T_jT_1 are self–mappings on C for $j \geq 2$.*
(b) *$I - T_j$ is demiclosed at 0 for $j \in N$.*

Let $V_1 = T_1$ and $V_j = T_jT_1$ for $j \geq 2$. Set $F = \cap_{j\in N} F(T_j)$ and $A' = \cap_{j\in N} A(V_j)$. Assume $\neq F \subset A'$. Let $S_n = \sum_{j=1}^{n} c_{n,j} V_j$ for $n \in N$and $U_n = bI + (1-b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1-a_n)U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_F q \in F$.

Proof. Fix any $j \geq 2$. Then, $T_jT_1u = T_ju = u$ for $u \in F(T_1) \cap F(T_j)$ and $T_jw = T_jT_1w = w$ for $w \in F(T_1) \cap F(T_jT_1)$. So, $F(T_1) \cap F(T_jT_1) = F(T_1) \cap F(T_j)$ for $j \geq 2$. Set $F' = \cap_{j\in N} F(V_j)$. Then we have

$$\begin{aligned} F' &= \cap_{j\in N} F(V_j) = F(T_1) \cap (\cap_{j\geq 2} F(T_jT_1)) \\ &= \cap_{j\geq 2}(F(T_1) \cap F(T_jT_1)) = \cap_{j\geq 2}(F(T_1) \cap F(T_j)) = \cap_{j\in N} F(T_j) = F. \end{aligned}$$

By our assumption, we see $\neq F = F' = \cap_{j\in N} F(V_j) \subset A' = \cap_{j\in N} A(V_j)$.

Then, replace T_j by V_j in the proof of Theorem 4.15. So, the rest of our proof and the proof of Theorem 4.15 are the same without the part below.

Let $\{u_{n_l}\}$ be a sequence in C. Suppose $\{u_{n_l}\}$ converges weakly to $u \in C$ and $\lim_l \|V_j u_{n_l} - u_{n_l}\| = 0$ for $j \in N$. However, from this, $u \in F' = \cap_{j\in N} F(V_j)$ does not follows directly. Because we do not know whether $I - V_j$ is demiclosed at 0 for $j \geq 2$. Instead, we know that $I - T_j$ is demiclosed at 0 for $j \in N$.

We show $u \in F' = \cap_{j\in N} F(V_j)$. Since $\lim_l \|V_j u_{n_l} - u_{n_l}\| = 0$ for $j \in N$, we know

$$\text{(i)}\ \lim_l \|T_1 u_{n_l} - u_{n_l}\| = 0, \qquad \text{(ii)}\ \lim_l \|T_jT_1 u_{n_l} - u_{n_l}\| = 0 \quad \text{for} \quad j \geq 2.$$

Furthermore, by (i) and (ii), we see

$$\lim_l \|T_jT_1 u_{n_l} - T_1 u_{n_l}\| = 0 \quad \text{for} \quad j \geq 2. \tag{4.8}$$

Thus, by (i) and (4.8), we see $u \in F = \cap_{j\in N} F(T_j) = \cap_{j\in N} F(V_j) = F'$.

□

5. Existence theorems and convergence theorems

The authors think that Theorems 4.2 and 4.15 are interesting. The theorems may have many useful applications because they are expressed in so wide setting. However, to guaranty $\neq A$ in theory, maybe $\{T_j\}$ need satisfy some strict constraints. Even so, we are interested in finding such $\{T_j\}$ and having related results.

We begin our argument with presenting two lemmas: for details, see Takahashi and Takeuchi [40], and Ibaraki and Takeuchi [13].

Lemma 5.1. *Let $x, v, w \in H$. Then the following equality holds:*

$$\langle (x-v) + (x-w), v-w \rangle = \|x-w\|^2 - \|x-v\|^2.$$

Remark 5.2. Let $v, w \in H$ and let $\{z_i\}$ be a sequence in H. Set $s_n = \frac{1}{n}\sum_{i=1}^{n} z_i$ for $n \in N$. Then, by Lemma 5.1, the following is immediate: For each $n \in N$,

$$\langle (s_n - v) + (s_n - w), v - w \rangle = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} \|z_i - w\|^2 - \tfrac{1}{n}\sum_{i=1}^{n} \|z_i - v\|^2.$$

Lemma 5.3. *Let C be a subset of H and let T be a mapping from C into H. Let $\{u_n\}$ be a sequence in H which satisfies*

$$\limsup_n \sup_{y\in C} \langle (u_n - y) + (u_n - Ty), y - Ty \rangle \leq 0.$$

Suppose $\{u_n\}$ converges weakly to $u \in H$. Then, $u \in A(T)$.

In the rest of this section, we deal with (λ)–hybrid mappings. We prepare the following lemma. For the lemma, there are previous studies; refer to Kohsaka [20], Brézis and Browder [7], Shimizu and Takahashi [32], and Takahashi and Takeuchi [40].

Lemma 5.4. *Let $k \in N$. Let C be a bounded subset of H. Set $L = \sup_{x,y\in C} \|x-y\|$. Let $\{T_j\}_{j\in N(1,k)}$ be a finite sequence of self–mappings on C. Assume that T_1 is (λ)–hybrid with λ. For $n \in N$, define a mapping S_n from C into H by*

$$S_n = \tfrac{1}{n^k} \textstyle\sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T_1^{i_1} \cdots T_k^{i_k}.$$

Then, for $n \in N$, the following holds:

$$\sup_{x,y\in C} \langle (S_n x - y) + (S_n x - T_1 y), y - T_1 y\rangle \le \tfrac{1+2|1-\lambda|}{n} L^2.$$

Remark. *Each S_n need not be a self–mapping on C.*

Proof. Fix any $x, y \in C$ and $n \in N$. We easily have

$$\begin{aligned} |\textstyle\sum_{i_1=1}^{n-1} \langle T_1^{i_1-1}x - T_1^{i_1}x, y - T_1 y\rangle| &= |\langle x - T_1^{n-1}x, y - T_1 y\rangle| \\ &\le \|x - T_1^{n-1}x\|\|y - T_1 y\| \le L^2. \end{aligned}$$

Then, since T_1 is (λ)–hybrid with λ, we have

$$\begin{aligned} \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - T_1 y\|^2 &= \tfrac{1}{n}\|x - T_1 y\|^2 + \tfrac{1}{n} \textstyle\sum_{i_1=1}^{n-1} \|T_1^{i_1}x - T_1 y\|^2 \\ &\le \tfrac{1}{n} L^2 + \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-2} \|T_1^{i_1}x - y\|^2 \\ &\quad + \tfrac{2(1-\lambda)}{n} \textstyle\sum_{i_1=1}^{n-1} \langle T_1^{i_1-1}x - T_1^{i_1}x, y - T_1 y\rangle \\ &\le \tfrac{1}{n} L^2 + \tfrac{2|1-\lambda|}{n} \times L^2 + \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - y\|^2. \end{aligned} \tag{5.1}$$

In Remark 5.2, set $z_i = T_1^{i_1-1}x \in C$, $w = T_1 y$ and $v = y$. Then, by (5.1), we have

$$\begin{aligned} &\langle (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1}x - y) + (\tfrac{1}{n} \sum_{i_1=0}^{n-1} T_1^{i_1}x - T_1 y), y - T_1 y\rangle \\ &\quad = \tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} \|T_1^{i_1}x - T_1 y\|^2 - \tfrac{1}{n} \sum_{i_1=0}^{n-1} \|T_1^{i_1}x - y\|^2 \le \tfrac{1+2|1-\lambda|}{n} L^2. \end{aligned} \tag{5.2}$$

Fix any $i_2, \cdots, i_k \in N(0, n-1)$. By replacing x by $T_2^{i_2} \cdots T_k^{i_k} x$ in (5.2), we have

$$\begin{aligned} &\langle (\tfrac{1}{n} \textstyle\sum_{i_1=0}^{n-1} T_1^{i_1} T_2^{i_2} \cdots T_k^{i_k} x - y) + (\tfrac{1}{n} \sum_{i_1=0}^{n-1} T_1^{i_1} T_2^{i_2} \cdots T_k^{i_k} x - T_1 y), y - T_1 y\rangle \\ &\qquad \le \tfrac{1+2|1-\lambda|}{n} L^2. \end{aligned} \tag{5.3}$$

Since $i_2, \cdots, i_k \in N(0, n-1)$ are arbitrary, the following holds:

$$\begin{aligned} &\tfrac{1}{n^{k-1}} \textstyle\sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} (\tfrac{1}{n} \sum_{i_1=0}^{n-1} T_1^{i_1} T_2^{i_2} \cdots T_k^{i_k} x) \\ &\quad = \tfrac{1}{n^k} \textstyle\sum_{i_1=0}^{n-1} \sum_{i_2=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} T_1^{i_1} T_2^{i_2} \cdots T_k^{i_k} x = S_n x. \end{aligned} \tag{5.4}$$

By (5.3) and (5.4), we have

$$\langle (S_n x - y) + (S_n x - T_1 y), y - T_1 y\rangle \le \tfrac{1+2|1-\lambda|}{n} L^2.$$

Finally, since x, y, n are arbitrary, we see that, for $n \in N$,

$$\sup_{x,y\in C} \langle (S_n x - y) + (S_n x - S y), y - S y\rangle \le \tfrac{1+2|1-\lambda|}{n} L^2.$$

□

We denote by $\lambda(C)$ the set of all (λ)–hybrid self–mappings on a subset C of a Hilbert space H. Also, we denote by $\lambda_1(C)$ the subset of $\lambda(C)$ such that a (λ)–hybrid self–mapping on C with λ is an element of $\lambda_1(C)$ if and only if $|1-\lambda| \le 1$. Then $\lambda_1(C)$ is the principal part of $\lambda(C)$; refer to Aoyama and co–authors [2].

Remark 5.5. Let C be a bounded convex subset of H. Under this setting, consider (5.2) in the proof of Lemma 5.4. Fix any $S \in \lambda_1(C)$ and $x \in C$. Set $T_1 = S$ and $y = \frac{1}{n}\sum_{i=0}^{n-1} S^i x \in C$. Then, (5.2) becomes

$$\langle \tfrac{1}{n}\sum_{i=0}^{n-1} S^i x - S(\tfrac{1}{n}\sum_{i=0}^{n-1} S^i x), \tfrac{1}{n}\sum_{i=0}^{n-1} S^i x - S(\tfrac{1}{n}\sum_{i=0}^{n-1} S^i x)\rangle \le \tfrac{3}{n}L^2.$$

Then we easily see that the following equality holds without closeness of C:

$$\lim_n \sup_{S\in\lambda_1(C),\, x\in C} \left\| S\left(\tfrac{1}{n}\sum_{i=0}^{n-1} S^i x\right) - \tfrac{1}{n}\sum_{i=0}^{n-1} S^i x \right\| = 0.$$

We know that nonexpansive self–mappings on C are elements of $\lambda_1(C)$. So, in the Hilbert space setting, we obtained an extension of Bruck's well known lemma [10].

Lemma 5.6. *Let $k \in N$. Let C be a bounded subset of H. Set $L = \sup_{x,y\in C}\|x-y\|$. Let $\{T_j\}_{j\in N(1,k)}$ be a finite sequence of self–mappings on C. Assume that T_1 is (λ)–hybrid with λ. For $n \in N$, define a mapping S_n from C into H by*

$$S_n = \tfrac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T_1^{i_1}\cdots T_k^{i_k}.$$

Then, the following hold:

(1) $\limsup_n \sup_{x,y\in C}\langle (S_n x - y) + (S_n x - T_1 y), y - T_1 y\rangle \le 0$.
(2) *For $x_1 \in C$, $\{S_n x_1\}$ is bounded.*
(3) *For $x_1 \in C$, every weak cluster point of $\{S_n x_1\}$ is a point of $A(T_1)$.*
(4) *$A(T_1)$ is non–empty closed and convex.*

Suppose further that C is closed and convex. Then the following hold:

(5) *For $x_1 \in C$, every weak cluster point of $\{S_n x_1\}$ is a point of $F(T_1)$.*
(6) *$F(T_1)$ is non–empty bounded closed and convex.*

Proof. By $\limsup_n \frac{1+2|1-\lambda|}{n}L^2 = 0$ and Lemma 5.4, we immediately see that (1) holds. We show (2)–(4). Fix any $x_1 \in C$ and consider $\{S_n x_1\}$.

Fix any $y \in C$. Then, by $T_1^{i_1}\cdots T_k^{i_k}x_1 \in C$ for $i_1,\cdots,i_k \in N_0$, we see that

$$\|S_n x_1 - y\| \le \tfrac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1}\|T_1^{i_1}\cdots T_k^{i_k}x_1 - y\| \le L.$$

Then $\{S_n x_1\}$ is bounded and has a weakly convergent subsequence. Let $\{S_{n_l}x_1\}$ be a subsequence of $\{S_n x_1\}$ which converges weakly to $u \in H$. By (1), we know

$$\limsup_l \sup_{y\in C}\langle (S_{n_l}x_1 - y) + (S_{n_l}x_1 - T_1 y), y - T_1 y\rangle \le 0.$$

Then, by Lemma 5.3, we know $u \in A(T_1)$. We confirmed that (3) holds. We also confirmed $A(T_1) \ne$. By Lemma 3.6, $A(T_1)$ is closed and convex. Then (4) holds.

Suppose further that C is closed and convex. We show (5) and (6). In the same way as in the proof of (3), we know $u \in A(T_1)$. Also $\{S_{n_l}x_1\}$ is in the weakly closed set C. Then, $u \in A(T_1)\cap C$. By Lemma 3.7, we see $u \in A(T_1)\cap C \subset F(T_1)$. So, we confirmed that (5) holds. Also we confirmed $F(T_1) \ne$. Since T_1 is (λ)–hybrid, by Lemma 3.7, we have $\ne F(T_1) = A(T_1)\cap C$. Then, (6) follows from (4). □

The following is a direct consequence of Lemma 5.6.

Lemma 5.7. *Let $k \in N$. Let C be a bounded subset of H and let $\{T_j\}_{j\in N(1,k)}$ be a finite family of commuting (λ)–hybrid self–mappings on C. Set $F = \cap_{j\in N(1,k)}F(T_j)$ and $A = \cap_{j\in N(1,k)}A(T_j)$. Then, A is non–empty closed and convex. Suppose further that C is closed and convex. Then F is non–empty bounded closed and convex.*

Proof. Since $\{T_j\}_{j\in N(1,k)}$ is commuting, for example, $T_1^{i_1}T_2^{i_2}\cdots T_k^{i_k} = T_2^{i_2}T_1^{i_1}\cdots T_k^{i_k}$. Since each T_j is (λ)–hybrid with λ_j, by Lemma 5.6 (4)–(6), the proof is trivial. □

In the Hilbert space setting, by using Lemma 5.7, we can have an extension of DeMarr's well–known common fixed point theorem; see DeMarr [11].

Theorem 5.8. *Let C be a bounded closed convex subset of H and let $\{T_j\}_{j\in J}$ be a family of commuting (λ)–hybrid self–mappings on C. Then $F = \cap_{j\in J} F(T_j)$ is non–empty bounded closed and convex.*

Proof. Since each T_j is (λ)–hybrid, we already know that $F(T_j)$ is closed and convex for $j \in J$. So $\{F(T_j)\}_{j\in J}$ consists of weakly closed subsets of C. By Lemma 5.7, $\{F(T_j)\}_{j\in J}$ has the finite intersection property. Thus, since C is weakly compact, we see $F = \cap_{j\in J} F(T_j) \neq$. It is obvious that F is bounded closed and convex. □

By Theorems 4.2 and 5.8, we have the following weak convergence theorem.

Theorem 5.9. *Let $a, b \in (0, 1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a, b]$. Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a bounded closed convex subset of H and let $\{T_j\}$ be a sequence of commuting (λ)–hybrid self–mappings on C. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ for $n \in N$. Define a sequence $\{x_n\}$ by $x_1 \in C$ and*

$$x_{n+1} = a_n x_n + (1 - a_n) S_n x_n \qquad \text{for } n \in N.$$

Then $\{x_n\}$ converges weakly to some $z \in F = \cap_{j\in N} F(T_j)$.

Proof. Set $A = \cap_{j\in N} A(T_j)$. By Lemma 3.4, $I - T_j$ is demiclosed at 0 for $j \in N$. Since T_j is (λ)–hybrid for $j \in N$, by Theorem 5.8, we know

$$\neq F = \cap_{j\in N} F(T_j) \subset \cap_{j\in N} A(T_j) = A.$$

Thus, by Theorems 4.2 (2), $\{x_n\}$ converges weakly to some $z \in F$. □

By Theorems 4.15 and 5.8, we have the following strong convergence theorem.

Theorem 5.10. *Let $b \in (0, 1)$ and let $\{a_n\}$ be a sequence such that $a_n \in (0, 1)$,*

$$\lim_n a_n = 0, \qquad \textstyle\sum_{n=1}^{\infty} a_n = \infty.$$

Let $\{c_j\}$ be a sequence satisfying (s) and let $\{c_{n,j}\}$ be the double sequence satisfying (ds). Let C be a bounded closed convex subset of H and let $\{T_j\}$ be a sequence of commuting (λ)–hybrid self–mappings on C. Set $F = \cap_{j\in N} F(T_j)$ and $A = \cap_{j\in N} A(T_j)$. Let $S_n = \sum_{j=1}^{n} c_{n,j} T_j$ and $U_n = bI + (1 - b)S_n$ for $n \in N$. Define a sequence $\{u_n\}$ by $q, u_1 \in C$ and

$$u_{n+1} = a_n q + (1 - a_n) U_n u_n \quad \text{for} \quad n \in N.$$

Then $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$.

Proof. By Lemma 3.4, for $j \in N$, $I - T_j$ is demiclosed at 0. Since T_j is (λ)–hybrid for $j \in N$, by Theorem 5.8, we know

$$\neq F = \cap_{j\in N} F(T_j) \subset \cap_{j\in N} A(T_j) = A.$$

Thus, by Theorem 4.15, $\{u_n\}$ converges strongly to $v = P_A q = P_F q \in F$. □

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