Journal of Nonlinear Analysis and Optimization
Volume 9(2) (2018)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

SOME INTEGRAL INEQUALITIES OF THE HADAMARD AND THE FEJÉR-HADAMARD TYPE VIA GENERALIZED FRACTIONAL INTEGRAL OPERATOR

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ABSTRACT.

In this paper we give the Hadamard and the Fejér-Hadamard type integral inequalities for convex and relative convex functions by involving a generalization of the Riemann-Liouville fractional integral. Also some connections with known results have been obtained.
KEYWORDS: Convex function; Hadamad inequality; Fejér-Hadamard inequality; Fractional integral operators.
AMS Subject Classification: Primary 26A51; Secondary 26A33, 33E12.

1. Preliminaries

Convex functions are very useful for diverse fields of Mathematics, a rich literature has been built since their discovery [15].

Definition 1.1. Let I be an interval of real numbers. Then a function $f : I \to \mathbb{R}$ is said to be convex function if for all $x, y \in I$ and $0 \le \lambda \le 1$ the following inequality holds

$$f(x\lambda + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y).$$

Convex functions are naturally obey the following inequality which is well known as the Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

where $f : I \to \mathbb{R}$ is a convex function on I and $a, b \in I, a < b$.
Following definitions are given in [14].

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Definition 1.2. Let T_g be a set of real numbers. This set T_g is said to be relative convex with respect to an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ if

$$(1-t)x + tg(y) \in T_g$$

where $x, y \in \mathbb{R}$ such that $x, g(y) \in T_g$, $0 \le t \le 1$.

Note that every convex set is relative convex, but the converse is not true. For example $T_g = [-1, \frac{-1}{2}] \bigcup [0,1]$ and $g(x) = x^2$, for all $x \in \mathbb{R}$. This set is relative convex but not convex set. Another possibility may be occure that a realtive covex set is convex set for example if $T_g = [-1,1]$ and $g(x) = (|x|)^{\frac{1}{4}}$ for all $x \in \mathbb{R}$ (see[9]). If $g = I$ the identity function, then the definition of relative convex set recaptures the definition of classical convex set.

Definition 1.3. A function $f : T_g \to \mathbb{R}$ is said to be relative convex, if there exists an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f((1-t)x + tg(y)) \le (1-t)f(x) + tf(g(y)),$$

holds, where $x, y \in \mathbb{R}$ such that $x, g(y) \in T_g$, $0 \le t \le 1$.

Noor et al proved the following Hadamard type integral inequality in [14] for relative convex functions via Riemann-Liouville fractional integral operators.

Theorem 1.4. *Let f be a positive relative convex function and integrable on $[a, g(b)]$. Then the following inequality holds*

$$f\left(\frac{a+g(b)}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(g(b)-a)^{\alpha}}[I^{\alpha}_{a^+} fg(b) + I^{\alpha}_{b_-} f(a)] \le \frac{f(a) + f(g(b))}{2}$$

$\alpha > 0$.

In the following we give some definitions and known facts about fractional integral operators [17].

Definition 1.5. Let $\omega \in \mathbb{R}$ and $\alpha, \beta, k, l, \gamma$ be positive real numbers. The generalized fractional integral operators $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ and $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}$ for a real valued continuous function f are defined as follows

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} f\right)(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha}) f(t)dt, \tag{1.1}$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} f\right)(x) = \int_x^b (t-x)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(t-x)^{\alpha}) f(t)dt,$$

where the function $E^{\gamma,\delta,k}_{\alpha,\beta,l}$ is the generalized Mittag-Leffler function defined as

$$E^{\gamma,\delta,k}_{\alpha,\beta,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\alpha n + \beta)(\delta)_{ln}}, \tag{1.2}$$

the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)(a+2)...(a+n-1)$, $(a)_0=1$.

For $\omega = 0$, (1.1) produces the definition of Riemann-Liouville fractional integral operators [17]

$$I^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)dt, \ x > a$$

and

$$I^{\beta}_{b_-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t)dt, \ x < b.$$

In [17] properties of the generalized Mittag-Leffler function are discussed and it is given that $E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)$ is absolutely convergent for $k < l + \alpha$. Let S be the sum of series of absolute terms of the Mittag-Leffler function $E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)$, then we have $\left|E^{\gamma,\delta,k}_{\alpha,\beta,l}(t)\right| \leq S$. We use this property of Mittag-Leffler function in our results where we need.

In [10] the following Hadamard and the Fejér-Hadamard inequalities for convex functions via generalized fractional integral operator containing the Mittag-Leffler function have been proved.

Theorem 1.6. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is convex on $[a, b]$, then the following inequality for generalized fractional integrals holds*

$$f\left(\frac{a+b}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1)(b) \leq \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}f)(a)}{2} \tag{1.3}$$
$$\leq \frac{f(a)+f(b)}{2}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}1)(a),$$

where $\omega' = \frac{w}{(b-a)^\alpha}$.

Theorem 1.7. *Let $f : [a, b] \to \mathbb{R}$ be a convex function with $0 \leq a < b$ and $f \in L_1[a, b]$. Also, let $g : [a, b] \to \mathbb{R}$ be a function which is non-negative, integrable and symmetric about $\frac{a+b}{2}$. Then the following inequality for generalized fractional integrals holds*

$$f\left(\frac{a+b}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}g)(b) \leq \frac{(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}fg)(b) + (\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}fg)(a)}{2} \tag{1.4}$$
$$\leq \frac{f(a)+f(b)}{2}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',b^-}g)(a),$$

where $\omega' = \frac{w}{(b-a)^\alpha}$.

In [12, 14] the Hadamard and the Fejér-Hadamard type inequalities for convex and relative convex functions via Riemann-Liouville fractional integral operators have been proved. In this paper we give fractional integral inequalities of the Hadamard and the Fejér-Hadamard type for convex and relative convex functions by using the fractional integral operators involving the generalized Mittag-Leffler function. We also produce the results which are given in [12, 14] by setting particular values of parameters.

2. MAIN RESULTS

Following lemmas are useful to establish new results.

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be an integrable and symmetric function about $\frac{a+b}{2}$. Then the following equality holds*

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b) = \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(a) = \frac{\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(a)}{2}. \tag{2.1}$$

Proof. As f is symmetric about $\frac{a+b}{2}$, therefore $f(a + b - t) = f(t)$. By definition we have

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b) = \int_a^b (b-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-t)^\alpha)f(t)dt, \tag{2.2}$$

replacing t by $a+b-t$ in equation (2.2) we have

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b)=\int_a^b(t-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(t-a)^{\alpha})f(t)dt.$$

This implies

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}f\right)(b)=\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}f\right)(a). \tag{2.3}$$

Therefore we get (2.1).

□

Lemma 2.2. *Let $f:[a.b]\to\mathbb{R}$ be a differentiable function on (a,b) and $f'\in L[a,b]$. If $g:[a.b]\to\mathbb{R}$ is integrable and symmetric about $\frac{a+b}{2}$, then we have the following equality*

$$\begin{aligned}&\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]\\&-\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}gf\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}gf\right)(a)\right]\\&=\int_a^b\left[\int_a^t(b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right.\\&\left.-\int_t^b(s-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^{\alpha})g(s)ds\right]f'(t)dt.\end{aligned}$$

Proof. To prove this lemma we take terms of the right hand side, on integrating by parts and after simplification we have

$$\begin{aligned}&\int_a^b\left[\int_a^t(b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right]f'(t)dt\\&=f(b)\int_a^b(b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds-\int_a^b(b-t)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-t)^{\alpha})gf(t)dt\\&=f(b)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)-\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}gf\right)(b).\end{aligned}$$

By using Lemma 2.1 we have

$$\begin{aligned}&\int_a^b\left[\int_a^t(b-s)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^{\alpha})g(s)ds\right]f'(t)dt \qquad (2.4)\\&=\frac{f(b)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]-\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}gf\right)(b).\end{aligned}$$

Similarly

$$\begin{aligned}&-\int_t^b\left[(s-a)^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^{\alpha})g(s)ds\right]f'(t)dt \qquad (2.5)\\&=\frac{f(a)}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]-\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}gf\right)(a).\end{aligned}$$

Adding (2.4) and (2.5) we get the left hand side. □

In the following we give our first integral inequality of the Hadamard type.

Theorem 2.3. *Let $f:I\to\mathbb{R}$ be a differentiable mapping in the interior of I with $f'\in L[a,b]$, $a<b$. If $|f'|$ is convex on $[a,b]$ and $g:I\to\mathbb{R}$ is continuous and symmetric function about $\frac{a+b}{2}$, then we have the following inequality*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}g\right)(b)+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-}g\right)(a)\right]\right.$$

$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Big|$$
$$\leq \frac{\parallel g \parallel_\infty S(b-a)^{\beta+1}}{\beta(\beta+1)} \left(1 - \frac{1}{2^\beta} \right) [|f'(a) + f'(b)|],$$

for $k < l + \alpha$ *and* $\parallel g \parallel_\infty = \sup_{t \in [a,b]} |g(t)|$.

Proof. By using Lemma 2.2 we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right. \tag{2.6}$$
$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Big|$$
$$\leq \int_a^b \Bigg| \Bigg[\int_a^t (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds$$
$$- \int_t^b (s-a)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^\alpha) g(s) ds \Bigg] \Bigg| |f'(t)| dt.$$

Using the convexity of $|f'|$ we have

$$|f'(t)| \leq \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)|; \ t \in [a,b]. \tag{2.7}$$

By using symmetry of function g we have

$$\int_t^b (s-a)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^\alpha) g(s) ds$$
$$= \int_a^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(a+b-s) ds$$
$$= \int_a^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds.$$

This implies

$$\left| \int_a^t (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds - \int_t^b (s-a)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(s-a)^\alpha) g(s) ds \right| \tag{2.8}$$
$$= \left| \int_t^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds \right|$$
$$\leq \begin{cases} \int_t^{a+b-t} |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds, t \in [a, \frac{a+b}{2}] \\ \int_{a+b-t}^t |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds, t \in [\frac{a+b}{2}, b]. \end{cases}$$

By (2.6), (2.7), (2.8) and absolute convergence of Mittag-Leffler function, we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right. \tag{2.9}$$
$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Big|$$
$$\leq \int_a^{\frac{a+b}{2}} \left(\int_a^{a+b-t} |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds \right) \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt$$
$$+ \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t |(b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s)| ds \right) \left(\frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt.$$

$$\leq \frac{\| g \|_\infty S}{\beta(b-a)} \left[\int_a^{\frac{a+b}{2}} ((b-t)^\beta - (t-a)^\beta (b-t)|f'(a)|)dt + \int_a^{\frac{a+b}{2}} ((b-t)^\beta - (t-a)^\beta (t-a)|f'(b)|)dt \right.$$
$$\left. + \int_{\frac{a+b}{2}}^b ((t-a)^\beta - (b-t)^\beta (b-t)|f'(a)|)dt + \int_{\frac{a+b}{2}}^b ((t-a)^\beta - (b-t)^\beta (t-a)|f'(b)|)dt \right].$$

Since we have

$$\int_a^{\frac{a+b}{2}} \left((b-t)^\beta - (t-a)^\beta\right)(b-t)dt = \frac{(b-a)^{\beta+2}}{\beta+1}\left(\frac{\beta+1}{\beta+2} - \frac{1}{2^{\beta+1}}\right)$$

and

$$\int_a^{\frac{a+b}{2}} \left((b-t)^\beta - (t-a)^\beta\right)(t-a)dt = \frac{(b-a)^{\beta+2}}{\beta+1}\left(\frac{1}{\beta+2} - \frac{1}{2^{\beta+1}}\right).$$

Using the above calculations in (2.9) we have

$$\left| \left(\frac{f(a)+f(b)}{2}\right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a) \right] \right.$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a) \right] \right|$$
$$\leq \frac{\| g \|_\infty S}{\beta(b-a)} \frac{(b-a)^{\beta+2}}{\beta+1} \left[\left(\frac{\beta+1}{\beta+2} - \frac{1}{2^{\beta+1}}\right) + \left(\frac{1}{\beta+2} - \frac{1}{2^{\beta+1}}\right) \right] [|f'(a)| + |f'(b)|]$$
$$= \frac{\| g \|_\infty S}{\beta(\beta+1)} (b-a)^{\beta+1} \left(1 - \frac{1}{2^\beta}\right) [|f'(a)| + |f'(b)|].$$

□

A special case is stated in the following, which is inequality of the Hadamard type for Riemann-Liouville fractional integrals.

Corollary 2.4. *Setting $\omega = 0$ in Theorem 2.3 we have the following inequality for Riemann-Liouville fractional integral operators*

$$\left| \left(\frac{f(a)+f(b)}{2}\right) \left[I^\beta_{a^+} g(b) + I^\beta_{b_-} g(a) \right] - \left[I^\beta_{a^+} fg(b) + I^\beta_{b_-} fg(a) \right] \right| \tag{2.10}$$
$$\leq \frac{\| g \|_\infty (b-a)^{\beta+1}}{\Gamma(\beta+2)} \left(1 - \frac{1}{2^\beta}\right) [|f'(a)| + |f'(b)|].$$

Remark 2.5. The above inequality (2.10) is proved in [12].

Theorem 2.6. *Let $f : I \to \mathbb{R}$ be a differentiable function in the interior of I, also let $f' \in L[a,b]$, $a < b$. If $|f'|^q$, $q > 0$ is convex on $[a,b]$ and $g : I \to \mathbb{R}$ is continuous and symmetric function about $\frac{a+b}{2}$, then we have the following inequality*

$$\left| \left(\frac{f(a)+f(b)}{2}\right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a) \right] \right. \tag{2.11}$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf\right)(a) \right] \right|$$
$$\leq \frac{2 \| g \|_\infty S(b-a)^{\beta+\frac{1}{p}}}{\beta(\beta+1)} \left(1 - \frac{1}{2^\beta}\right) \left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}},$$

for $k < l + \alpha$ and $\| g \|_\infty = \sup_{t\in[a,b]} |g(t)|$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.2, Hölder inequality, inequality (2.8) one can has

$$\left| \left(\frac{f(a)+f(b)}{2}\right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g\right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g\right)(a) \right] \right. \tag{2.12}$$

$$- \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \Bigg|$$
$$\leq \left[\int_a^b \left| \int_t^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds \right| dt \right]^{1-\frac{1}{q}}$$
$$\left[\int_a^b \left| \int_t^{a+b-t} (b-s)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(b-s)^\alpha) g(s) ds \right| |f'(t)|^q dt \right]^{\frac{1}{q}}.$$

Using absolute convergence of Mittag-Leffler function and $\parallel g \parallel_\infty = \sup_{t\in[a,b]} |g(t)|$ we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right.$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \right|$$
$$\leq \parallel g \parallel_\infty^{1-\frac{1}{q}} S^{1-\frac{1}{q}} \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} (b-s)^{\beta-1} ds \right) dt + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t (b-s)^{\beta-1} ds \right) dt \right]^{1-\frac{1}{q}}$$
$$\times \parallel g \parallel_\infty^{\frac{1}{q}} S^{\frac{1}{q}} \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} (b-s)^{\beta-1} ds \right) |f'(t)|^q dt \right.$$
$$\left. + \int_{\frac{a+b}{2}}^b \left(\int_{a+b-t}^t (b-s)^{\beta-1} ds \right) |f'(t)|^q dt \right]^{\frac{1}{q}}.$$

By some calculation we have

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right.$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \right|$$
$$\leq \parallel g \parallel_\infty S \left[\frac{(b-a)^{\beta+1}}{\beta+1}(1-\frac{1}{2^\beta}) + \frac{(b-a)^{\beta+1}}{\beta+1}(1-\frac{1}{2^\beta}) \right]^{1-\frac{1}{q}}$$
$$\times \left[\int_a^{\frac{a+b}{2}} \left((b-t)^\beta - (t-a)^\beta \right) |f'(t)|^q dt + \int_{\frac{a+b}{2}}^b \left((b-t)^\beta - (t-a)^\beta . \right) |f'(t)|^q dt \right]^{\frac{1}{q}}.$$

Since $|f'|^q$ is convex on $[a,b]$, therefore we have

$$|f'(t)|^q \leq \frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q. \tag{2.13}$$

Hence

$$\left| \left(\frac{f(a)+f(b)}{2} \right) \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} g \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} g \right)(a) \right] \right.$$
$$\left. - \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} gf \right)(b) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b_-} gf \right)(a) \right] \right|$$
$$\leq \parallel g \parallel_\infty S \left[2\frac{(b-a)^{\beta+1}}{\beta+1}(1-\frac{1}{2^\beta}) \right]^{1-\frac{1}{q}}$$
$$\times \left[\int_a^{\frac{a+b}{2}} \left((b-t)^\beta - (t-a)^\beta \right) \left(\frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q \right) dt \right.$$

$$+\int_{\frac{a+b}{2}}^{b}\left((b-t)^{\beta}-(t-a)^{\beta}\right)\left(\frac{b-t}{b-a}|f'(a)|^{q}+\frac{t-a}{b-a}|f'(b)|^{q}\right)dt\Bigg]^{\frac{1}{q}}.$$

From which one can have (2.11). □

Corollary 2.7. *Setting $\omega = 0$ in Theorem 2.6 we have the following result for Riemann-Liouville fractional integral operators*

$$\left|\left(\frac{f(a)+f(b)}{2}\right)[I_{a^+}^{\beta}g(b)+I_{b_-}^{\beta}g(a)]-[I_{a^+}^{\beta}fg(b)+I_{b_-}^{\beta}fg(a)]\right|$$
$$\leq\frac{2\parallel g\parallel_{\infty}(b-a)^{\beta+1-\frac{1}{q}}}{\Gamma(\beta+2)}\left(1-\frac{1}{2^{\beta}}\right)(|f'(a)|^{q}+|f'(b)|^{q})^{\frac{1}{q}},$$

$\beta > 0$.

In the following we give the Hadamard inequality for relative convex functions via generalized fractional integral operators.

Theorem 2.8. *Let $f : [a, g(b)] \to \mathbb{R}$ be a positive relative convex function and $f \in L[a, g(b)]$. Then the following inequalities for generalized fractional integral operators hold*

$$f\left(\frac{a+g(b)}{2}\right)\left(\epsilon_{\alpha,\beta,l,\omega',a^+}^{\gamma,\delta,k}1\right)(g(b))\leq\frac{1}{2}\left[\left(\epsilon_{\alpha,\beta,l,\omega',a^+}^{\gamma,\delta,k}f\right)(g(b))+\left(\epsilon_{\alpha,\beta,l,\omega',g(b)_-}^{\gamma,\delta,k}f\right)(a)\right]$$
$$\leq\frac{f(a)+f(g(b))}{2}\left(\epsilon_{\alpha,\beta,l,\omega',g(b)_-}^{\gamma,\delta,k}1\right)(a),$$

where $\omega' = \frac{\omega}{(g(b)-a)^{\alpha}}$.

Proof. Since f is relative convex on $[a, g(b)]$, we have

$$f\left(\frac{a+g(b)}{2}\right)=f\left[\left(\frac{1}{2}(ta+(1-t)g(b)\right)+\left(1-\frac{1}{2}\right)((1-t)a+tg(b))\right]$$
$$\leq\frac{1}{2}f\left(ta+(1-t)g(b)\right)+\frac{1}{2}f\left((1-t)a+tg(b)\right).$$

Multiplying both sides by $2t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})$ and integrating over $[0, 1]$ we have

$$2f\left(\frac{a+g(b)}{2}\right)\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})dt\leq\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})f\left(ta+(1-t)g(b)\right)dt$$
$$+\int_0^1 t^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}(\omega t^{\alpha})f\left((1-t)a+tg(b)\right)dt. \quad (2.14)$$

Setting $ta + (1 - t)g(b) = x$ that is $t = \frac{g(b)-x}{g(b)-a}$ and $(1 - t)a + tg(b) = y$ that is $t = \frac{y-a}{g(b)-a}$ we have

$$2f\left(\frac{a+g(b)}{2}\right)\int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}\left(\omega\left(\frac{g(b)-x}{g(b)-a}\right)^{\alpha}\right)\left(\frac{-dx}{g(b)-a}\right) \quad (2.15)$$
$$\leq\int_{g(b)}^{a}\left(\frac{g(b)-x}{g(b)-a}\right)^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}\left(\omega\left(\frac{g(b)-x}{g(b)-a}\right)^{\alpha}\right)f(x)\left(\frac{-dx}{g(b)-a}\right)$$
$$+\int_{a}^{g(b)}\left(\frac{y-a}{g(b)-a}\right)^{\beta-1}E_{\alpha,\beta,l}^{\gamma,\delta,k}\left(\omega\left(\frac{y-a}{g(b)-a}\right)^{\alpha}\right)f(y)\left(\frac{dy}{g(b)-a}\right).$$

After simplification we get

$$2f\left(\frac{a+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}1\right)(g(b)) \leq \left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f\right)(g(b))+\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}f\right)(a)\right]. \tag{2.16}$$

By using the relative convexity of f on $[a, g(b)]$ one can has

$$f(ta+(1-t)g(b))+f((1-t)a+tg(b)) \leq tf(a)+(1-t)f(g(b))+(1-t)f(a)+tf(g(b)).$$

Multiplying $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})$ on both sides and integrating over $[0,1]$ we have

$$\begin{aligned}&\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f(ta+(1-t)g(b))dt + \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f((1-t)a+tg(b))dt \\ &\leq \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})(tf(a)+(1-t)f(g(b)))dt + \int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})((1-t)f(a)+tf(g(b)))dt.\end{aligned}$$

Setting $ta+(1-t)g(b) = x$ that is $t = \frac{g(b)-x}{g(b)-a}$ and $(1-t)a+tg(b) = y$ that is $t = \frac{y-a}{g(b)-a}$ and after simple calculation we have

$$\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',a^+}f\right)(g(b)) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}f\right)(a)\right] \leq [f(a)+f(g(b))]\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}1\right)(a). \tag{2.17}$$

Combinig (2.16) and (2.17) we get the result. □

Remark 2.9. (i) If we put $\omega = 0$ and $k = 1$ in Theorem 2.8 we obtain Theorem 1.4.
(ii) If we put $\omega = 0$ and $\beta = \frac{\alpha}{k}$ in Theorem 2.8, then we get [11, Theorem 3].

In the upcoming theorem we give the generalization of previous result.

Theorem 2.10. *Let $f : [g(a), g(b)] \to \mathbb{R}$ be a positive relative convex function and $f \in L[g(a), g(b)]$. Then the following inequalities for generalized fractional integral operator holds*

$$\begin{aligned}&f\left(\frac{g(a)+g(b)}{2}\right)\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(a)^+}1\right)(g(b)) \\ &\leq \frac{1}{2}\left[\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(a)^+}f\right)(g(b)) + \left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}f\right)(a)\right] \\ &\leq \frac{f(g(a))+f(g(b))}{2}\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega',g(b)_-}1\right)(g(a)),\end{aligned}$$

where $\omega' = \frac{\omega}{(g(b)-g(a))^{\alpha}}$.

Proof. Proof of this theorem is on the same lines of the proof of Theorem 2.8. □

Corollary 2.11. *For $\omega = 0$ we obtain the following inequality for Riemann-Liouville integral operator from Theorem 2.10*

$$\begin{aligned}f\left(\frac{g(a)+g(b)}{2}\right) &\leq \frac{\Gamma(\beta+1)}{2(g(b)-g(a))^{\beta}}[I^{\beta}_{g(a)^+}f(g(b)) + I^{\beta}_{g(b)_-}f(g(a))] \\ &\leq \frac{f(g(a))+f(g(b))}{2},\end{aligned}$$

with $\beta > 0$.

Remark 2.12. In Theorem 2.10 if we take $\omega = 0$, $\beta = \frac{\alpha}{k}$, then we get [11, Theorem 5].

3. Acknowledgments

The research work of Ghulam Farid is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

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