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EXISTENCE AND STABILITY OF A DAMPED WAVE EQUATION WITH TWO DELAYED TERMS IN BOUNDARY

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ABSTRACT. This paper considers a linear damped wave equation with dynamic bouandary conditions where two feedback terms have a delay. In bounded domain, we first establish the question of well-posedness and uniqueness of the solution for the initial-boundary value problem, using semigroup arguments in [13, 14, 29]. Next, by introducing suitable Lyapunov functionals, exponential stability estimates are obtained under conditions on the delay terms.

KEYWORDS: Damped wave equation; delay feedback; stabilization; semigroup formulation.
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1. INTRODUCTION

It is well known that the PDEs with time delay have been much studied during the last years and their results is by now rather developed. See [5, 7, 1, 26, 24, 32, 31]

In the classical theory of delayed wave equations, several main parts are joined in a fruitful way, it is very remarkable that the damped wave equation with two delays occupies a similar position and arise in many applied problems, when it comes to boundary conditions.

Dynamic boundary conditions arise in many physical applications, in particular they occur in elastic models. These conditions appear in modelling dynamic vibrations of linear viscoelastic rods and beams which have attached tip masses at their free ends. See [2, 4, 6, 22, 10].

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In this paper, we consider n-dimensional wave equation with strong damping and boundary conditions when two terms acting on the boundary are delayed in the following problem

$$u'' - \Delta u - a\Delta u' = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$
$$u = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \tag{1.2}$$
$$\mu u'' + \tfrac{\partial(u+au')}{\partial\nu} = -k_1 u'(x,t-\tau_1) - k_2 u'(x,t-\tau_2) \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.3}$$
$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.4}$$
$$u' = f_0 \quad \text{in } \Gamma_1 \times (-\max(\tau_1(0), \tau_2(0)), 0), \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set with boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_0 and Γ_1; i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and $meas\Gamma_0 \neq 0$.

The vector $\nu(x)$ denotes the outer unit normal vector to the point $x \in \Gamma$ and $\frac{\partial u}{\partial \nu}$ is the normal derivative. Moreover, $\tau_i = \tau_i(t)$, $i = 1, 2$ is the time delay, μ, a, k are real numbers, with $\mu \geq 0, a > 0$, and the initial datum (u_0, u_1, f_0) belongs to a suitable space.

We define the energy of system (1.1)–(1.5) as

$$E(t) := \frac{1}{2}\int_\Omega \{u'^2 + |\nabla u|^2\} dx + \frac{\xi_1}{2}\int_{t-\tau_1}^{t}\int_{\Gamma_1} e^{\lambda(s-t)} u'^2(x,s) d\Gamma ds$$
$$+ \frac{\xi_2}{2}\int_{t-\tau_2}^{t}\int_{\Gamma_1} e^{\lambda(s-t)} u'^2(x,s) d\Gamma ds + \frac{\mu}{2}\int_{\Gamma_1} u'^2 d\Gamma, \tag{1.6}$$

where ξ_i, λ are suitable positive constants.

To motivate our work, let us mention the major work [25], when the authors studied well-posedness and exponential stability of the problem (1.1)–(1.5) with structural damping and boundary delay in both cases $\mu > 0$ and $\mu = 0$ in a bounded and smooth domain, where $k_2 = 0$. The analogous problem with boundary feedback has been introduced and studied by Xu, Yung, Li [31] in one-space dimension using a fine spectral analysis and in higher space dimension by the authors [26]. The case of time-varying delay has been already studied in [28] in one space dimension and in general dimension, with a possibly degenerate delay, in [27]. Both these papers deal with boundary feedback.

When $\tau_1(t) \equiv \tau_2(t) \equiv 0$ (in absence of delays), it is well-known that the above problem is exponentially stable. See in this direction [3, 19, 18, 20, 15, 17, 16, 33, 12, 23, 8, 30, 10]. When $\mu = 0, k_2 = 0$, in presence of a constant delay, and the condition (1.3) is substituted by

$$\frac{\partial u}{\partial \nu} = -k u_t(x, t-\tau), \quad \Gamma_1 \times (0, +\infty),$$

the system becomes unstable for arbitrarily small delays (see [6]).

The above model without delay (e.g. $\tau = 0$) has been proposed in one dimension by Pellicer and Sòla-Morales [30] as an alternative model for the classical spring-mass damper system. In both cases, no rates of convergence are proved. In dimension higher than 1, we refer to Gerbi and Said-Houari [10] where a nonlinear boundary feedback is even considered and the exponential growth of the energy is proved if the initial data are large enough. A different problem with a dynamic boundary condition (without delay), motivated by the study of flows of gas in a channel with porous walls, is analyzed in [8] where exponential decay is proved.

2. Assumptions

We assume, on the time-delay functions, that there exist positive constants $\overline{\tau}_0, \widetilde{\tau}_0, \overline{\tau}, \widetilde{\tau}$ such that

$$0 < \overline{\tau}_0 \leq \tau_1 \leq \overline{\tau}, \quad \forall t > 0, \tag{2.1}$$

$$0 < \widetilde{\tau}_0 \leq \tau_2 \leq \widetilde{\tau}, \quad \forall t > 0. \tag{2.2}$$

Moreover, we assume

$$\tau_i \in W^{2,\infty}([0,T]), \quad \forall T > 0, i = 1, 2, \tag{2.3}$$

$$\max\{\tau_1', \tau_2'\} \leq d < 1, \quad \forall t > 0, \tag{2.4}$$

where d is the positive constant.

Under (2.3)-(2.4) we will prove that an exponential stability result holds under a suitable assumption between the coefficients a and k_1, k_2.

Let C^* be a Poincaré's type constant defined as the smallest positive constant such that

$$\int_{\Gamma_1} |v|^2 d\Gamma \leq C^* \int_\Omega |\nabla v|^2 dx, \quad \forall v \in H^1_{\Gamma_0}(\Omega), \tag{2.5}$$

where, as usual,

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u = 0 \quad \text{on } \Gamma_0\}.$$

We will give a well-posedness result under the assumption

$$\frac{|k|C^*}{\sqrt{1-d}} \leq \frac{a}{2}, \quad k = \max\{k_1, k_2\}. \tag{2.6}$$

We omit the space variable x of $u(x,t)$, $u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

3. EXISTENCE OF SOLUTION

First as in [26] we introduce the new variables

$$z(x,\rho,t) = u'(x, t - \tau_1\rho) \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{3.1}$$

$$w(x,\rho,t) = u'(x, t - \tau_2\rho) \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty). \tag{3.2}$$

Then we have

$$\tau_1 z'(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty), \tag{3.3}$$

$$\tau_2 w'(x,\rho,t) + w_\rho(x,\rho,t) = 0 \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{3.4}$$

Therefore problem (1.1)–(1.5) is equivalent to

$$u'' - \Delta u - a\Delta u' = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{3.5}$$

$$\tau_1 z'(x,\rho,t) + (1 - \tau_1'\rho) z_\rho(x,\rho,t) = 0 \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{3.6}$$

$$\tau_2 w'(x,\rho,t) + (1 - \tau_2'\rho) z_\rho(x,\rho,t) = 0 \quad \text{in } \Gamma_1 \times (0,1) \times (0,+\infty), \tag{3.7}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,+\infty), \tag{3.8}$$

$$\mu u'' = -\tfrac{\partial(u + au')}{\partial\nu} - k_1 z(x,1,t) - k_2 w(x,1,t) \quad \text{on } \Gamma_1 \times (0,+\infty), \tag{3.9}$$

$$z(x,0,t) = w(x,0,t) = u' \quad \text{on } \Gamma_1 \times (0,\infty), \tag{3.10}$$

$$u(x,0) = u_0(x) \text{ and } u'(x,0) = u_1(x) \quad \text{in } \Omega, \tag{3.11}$$

$$z(x,\rho,0) = f_0(x, -\rho\tau_1(0)) \quad \text{in } \Gamma_1 \times (0,1), \tag{3.12}$$

$$w(x,\rho,0)=f_0(x,-\rho\tau_2(0)) \quad \text{in } \Gamma_1\times(0,1). \tag{3.13}$$

Let us denote

$$U=(u,u',\gamma u',z,w)^T,$$

where γ is the trace operator on Γ_1. Then problem (1.1)–(1.5) equivalent to

$$U'=(u',u'',\gamma_1 u'',z',w')^T$$
$$=(u',\Delta u+a\Delta u',-\mu^{-1}(\tfrac{\partial(u+au')}{\partial\nu}+k_1z(\cdot,1,\cdot)+k_2w(\cdot,1,\cdot)),\tfrac{\tau_1'(t)\rho-1}{\tau_1}z_\rho,\tfrac{\tau_2'(t)\rho-1}{\tau_2}w_\rho)^T.$$

Therefore, problem (1.1)–(1.5) can be rewritten as

$$U'=A(t)U,$$
$$U(0)=(u_0,u_1,\gamma_1u_1,f_0(\cdot,-\cdot\,\tau_1),f_0(\cdot,-\cdot\,\tau_2))^T, \tag{3.14}$$

where $A(t)$ is defined by

$$A(t)(u,v,v_1,z,w)^T$$
$$=\left(v,\Delta(u+av),-\mu^{-1}\left(\tfrac{\partial(u+av)}{\partial\nu}+k_1z(\cdot,1)+k_2w(\cdot,1)\right),\tfrac{\tau_1'(t)\rho-1}{\tau_1(t)}z_\rho,\tfrac{\tau_2'(t)\rho-1}{\tau_2(t)}w_\rho\right)^T,$$

with domain of $A(t)$ given by

$$\begin{aligned} &D(A(t)) \\ &=\Big\{(u,v,v_1,z,w)^T\in H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1) \\ &\qquad\qquad\times(L^2(\Gamma_1\times(0,1)))^2\times L^2(\Gamma_1\times(0,1)), \\ &u+av\in E\left(\Delta,L^2(\Omega)\right),\tfrac{\partial(u+av)}{\partial\nu}\in L^2(\Gamma_1),v=v_1=z\left(\cdot,0\right)=w\left(\cdot,0\right)\quad\text{on }\Gamma_1\Big\}\end{aligned} \tag{3.15}$$

is independent of the time t, i.e.,

$$D(A(t))=D(A(0)),\quad t>0. \tag{3.16}$$

where

$$E(\Delta,L^2(\Omega))=\{u\in H^1(\Omega):\Delta u\in L^2(\Omega)\}.$$

For a function $u\in E(\Delta,L^2(\Omega))$, $\frac{\partial u}{\partial\nu}$ belongs to $H^{-1/2}(\Gamma_1)$ and the next Green formula

$$\int_\Omega\nabla u\nabla q dx=-\int_\Omega\Delta uqdx+\langle\frac{\partial u}{\partial\nu};q\rangle_{\Gamma_1},\ \forall q\in H^1_{\Gamma_0}(\Omega), \tag{3.17}$$

is valid (see [11]), where $\langle\cdot;\cdot\rangle_{\Gamma_1}$ means the duality pairing between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

Let us introduce a Hilbert space $\tilde H$ defined by

$$\tilde H=H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)\times L^2(\Gamma_1)\times(L^2(\Gamma_1\times(0,1)))^2,$$

equipped with the standard inner product

$$\begin{aligned}&\left\langle(u,v,v_1,z,w)^T,(\tilde u,\tilde v,\tilde v_1,\tilde z,\widetilde w)^T\right\rangle_{\tilde H}\\ &=\int_\Omega\{\nabla u(x)\nabla\tilde u(x)+v(x)\tilde v(x)\}dx\\ &+\mu\int_{\Gamma_1}v_1(x)\tilde v_1(x)d\Gamma+\xi_1\tau_1\left(t\right)\int_{\Gamma_1}\int_0^1z(x,\rho)\tilde z(x,\rho)d\rho d\Gamma\\ &+\xi_2\tau_2\left(t\right)\int_{\Gamma_1}\int_0^1w\left(x,\rho\right)\widetilde w\left(x,\rho\right)d\rho d\Gamma.\end{aligned} \tag{3.18}$$

Remark 3.1. The time varying operator $A(t)$ is an unbounded in $\tilde H$.

The next theorem is our main tool to prove well-posedness results, its proof is similar in [13].

Theorem 3.1. *Assume that*

(i) *$D(A(0))$ is a dense subset of $\tilde{H}$,*
(ii) *$D(A(t)) = D(A(0))$ for all $t > 0$,*
(iii) *for all $t \in [0,T]$, $A(t)$ generates a strongly continuous semigroup on $\tilde{H}$ and the family $A = \{A(t) : t \in [0,T]\}$ is stable with stability constants C and m independent of t (i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $A(t)$ satisfies $\|S_t(s)u\|_{\tilde{H}} \leq Ce^{ms}\|u\|_{\tilde{H}}$, for all $u \in \tilde{H}$ and $s \geq 0$),*
(iv) *$\partial_t A$ belongs to $L^\infty_*([0,T], B(D(A(0)), \tilde{H}))$, the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(D(A(0)), \tilde{H})$ of bounded operators from $D(A(0))$ into $\tilde{H}$.*

Then, problem (3.14) has a unique solution $U \in C([0,T], D(A(0))) \cap C^1([0,T], \tilde{H})$ for any initial datum in $D(A(0))$.

Let ξ_1 and ξ_2 are a positive constants such that

$$\frac{|k|}{\sqrt{1-d}} \leq \xi_i \leq \frac{a}{C^*} - \frac{|k|}{\sqrt{1-d}}, \quad \text{for } i = 1, 2. \tag{3.19}$$

In order to deduce a well-posedness result, we define on $\tilde{H}$ the time dependent inner product

$$\begin{aligned}
&\left\langle (u, v, v_1, z, w)^T, (\tilde{u}, \tilde{v}, \tilde{v}_1, \tilde{z}, \widetilde{w})^T \right\rangle_t \\
&= \int_\Omega \{\nabla u(x) \nabla \tilde{u}(x) + v(x)\tilde{v}(x)\} dx \\
&+ \mu \int_{\Gamma_1} v_1(x)\tilde{v}_1(x) d\Gamma \\
&+ \xi_1 \tau_1(t) \int_{\Gamma_1} \int_0^1 z(x,\rho)\tilde{z}(x,\rho) d\rho \, d\Gamma \\
&+ \xi_2 \tau_2(t) \int_{\Gamma_1} \int_0^1 w(x,\rho)\, \widetilde{w}(x,\rho)\, d\rho d\Gamma.
\end{aligned} \tag{3.20}$$

and using Theorem 3.1.

Theorem 3.2. *Assume that (2.1)–(2.4) and (2.6) hold. Then for any initial datum $U_0 \in \tilde{H}$ there exists a unique solution $U \in C([0,+\infty), \tilde{H})$ of problem (3.14). Moreover, if $U_0 \in D(A(0))$, then*

$$U \in C([0,+\infty), D(A(0))) \cap C^1([0,+\infty), \tilde{H}).$$

We need to check assumptions of Theorem 3.1 for problem (3.14).

Lemma 3.2. *$D(A(0))$ is dense in $\tilde{H}$.*

Proof. Let $(f, g, g_1, h_1, h_2)^T \in \tilde{H}$ be orthogonal to all elements of $D(A(0))$, that is,

$$\begin{aligned}
0 &= \left\langle (u, v, v_1, z, w)^T, (f, g, g_1, h_1, h_2)^T \right\rangle_{\tilde{H}} \\
&= \int_\Omega \{\nabla u(x) \nabla f(x) + v(x) g(x)\} dx \\
&\quad + \int_{\Gamma_1} v_1 g_1 d\Gamma + \int_{\Gamma_1} \int_0^1 z(x,\rho) h_1(x,\rho) d\rho d\Gamma
\end{aligned}$$

$$+\int_{\Gamma_1}\int_0^1 w(x,\rho)h_2(x,\rho)d\rho d\Gamma$$

$\forall (u,v,v_1,z,w)^T \in D(A(0))$.

Taking $u=v=0$ (then $v_1=0$) , $z=0$ and $w\in D(\Gamma_1\times(0,1))$. As $(0,0,0,0,w)^T\in D(A(0))$, we obtain

$$\int_{\Gamma_1}\int_0^1 w(x,\rho)h_2(x,\rho)d\rho d\Gamma=0.$$

Since $D(\Gamma_1\times(0,1))$ is dense in $L^2(\Gamma_1\times(0,1))$, we deduce that $h_2=0$.

In the same way, by taking $u=v=0$ (then $v_1=0$), $w=0$ and $z\in D(\Gamma_1\times(0,1))$. As $(0,0,0,z,0)^T\in D(A(0))$, we obtain

$$\int_{\Gamma_1}\int_0^1 z(x,\rho)h_1(x,\rho)d\rho d\Gamma=0.$$

Since $D(\Gamma_1\times(0,1))$ is dense in $L^2(\Gamma_1\times(0,1))$, we deduce that $h_1=0$. Also for $u=z=w=0$ and $v\in D(\Omega)$ (then $v_1=0$) we see that $g=0$. Therefore, for $u=0$, $z=0$ and $w=0$, we deduce also

$$\int_{\Gamma_1} g_1 v_1 d\Gamma=0,\ \forall v_1\in D(\Gamma_1),$$

and so $g_1=0$.

The above orthogonality condition is then reduced to

$$0=\int_\Omega \nabla u\nabla f dx,\ \forall (u,v,v_1,z,w)^T\in D(A(0)).$$

By restricting ourselves to $v=z=w=0$, we obtain

$$\int_\Omega \nabla u(x)\nabla f(x)dx=0,\ \forall (u,0,0,0,0)^T\in D(A(0)).$$

But we easily see that $(u,0,0,0,0)^T\in D(A(0))$ if and only if $u\in E(\Delta,L^2(\Omega))\cap H^1_{\Gamma_0}(\Omega)$. This set is dense in $H^1_{\Gamma_0}(\Omega)$ (equipped with the inner product $\langle .,.\rangle_{H^1_{\Gamma_0}(\Omega)}$), thus we conclude that $f=0$. □

Lemma 3.3. *Let* $\Phi=(u,v,v_1,z,w)^T$*, then*

$$\|\Phi\|_t\le \|\Phi\|_s e^{\left(\frac{d(\tilde\tau_0+\overline\tau_0)}{\overline\tau_0\tilde\tau_0}\right)|t-s|},\ \forall t,s\in[0,T], \tag{3.21}$$

where d is a positive constant.

Proof. For all $s,t\in[0,T]$, we have

$$\begin{aligned}&\|\Phi\|_t^2-\|\Phi\|_s^2 e^{\left(\frac{d(\tilde\tau_0+\overline\tau_0)}{\overline\tau_0\tilde\tau_0}\right)|t-s|}\\ &=\left(1-e^{\left(\frac{d(\tilde\tau_0+\overline\tau_0)}{\overline\tau_0\tilde\tau_0}\right)|t-s|}\right)\left(\int_\Omega(|\nabla u(x)|^2+v^2)dx+\mu\int_{\Gamma_1}v_1^2d\Gamma\right)\\ &+\xi_1\left(\tau_1(t)-\tau_1(s)e^{\left(\frac{d(\tilde\tau_0+\overline\tau_0)}{\overline\tau_0\tilde\tau_0}\right)|t-s|}\right)\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma\\ &+\xi_2\left(\tau_2(t)-\tau_2(s)e^{\left(\frac{d(\tilde\tau_0+\overline\tau_0)}{\overline\tau_0\tilde\tau_0}\right)|t-s|}\right)\int_{\Gamma_1}\int_0^1 w^2(x,\rho)d\rho d\Gamma.\end{aligned}$$

We notice that

$$e^{\left(\frac{d(\tilde\tau_0+\overline\tau_0)}{\overline\tau_0\tilde\tau_0}\right)|t-s|}\ge 1.$$

Moreover

$$\tau_1(t) - \tau_1(s)e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|} \le 0,$$

and

$$\tau_2(t) - \tau_2(s)e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|} \le 0,$$

for some $d > 0$.

Indeed,

$$\tau_1(t) = \tau_1(s) + \tau_1'(a)(t-s),$$

and

$$\tau_2(t) = \tau_2(s) + \tau_2'(b)(t-s),$$

where $a, b \in (s,t)$, and thus,

$$\begin{aligned}
\frac{\tau_1(t)}{\tau_1(s)} &= 1 + \frac{|\tau_1'(a)|}{\tau_1(s)}|t-s|, \\
\frac{\tau_2(t)}{\tau_2(s)} &= 1 + \frac{|\tau_2'(b)|}{\tau_2(s)}|t-s|.
\end{aligned}$$

By (2.3), τ_1' and τ_2' are bounded on $[0,T]$ and therefore, recalling also (2.1), (2.2),

$$\begin{aligned}
\frac{\tau_1(t)}{\tau_1(s)} &\le 1 + \frac{d}{\overline{\tau}_0}|t-s| \le e^{\frac{d}{\overline{\tau}_0}|t-s|}, \\
\frac{\tau_2(t)}{\tau_2(s)} &\le 1 + \frac{d}{\tilde{\tau}_0}|t-s| \le e^{\frac{d}{\tilde{\tau}_0}|t-s|},
\end{aligned}$$

thus

$$\frac{\tau_1(t)}{\tau_1(s)} \le e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|},$$

and

$$\frac{\tau_2(t)}{\tau_2(s)} \le e^{\left(\frac{d(\tilde{\tau}_0+\overline{\tau}_0)}{\overline{\tau}_0\tilde{\tau}_0}\right)|t-s|}.$$

This complete the proof. □

Lemma 3.4. *Under condition (3.19), the operator $\tilde{A}(t) = A(t) - \kappa(t)I$ is dissipative, and*

$$\frac{d}{dt}\tilde{A}(t) \in L^\infty_*([0,T], B(D(A(0)), \tilde{H})),$$

where

$$\kappa(t) = \frac{\sqrt{\tau_1'^2(t)+1}}{2\tau_1(t)} + \frac{\sqrt{\tau_2'^2(t)+1}}{2\tau_2(t)}. \tag{3.22}$$

Proof. Taking $U = (u, v, v_1, z, w)^T \in D(A(t))$. Then, for a fixed t,

$$\begin{aligned}
\langle A(t)U, U\rangle_t = &\int_\Omega \{\nabla v(x)\nabla u(x) + v(x)\Delta(u(x) + av(x))\}dx \\
&- \xi_1 \int_{\Gamma_1}\int_0^1 (1 - \tau_1'(t)\rho) z_\rho(x,\rho) z(x,\rho)\, d\rho\, d\Gamma \\
&- \xi_2 \int_{\Gamma_1}\int_0^1 (1 - \tau_2'(t)\rho) w_\rho(x,\rho) w(x,\rho)\, d\rho\, d\Gamma \\
&- \int_{\Gamma_1} \left(\frac{\partial(u+av)}{\partial\nu}(x) + k_1 z(x,1) + k_2 w(x,1)\right) v(x)\, d\Gamma.
\end{aligned}$$

By Green's formula,

$$\begin{aligned}\langle A(t)U,U\rangle_t &= -k\int_{\Gamma_1} z(x,1)v(x)d\Gamma - a\int_\Omega |\nabla v(x)|^2dx\\ &- \xi_1\int_{\Gamma_1}\int_0^1 (1-\tau_1'(t)\rho)z_\rho(x,\rho)z(x,\rho)d\rho d\Gamma\\ &- \xi_2\int_{\Gamma}\int_0^1 (1-\tau'(t)\rho)\,w_\rho(x,\rho)\,w(x,\rho)\,d\rho d\Gamma. \end{aligned}\tag{3.23}$$

Integrating by parts in ρ and ρ we obtain

$$\begin{aligned}&\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)z(x,\rho)(1-\tau_1'(t)\rho)d\rho d\Gamma\\ &= \int_{\Gamma_1}\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}z^2(x,\rho)(1-\tau_1'(t)\rho)d\rho d\Gamma\\ &= \frac{\tau_1'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma + \frac{1}{2}\int_{\Gamma_1}\{z^2(x,1)(1-\tau_1'(t)) - z^2(x,0)\}d\Gamma, \end{aligned}\tag{3.24}$$

and

$$\begin{aligned}&\int_{\Gamma_1}\int_0^1 w_\rho(x,\rho)\,w(x,\rho)\,(1-\tau_2'(t))\,d\rho d\Gamma\\ &= \int_{\Gamma_1}\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}w^2(x,\rho)\,(1-\tau_2'(t))\,d\rho d\Gamma\\ &= \frac{\tau_2'(t)}{2}\int_{\Gamma_1}\int_0^1 w^2(x,\rho)\,d\rho d\Gamma + \int_{\Gamma_1}\left\{w^2(x,1)\,(1-\tau_2'(t)) - w^2(x,0)\right\}d\Gamma. \end{aligned}\tag{3.25}$$

Therefore, from (3.23), (3.24) and (3.25),

$$\begin{aligned}&\langle A(t)U,U\rangle_t\\ &= -k_1\int_{\Gamma_1} z(x,1)v(x)d\Gamma - k_2\int_{\Gamma_1} w(x,1)v(x)d\Gamma - a\int_\Omega|\nabla v(x)|^2dx\\ &- \frac{\xi_1}{2}\int_{\Gamma_1}\{z^2(x,1)(1-\tau_1'(t)) - z^2(x,0)\}d\Gamma - \frac{\xi_1\tau_1'(t)}{2}\int_{\Gamma_1}\int_0^1 Z^2(x,\rho)d\rho d\Gamma\\ &- \frac{\xi_2}{2}\int_{\Gamma_1}\{w^2(x,1)(1-\tau_2'(t)) - w^2(x,0)\}d\Gamma - \frac{\xi_2\tau_2'(t)}{2}\int_{\Gamma_1}\int_0^1 w^2(x,\rho)d\rho d\Gamma\\ &= -k_1\int_{\Gamma_1} z(x,1)v(x)d\Gamma - k_2\int_{\Gamma_1} w(x,1)v(x)d\Gamma\\ &- a\int_\Omega|\nabla v(x)|^2dx - \frac{\xi_1}{2}\int_{\Gamma_1} z^2(x,1)(1-\tau_1'(t))d\Gamma\\ &- \frac{\xi_2}{2}\int_{\Gamma_1} w^2(x,1)(1-\tau_2'(t))d\Gamma + \frac{\xi_1}{2}\int_{\Gamma_1} v^2(x)d\Gamma - \frac{\xi_1\tau_1'(t)}{2}\int_{\Gamma_1}\int_0^1 z^2(x,\rho)d\rho d\Gamma\\ &+ \frac{\xi_2}{2}\int_{\Gamma_1} v^2(x)d\Gamma - \frac{\xi_2\tau_2'(t)}{2}\int_{\Gamma_1}\int_0^1 w^2(x,\rho)d\rho d\Gamma. \end{aligned}$$

Using Cauchy-Schwarz's and Poincaré's inequalities, a trace estimate, it follows that

$$\begin{aligned}&\langle A(t)U,U\rangle_t\\ &\le -\left[\left(\frac{a}{2} - \frac{|k|C^*}{2\sqrt{1-d}} - \frac{\xi_1}{2}C^*\right) + \left(\frac{a}{2} - \frac{|k|\,C^*}{2\sqrt{1-d}} - \frac{\xi_2}{2}C^*\right)\right]\int_\Omega|\nabla v(x)|^2dx \end{aligned}$$

$$- \left(\frac{\xi_1}{2}(1-d) - \frac{|k|}{2}\sqrt{1-d} \right) \int_{\Gamma_1} z^2(x,1) d\Gamma$$
$$- \left(\frac{\xi_2}{2} (1-d) - \frac{|k|}{2}\sqrt{1-d} \right) \int_{\Gamma_1} w^2 (x,1) \, d\Gamma + \kappa(t) \langle U, U \rangle_t, \tag{3.26}$$

where

$$\kappa(t) = \frac{(\tau_1'^2 (t) + 1)^{\frac{1}{2}}}{2\tau_1(t)} + \frac{\left(\tau_2'^2 (t) + 1\right)^{\frac{1}{2}}}{2\tau_2 (t)}.$$

then, from (3.19),

$$\langle A(t)U, U \rangle_t \leq \kappa(t) \langle U, U \rangle_t. \tag{3.27}$$

Moreover,

$$\kappa'(t) = \frac{\tau_1''(t)\tau_1'(t)}{2\tau_1(t)(\tau_1'^2 + 1)^{\frac{1}{2}}} - \frac{\tau_1'(t)(\tau_1'^2 (t) + 1)^{\frac{1}{2}}}{2\tau_1(t)^2}$$
$$+ \frac{\tau_2''(t)\tau_2'(t)}{2\tau_2(t)(\tau_2'^2 (t) + 1)^{\frac{1}{2}}} - \frac{\tau_2'(t)(\tau_2'^2 (t) + 1)^{\frac{1}{2}}}{2\tau_2(t)^2},$$

is bounded on $[0,T]$ for all $T > 0$ (by (2.1) and (2.3) and we have

$$\frac{d}{dt} A(t)U$$
$$= (0, 0, 0, \frac{\tau_1''(t)\tau_1(t)\rho - \tau_1'(t)(\tau_1'(t)\rho - 1)}{\tau_1(t)^2} z_\rho, \frac{\tau_2''(t)\tau_2(t)\rho - \tau_2'(t) \left(\tau_2'(t)\rho - 1\right)}{\tau_2(t)^2} w_\rho)^T \tag{3.28}$$

with $\frac{\tau_1''(t)\tau_1(t)\rho - \tau_1'(t)(\tau_1'(t)\rho - 1)}{\tau_1(t)^2}$ and $\frac{\tau_2''(t)\tau_2(t)\rho - \tau_2'(t)\left(\tau_2'(t)\rho - 1\right)}{\tau_2(t)^2}$ are bounded on $[0,T]$. Thus

$$\frac{d}{dt}\tilde{A}(t) \in L_*^\infty([0,T], B(D(A(0)), \tilde{H})), \tag{3.29}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0,T]$ into $B(D(A(0)), \tilde{H})$. □

Lemma 3.5. *For fixed $t > 0$ and $\lambda > 0$, the operator $\lambda I - A(t)$ is surjective.*

Proof. Let $(f, g, g_1, h_1, h_2)^T \in \tilde{H}$, we seek $U = (u, v, v_1, z, w)^T \in D(A(t))$ solution of

$$(\lambda I - A(t))(u, v, v_1, z, w)^T = (f, g, g_1, h_1, h_2)^T,$$

that is verifying

$$\begin{aligned} \lambda u - v &= f, \\ \lambda v - \Delta (u + av) &= g, \\ \lambda v_1 + \mu^{-1} \left(\frac{\partial (u + av)}{\partial \nu} (x) + k_1 z (x,1) + k_2 w (x,1) \right) &= g_1, \\ \lambda z + \frac{1 - \tau_1' (t) \rho}{\tau_1 (t)} z_\rho &= h_1, \\ \lambda w + \frac{1 - \tau_2' (t) \rho}{\tau_2 (t)} w_\rho &= h_2. \end{aligned} \tag{3.30}$$

Suppose that we have found u with the appropriate regularity. Then

$$v = \lambda u - f, \tag{3.31}$$

and we can determine z, w. Indeed, by (3.15),

$$z(x,0) = v(x), \quad \text{for } x \in \Gamma_1, \tag{3.32}$$

and, from (3.30),

$$\lambda z(x,\rho) + \frac{1-\tau_1'(t)\rho}{\tau_1(t)} z_\rho(x,\rho) = h_1(x,\rho), \quad \text{for } x \in \Gamma_1,\ \rho \in (0,1). \tag{3.33}$$

Then, by (3.32) and (3.33), we obtain

$$z(x,\rho) = v(x)e^{-\lambda\rho\tau_1(t)} + \tau_1(t)e^{-\lambda\rho\tau_1(t)} \int_0^\rho h_1(x,\sigma)e^{\lambda\sigma\tau_1(t)}d\sigma,$$

if $\tau_1'(t) = 0$, and

$$z(x,\rho) = v(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} + e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} \int_0^\rho \frac{h_1(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\sigma)}d\sigma,$$

otherwise. From (3.31),

$$z(x,\rho) = \lambda u(x)e^{-\lambda\rho\tau_1(t)} - f(x)e^{-\lambda\rho\tau_1(t)} + \tau_1(t)e^{-\lambda\rho\tau_1(t)} \int_0^\rho h_1(x,\sigma)e^{\lambda\sigma\tau_1(t)}d\sigma, \tag{3.34}$$

on $\Gamma_1 \times (0,1)$.

If $\tau_1'(t) = 0$, and

$$z(x,\rho) = \lambda u(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} - f(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} + e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\rho)} \int_0^\rho \frac{h_1(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\sigma)}d\sigma, \tag{3.35}$$

on $\Gamma_1 \times (0,1)$ otherwise.

In particular, if $\tau_1'(t) = 0$,

$$z(x,1) = \lambda u(x)e^{-\lambda\tau_1(t)} + z_0(x), \quad x \in \Gamma_1, \tag{3.36}$$

with $z_0 \in L^2(\Gamma_1)$ defined by

$$z_0(x) = -f(x)e^{-\lambda\tau_1(t)} + \tau_1(t)e^{-\lambda\tau_1(t)} \int_0^1 h_1(x,\sigma)e^{\lambda\sigma\tau_1(t)}d\sigma, \quad x \in \Gamma_1, \tag{3.37}$$

and, if $\tau_1'(t) \neq 0$,

$$z(x,1) = \lambda u(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} + z_0(x), \quad x \in \Gamma_1, \tag{3.38}$$

with $z_0 \in L^2(\Gamma_1)$ defined by

$$z_0(x) = -f(x)e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} + e^{\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} \int_0^1 \frac{h_1(x,\sigma)\tau_1(t)}{1-\tau_1'(t)\sigma} e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t)\sigma)}d\sigma. \tag{3.39}$$

Now we will determine w, again by (3.15),

$$\lambda w(x,\rho) + \frac{1-\tau_2'(t)\rho}{\tau_2(t)} w_\rho(x,\rho) = h_2(x,\rho),$$

then

$$w(x,\rho) = v(x)e^{-\lambda\rho\tau_2(t)} + \tau_2(t)e^{-\lambda\rho\tau_2(t)} \int_0^\rho h_2(x,\sigma)e^{\lambda\sigma\tau_2(t)}d\sigma,$$

if $\tau_2'(t) = 0$, and

$$w(x,\rho) = v(x)e^{\lambda\frac{\tau_2(t)}{\tau_2'}\ln(1-\tau_2'(t)\rho)} + e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\rho)}\int_0^\rho \frac{h_2(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma}e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\sigma)}d\sigma,$$

otherwise. From (3.31),

$$w(x,\rho) = \lambda u(x)e^{-\lambda\rho\tau_2(t)} - f(x)e^{-\lambda\rho\tau_2(t)} + \tau_2(t)e^{-\lambda\rho\tau_2(t)}\int_0^\rho h_2(x,\sigma)e^{\lambda\sigma\tau_2(t)}d\sigma, \tag{3.40}$$

if $\tau_2'(t) = 0$, and

$$w(x,\rho) = \lambda u(x)e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\rho)} - f(x)e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\rho)} + e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\rho)}\int_0^\rho \frac{h_2(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma}e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\sigma)}d\sigma, \tag{3.41}$$

on $\Gamma_1 \times (0,1)$ otherwise.

In particular, if $\tau_2'(t) = 0$,

$$w(x,1) = \lambda u(x)e^{-\lambda\tau_2(t)} + w_0(x), \quad x \in \Gamma_1, \tag{3.42}$$

with $w_0 \in L^2(\Gamma_1)$ defined by

$$w_0(x) = -f(x)e^{-\lambda\tau_2(t)} + \tau_2(t)e^{-\lambda\tau_2(t)}\int_0^1 h_2(x,\sigma)e^{\lambda\sigma\tau_2(t)}d\sigma, \quad x \in \Gamma_1, \tag{3.43}$$

and, if $\tau_2'(t) \neq 0$,

$$w(x,1) = \lambda u(x)e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))} + w_0(x), \quad x \in \Gamma_1, \tag{3.44}$$

with $w_0 \in L^2(\Gamma_1)$ defined by

$$w_0(x) = -f(x)e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))} + e^{\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))}\int_0^1 \frac{h_2(x,\sigma)\tau_2(t)}{1-\tau_2'(t)\sigma}e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t)\sigma)}d\sigma, \tag{3.45}$$

for $x \in \Gamma_1$. Then, we have to find u. In view of the equation

$$\lambda v - \Delta(u + av) = g,$$

we set $s = u + av$ and look at s. Now according to (3.31), we may write

$$v = \lambda u - f = \lambda s - f - \lambda av,$$

or equivalently

$$v = \frac{\lambda}{1+\lambda a}s - \frac{1}{1+\lambda a}f. \tag{3.46}$$

Hence once s will be found, we will get v by (3.46) and then u by $u = s - av$, or equivalently

$$u = \frac{1}{1+\lambda a}s + \frac{a}{1+\lambda a}f. \tag{3.47}$$

By (3.46) and (3.30), the function s satisfies

$$\frac{\lambda^2}{1+\lambda a}s - \Delta s = g + \frac{\lambda}{1+\lambda a}f \quad \text{in } \Omega, \tag{3.48}$$

with the boundary conditions

$$s = 0 \quad \text{on } \Gamma_0, \tag{3.49}$$

as well as (at least formally)

$$\frac{\partial s}{\partial \nu} = \mu g_1 - \mu\lambda v_1 - k_1 z(\cdot,1) - k_2 w(\cdot,1) \quad \text{on } \Gamma_1,$$

which becomes due to (3.46), (3.47), (3.36), (3.38), (3.40) and the requirement that $v_1 = \gamma_1 v$ on Γ_1:

$$\frac{\partial s}{\partial \nu} = -\frac{\lambda(k_1 e^{-\lambda\tau_1(t)} + k_2 e^{-\lambda\tau_2(t)} + \mu\lambda)}{1+\lambda a} s + l \quad \text{on } \Gamma_1, \tag{3.50}$$

where

$$l = \mu g_1 + \frac{\lambda(\mu - k_1 a e^{-\lambda\tau_1(t)} - k_2 a e^{-\lambda\tau_2(t)})}{1+\lambda a} f - k_1 z_0 - k_2 w_0 \quad \text{on } \Gamma_1,$$

if $\tau_1'(t) = \tau_2'(t) = 0$, before $\tau_1'(t) \neq 0$ and $\tau_2'(t) \neq 0$ we obtain

$$\frac{\partial s}{\partial \nu} = -\frac{\lambda(k_1 e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} + k_2 e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))} + \mu\lambda)}{1+\lambda a} s + \tilde{l} \quad \text{on } \Gamma_1, \tag{3.51}$$

where

$$\tilde{l} = \mu g_1 + \frac{\lambda(\mu - k_1 a e^{-\lambda\frac{\tau_1(t)}{\tau_1'(t)}\ln(1-\tau_1'(t))} - k_2 a e^{-\lambda\frac{\tau_2(t)}{\tau_2'(t)}\ln(1-\tau_2'(t))})}{1+\lambda a} f - k_1 z_0 - k_2 w_0 \quad \text{on } \Gamma_1.$$

From (3.48), integrating by parts, and using (3.49), (3.50), (3.51) we find the variational problem

$$\begin{aligned}&\int_\Omega \left(\frac{\lambda^2}{1+\lambda a} sq + \nabla s\cdot\nabla q\right)dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\tau_1} + k_2 e^{-\lambda\tau_2} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega \left(g + \frac{\lambda}{1+\lambda a} f\right) q dx + \int_{\Gamma_1} lq d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega),\end{aligned} \tag{3.52}$$

if $\tau_1'(t) = \tau_2'(t) = 0$, before $\tau_1'(t) \neq 0$ and $\tau_2'(t) \neq 0$ we obtain

$$\begin{aligned}&\int_\Omega \left(\frac{\lambda^2}{1+\lambda a} sq + \nabla s\cdot\nabla q\right)dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\frac{\tau_1}{\tau_1'}\ln(1-\tau_1')} + k_2 e^{-\lambda\frac{\tau_2}{\tau_2'}\ln\left(1-\tau_2'\right)} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega \left(g + \frac{\lambda}{1+\lambda a} f\right) q dx + \int_{\Gamma_1} \tilde{l} q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega).\end{aligned} \tag{3.53}$$

If $\tau_1'(t) = 0$ and $\tau_2'(t) \neq 0$ we have

$$\begin{aligned}&\int_\Omega \left(\frac{\lambda^2}{1+\lambda a} sq + \nabla s\cdot\nabla q\right)dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\tau_1} + k_2 e^{-\lambda\frac{\tau_2}{\tau_2'}\ln\left(1-\tau_2'\right)} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega \left(g + \frac{\lambda}{1+\lambda a} f\right) q dx + \int_{\Gamma_1} \tilde{l} q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega).\end{aligned} \tag{3.54}$$

Otherwise, we get

$$\begin{aligned}&\int_\Omega \left(\frac{\lambda^2}{1+\lambda a} sq + \nabla s\cdot\nabla q\right)dx + \int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\frac{\tau_1}{\tau_1'}\ln(1-\tau_1')} + k_2 e^{-\lambda\tau_2} + \mu\lambda)}{1+\lambda a} sq d\Gamma \\ &= \int_\Omega \left(g + \frac{\lambda}{1+\lambda a} f\right) q dx + \int_{\Gamma_1} \tilde{l} q d\Gamma \quad \forall q \in H^1_{\Gamma_0}(\Omega),\end{aligned} \tag{3.55}$$

As the left-hand side of (3.52), (3.53), (3.54), (3.55) is coercive on $H^1_{\Gamma_0}(\Omega)$, the Lax-Milgram lemma guarantees the existence and uniqueness of a solution $s \in H^1_{\Gamma_0}(\Omega)$ of (3.52), (3.53), (3.54), (3.55).

If we consider $q \in D(\Omega)$ in (3.52), (3.53), we have that s solves (3.48) in $D'(\Omega)$ and thus $s = u + av \in E(\Delta, L^2(\Omega))$.

Using Green's formula (3.17) in (3.52) and using (3.48), we obtain

$$\int_{\Gamma_1} \frac{\lambda(k_1 e^{-\lambda\tau_1} + k_2 e^{-\lambda\tau_2} + \mu\lambda)}{1+\lambda a} sq d\Gamma + \langle \frac{\partial s}{\partial \nu}; q\rangle_{\Gamma_1} = \int_{\Gamma_1} lq \, d\Gamma,$$

leading to (3.50) and then to the third equation of (3.30) due to the definition of l and the relations between u, v and s. We find the same result if $\tau_i'(t) \neq 0$, $i = 1, 2$.

In conclusion, we have found $(u, v, v_1, z, w)^T \in D(A)$, which verifies (3.30), and thus $\lambda I - A(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{A}(t) = (\lambda + \kappa(t))I - A(t) \text{ is surjective}, \tag{3.56}$$

for any $\lambda > 0$ and $t > 0$. □

Proof. (of Theorem 3.2) Then, (3.21), (3.27) and (3.56) imply that the family $\tilde{A} = \{\tilde{A}(t) : t \in [0, T]\}$ is a stable family of generators in $\tilde{H}$ with stability constants independent of t. Therefore, all assumptions of Theorem 3.1 are satisfied by (3.16), Lemma3.2– Lemma3.5, and thus, the problem

$$\begin{aligned} \tilde{U}' &= \tilde{A}(t)\tilde{U}, \\ \tilde{U}(0) &= U_0, \end{aligned}$$

has a unique solution $\tilde{U} \in C([0, +\infty), D(A(0))) \cap C^1([0, +\infty), \tilde{H})$ for $U_0 \in D(A(0))$. The requested solution is then given by

$$U(t) = e^{\int_0^t \kappa(s)ds}\tilde{U}(t).$$

This concludes the proof. □

4. STABILITY RESULT

Now, we show that problem (1.1)–(1.5) is uniformly exponentially stable under the assumption

$$\frac{a}{2} > \frac{C^*|k|}{\sqrt{1-d}}. \tag{4.1}$$

We fix ξ_i, (given in (1.6)) such that

$$\frac{|k|}{\sqrt{1-d}} < \xi_i < \frac{a}{C^*} - \frac{|k|}{\sqrt{1-d}}, \;\; i = 1, 2. \tag{4.2}$$

Moreover, the parameter λ (given in (1.6)) is fixed to satisfy

$$\lambda < \min\left\{ \frac{1}{\overline{\tau}} \left| \log \frac{|k|}{\xi_1\sqrt{1-d}} \right|, \frac{1}{\overline{\tau}} \left| \log \frac{|k|}{\xi_2\sqrt{1-d}} \right| \right\}. \tag{4.3}$$

We start with giving an explicit formula for the derivative of the energy.

Lemma 4.1. *Assume (2.1)–(2.4) and (4.1). Then, for any regular solution of problem (1.1)–(1.5) the energy is decreasing and, for a suitable positive constant C, we have*

$$\begin{aligned} E'(t) \leq -C &\left\{ \int_\Omega |\nabla u'|^2 \, dx + \int_{\Gamma_1} u'^2(x, t - \tau_1(t)) d\Gamma + \int_{\Gamma_1} u'^2(x, t - \tau_2(t)) \, d\Gamma \right\} \\ &- C \left\{ \int_{t-\tau_1(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u'^2(x, s) d\Gamma ds + \int_{t-\tau_2(t)}^t \int_{\Gamma_1} e^{\lambda(s-t)} u'^2(x, s) \, d\Gamma ds \right\}. \end{aligned} \tag{4.4}$$

Proof. Differentiating (1.6), we obtain

$$\begin{aligned} E'(t) &= \int_\Omega \{u'u'' + \nabla u \nabla u'\} dx + \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2 d\Gamma + \mu \int_{\Gamma_1} u'u''\, d\Gamma \\ &- \frac{\xi_1(1-\tau_1')}{2}\int_{\Gamma_1} e^{-\lambda\tau_1} u'^2(x,t-\tau_1)d\Gamma - \frac{\xi_2(1-\tau_2')}{2}\int_{\Gamma_1} e^{-\lambda\tau_2} u'^2(x,t-\tau_2)d\Gamma \\ &- \lambda\frac{\xi_1}{2}\int_{t-\tau_1}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u'^2(x,s)d\Gamma ds - \lambda\frac{\xi_2}{2}\int_{t-\tau_2}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u'^2(x,s)d\Gamma ds, \end{aligned}$$

and then, applying Green's formula,

$$\begin{aligned} E'(t) &= \int_\Omega a u' \Delta u' dx + \int_{\Gamma_1} u' \frac{\partial u}{\partial \nu} d\Gamma \\ &+ \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2(x,t)\, d\Gamma - \frac{\xi_1(1-\tau_1'(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_1(t)} u'^2(x,t-\tau_1(t))d\Gamma \\ &- \frac{\xi_2(1-\tau_2'(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_2(t)} u'^2(x,t-\tau_2(t))\, d\Gamma \\ &- \lambda\frac{\xi_1}{2}\int_{t-\tau_1(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u'^2(x,s)d\Gamma ds \\ &- \lambda\frac{\xi_2}{2}\int_{t-\tau_2(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u'^2(x,s)\, d\Gamma ds + \mu\int_{\Gamma_1} u'u'' d\Gamma. \end{aligned} \tag{4.5}$$

Integrating once more by parts and using the boundary conditions we obtain

$$\begin{aligned} F'(t) &= -a\int_\Omega \left|\nabla u'\right|^2 dx - k_1\int_{\Gamma_1} u'u'(x,t-\tau_1(t))d\Gamma \\ &- k_2 \int_{\Gamma_1} u'(x,t)\, u'(x,t-\tau_2(t))\, d\Gamma \\ &- \frac{\xi_1(1-\tau_1'(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_1} u'^2(x,t-\tau_1(t))d\Gamma \\ &- \frac{\xi_2(1-\tau_2'(t))}{2}\int_{\Gamma_1} e^{-\lambda\tau_2(t)} u'^2(x,t-\tau_2(t))\, d\Gamma \\ &+ \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2 d\Gamma - \lambda\frac{\xi_1}{2}\int_{t-\tau_1(t)}^t \int_{\Gamma_1} e^{-\lambda(t-s)} u'^2(x,s)d\Gamma ds \\ &- \lambda\frac{\xi_2}{2}\int_{t-\tau_2(t)} \int_{\Gamma_1} e^{-\lambda(t-s)} u'^2(x,s)\, d\Gamma ds. \end{aligned} \tag{4.6}$$

Applying Cauchy-Schwarz's and Poincaré's inequalities, a trace estimate and recalling the assumptions (2.1)–(2.4), we obtain

$$\begin{aligned} E'(t) &\le -a\int_\Omega |\nabla u'|^2 dx + \frac{\xi_1+\xi_2}{2}\int_{\Gamma_1} u'^2 d\Gamma \\ &+ \frac{|k|}{2\sqrt{1-d}}\int_{\Gamma_1} u'^2 d\Gamma + \frac{|k|}{2\sqrt{1-d}}\int_{\Gamma_1} u'^2 d\Gamma \\ &+ \frac{|k|}{2}\sqrt{1-d}\int_{\Gamma_1} u'^2(t-\tau_1)d\Gamma + \frac{|k|}{2}\sqrt{1-d}\int_{\Gamma_1} u'^2(t-\tau_2)d\Gamma \\ &- \frac{\xi_1}{2}(1-d_1)e^{-\lambda\overline{\tau}}\int_{\Gamma_1} u'^2(x,t-\tau_1)d\Gamma - \frac{\xi_2}{2}(1-d_2)e^{-\lambda\widetilde{\tau}}\int_{\Gamma_1} u'^2(x,t-\tau_2)d\Gamma \end{aligned}$$

$$
\begin{aligned}
& -\lambda\frac{\xi_1}{2}\int_{t-\tau_1}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds-\lambda\frac{\xi_2}{2}\int_{t-\tau_2}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds\\
\le\ & -\left(\frac{a}{2}-\frac{|k|C^*}{2\sqrt{1-d}}-\frac{\xi_1}{2}C^*\right)\int_\Omega |\nabla u'|^2\,dx\\
& -\left(\frac{a}{2}-\frac{|k|C^*}{2\sqrt{1-d}}-\frac{\xi_2}{2}C^*\right)\int_\Omega |\nabla u'|^2\,dx\\
& -\left(e^{-\lambda\overline{\tau}}\frac{\xi_1}{2}(1-d)-\frac{|k|}{2}\sqrt{1-d}\right)\int_{\Gamma_1} u'^2(x,t-\tau_1)d\Gamma\\
& -\left(e^{-\lambda\widetilde{\tau}}\frac{\xi_2}{2}(1-d)-\frac{|k|}{2}\sqrt{1-d}\right)\int_{\Gamma_1} u'^2(x,t-\tau_2 d\Gamma\\
& -\lambda\frac{\xi_1}{2}\int_{t-\tau_1}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds-\lambda\frac{\xi_2}{2}\int_{t-\tau_2}^{t}\int_{\Gamma_1} e^{-\lambda(t-s)}u'^2(x,s)d\Gamma ds.
\end{aligned}
$$

Therefore, (4.4) immediately follows recalling (4.2) and (4.3). □

We will use an appropriate Lyapunov functional. For this purpose, let us define the Lyapunov functional

$$F(t)=E(t)+\varepsilon\left[\int_\Omega uu'dx+\mu\int_{\Gamma_1} uu'd\Gamma\right], \tag{4.7}$$

where ε is a positive small constant that we will choose later on.

Remark 4.2. From Poincaré's inequality, it is easy to verify that the functional F is equivalent to the energy E, that is, for ε small enough, there exist two positive constant $\varepsilon_1,\varepsilon_2$ such that

$$\varepsilon_1 F(t)\le E(t)\le \varepsilon_2 F(t),\quad \forall t\ge 0. \tag{4.8}$$

Lemma 4.3. *For any regular solution (u,z,w) of problem (1.1)–(1.5), we have*

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega uu'dxdt+\mu\int_{\Gamma_1} uu'd\Gamma\right\}\\
&\le C\left\{\int_\Omega |\nabla u'|^2\,dx+\int_{\Gamma_1} u'^2(x,t-\tau_1(t))d\Gamma+\int_{\Gamma_1} u'^2\left(x,t-\tau_2\left(t\right)\right)d\Gamma\right\}\\
&-\frac{1}{2}\int_\Omega |\nabla u|^2dx,
\end{aligned}
\tag{4.9}
$$

for a suitable positive constant C.

Proof. Differentiating and integrating by parts we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega uu'dx &= \int_\Omega u'^2dx+\int_\Omega u(\Delta u+a\Delta u')dx\\
&=\int_\Omega u'^2dx-\int_\Omega |\nabla u|^2dx-a\int_\Omega \nabla u\cdot\nabla u'dx\\
&+\int_{\Gamma_1} u(t)\frac{\partial(u+au')}{\partial\nu}(t)d\Gamma.
\end{aligned}
\tag{4.10}
$$

From (4.10), using the boundary condition on Γ_1, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega uu'dx+\mu\int_{\Gamma_1} uu'd\Gamma\right\}\\
&=\int_\Omega u'^2dx+\int_\Omega u(\Delta u+a\Delta u')dx+\mu\int_{\Gamma_1} u'^2d\Gamma+\mu\int_{\Gamma_1} uu''d\Gamma
\end{aligned}
$$

$$= \int_\Omega u'^2 dx - \int_\Omega |\nabla u|^2 dx - a \int_\Omega \nabla u \cdot \nabla u' dx$$
$$- k_1 \int_{\Gamma_1} u(t) u'(t - \tau_1(t)) d\Gamma - k_2 \int_{\Gamma_1} u\,(t)\, u'\,(t - \tau_2\,(t))\, d\Gamma + \mu \int_{\Gamma_1} u'^2 d\Gamma. \quad (4.11)$$

We can conclude by using Young's, Poincaré's inequalities and a trace estimate. □

Now we can deduce our last result.

Theorem 4.1. *Assume (2.1)–(2.4) and (4.1). Then there exist positive constants C_1, C_2 such that for any solution of problem (1.1)-(1.5),*

$$F(t) \le C_1 F(0) e^{-C_2 t}, \quad \forall t \ge 0. \quad (4.12)$$

Proof. From Lemma 4.3, taking ε sufficiently small in the definition of the Lyapunov functional F, we have

$$\frac{d}{dt} F(t) \le -C \left\{ \int_\Omega |\nabla u'|^2 dx + \int_{\Gamma_1} u'^2(x, t - \tau_1(t)) dx + \int_{\Gamma_1} u'^2\,(x; t - \tau_2\,(t))\, dx \right\}$$
$$- C \int_{t-\tau_1(t)}^t e^{-\lambda(t-s)} \int_{\Gamma_1} u'^2(x,s) d\Gamma ds - C \int_{t-\tau_2(t)}^t e^{-\lambda(t-s)} \int_{\Gamma_1} u'^2(x,s) d\Gamma ds$$
$$- \frac{\varepsilon}{2} \int_\Omega |\nabla u|^2 dx, \quad (4.13)$$

for a suitable positive constant C. Poincaré's inequality implying

$$\int_\Omega |u'|^2 dx + \int_{\Gamma_1} |u'|^2 ds \le C_1^* \int_\Omega |\nabla u'|^2 dx,$$

for some $C_1^* > 0$, we obtain

$$\frac{d}{dt} F(t) \le -C' E(t), \quad (4.14)$$

for a suitable positive constant C'. (4.8) permits us to conclude our result. □

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