H. Olaoluwa & J.O. Olaleru, J. Nonlinear Anal. Optim. Vol. 9(1) (2018), 39-48

Journal of Nonlinear Analysis and Optimization
Volume 9(1) (2018)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

GENERALIZED CONTRACTIVE MAPPINGS IN b-METRIC SPACES

ZAHRA ALINEJAD AND ALIREZA KAMEL MIRMOSTAFAEE*

Department of Mathematics, Ferdowsi University of Mashhad,
P. O. Box 1159, Mashhad 91775, Mashhad, Iran

ABSTRACT. We will apply a new method to generalize some fixed point theorems for a special class of contractive type mappings in complete b-metric spaces. We also discuss about the existence of a unique fixed point for a family of self mappings. Our results enable us to generalize some known fixed point theorems in the literature. Finally, we present some applications of our results.

KEYWORDS: Fixed points, b-meric spaces, contraction-type mappings.
AMS Subject Classification: Primary 47H10, 54E40; Secondary 47H9.

1. Introduction and preliminaries

Let (X, d) be a metric space and T be a self mapping on X. The function T is said to be a contraction on X if there is some $0 \leq r < 1$ such that

$$d\big(T(x), T(y)\big) \leq r d(x, y) \quad (x, y \in X). \tag{1.1}$$

A celebrated result due to Banach [3] states that every contraction function on a complete metric space has a unique fixed point. The Banach contraction mapping principle is considered to be the core of many extended fixed point theorems (see. e. g. [6, 8, 10, 14, 18, 20, 21, 22]).

Despite these important features, Banach fixed point theorem suffers from one serious drawback - the contractive condition (1.1) forces T to be continuous on the entire space X. It was then naturally to ask if there exist contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [13] in 1968, who proved a fixed point theorem which extends Banach's theorem to mappings that need not be continuous, by considering instead of (1.1) the next condition:

$$d\big(T(x), T(y)\big) \leq r\big\{d\big(x, T(x)\big) + d\big(y, T(y)\big)\big\} \text{ for some } 0 \leq r < \frac{1}{2} \text{ and all } x, y \in X. \tag{1.2}$$

* *Corresponding author.*
Email address : alinejad_z@yahoo.com, mirmostafaei@um.ac.ir.

Kannan's theorem has been generalized by some authors [16, 17]. In particular, G. Hardy and T. Rogers [12] proved the following results:

THEOREM 1.1. [12, Theorem 1] *Let X be a complete metric space with metric d, and let $T : X \longrightarrow X$ be a function with the following property:*

$$d\big(T(x), T(y)\big) \leq a\ d\big(x, T(x)\big) + b\ d\big(y, T(y)\big) + c\ d\big(x, T(y)\big) + e\ d\big(y, T(x)\big) + f\ d(x, y), \tag{1.3}$$

where $0 \leq a, b, c, e, f < 1$ and $a+b+c+e+f < 1$. Then T has a unique fixed point.

THEOREM 1.2. [12, Theorem 2] *Let (X, d) be a complete metric space, a, b, c, e, f be monotonically decreasing functions from $[0, \infty)$ to $[0, 1)$, and let the sum of these five functions be less than 1. Suppose $T : X \longrightarrow X$ satisfies condition (1.3) with $a = a(d(x, y)), \cdots, f = f(d(x, y))$ for all $x, y \in X$. Then T has a unique fixed point.*

The above results were extended for some metric-like spaces (see e.g. [1, 15, 19]). The following generalization of a metric is due to S. Czerwik [7]:

DEFINITION 1.3. Let X be a space and $d : X \times X \longrightarrow [0, \infty)$ be a function such that for each x, y and z in X,

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) for each $\varepsilon > 0$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.

Then d is called a b-metric on X. It is easy to verify that the condition (iii) is equivalent to the following (see e. g. [7, Lemma 1]).

(iv) For each $\varepsilon > 0$ and $x, y, z \in X$, if $d(x, y) \leq \varepsilon$ and $d(y, z) \leq \varepsilon$, then $d(x, z) \leq 2\varepsilon$.

Clearly, every metric space is a b-metric space. However, the converse is not true in general.

EXAMPLE 1.4. Let $X = \{1, 2, 3\}$. Define a symmetric function $d : X \times X \longrightarrow [0, \infty)$ by

$$d(1, 1) = d(2, 2) = d(3, 3) = 0,\ d(1, 2) = 3,\ d(2, 3) = 2,\ d(1, 3) = 6.$$

It is easy to verify that (iv) holds, hence (X, d) is a b-metric space but it is not a metric space, since $d(1, 3) \not\leq d(1, 2) + d(2, 3)$.

In order to prove some fixed point theorems of Banach type for b-metric spaces, S. Czerwik [7] replaced (iv) by the following weaker condition:

$$d(x, y) \leq 2d(x, z) + 2d(z, y) \quad (x, y, z \in X). \tag{1.4}$$

The following example shows that (1.4) is strictly weaker than (iv).

EXAMPLE 1.5. Let $X = \mathbb{R}$ and define $d : X \times X \longrightarrow [0, \infty)$ by

$$d(x, y) = |x - y|^2 \quad (x, y \in \mathbb{R}).$$

Then for $\varepsilon = 1$, we have $d(1, 0) = \varepsilon$, $d(0, -1) = \varepsilon$ but $d(1, -1) = 4 \nleq 2\varepsilon = 2$. Hence (iv) is not true. However, it is easy to verify that (1.4) holds.

Following [7], a few mathematicians investigated the existence of fixed points for self-mappings on special kind of b-metrics X; i. e. the triangle inequality is replaced by

$$d(x, y) \leq k\big(d(x, z) + d(z, y)\big) \quad (x, y, z \in X),$$

for some $k \geq 1$ (see e. g. [2, 4, 5, 22]).

In this paper, we will assume that our b-metric space satisfies (iv). We will show that Frink's lemma and the inequality (iv) enable us to improve some known fixed point theorems. More precisely, we will establish Theorems 1.1 and 1.2 for b-metric spaces, which improves some results in [7] and [16]. We also discuss about the existence of a unique fixed point for a family of self mappings on b-metric spaces.

2. Results

In order to state main results of this paper, we need to the following result.

LEMMA 2.1. *(see [9] or [11]). Suppose $d : X \times X \longrightarrow [0, \infty)$ satisfies the following condition:*
For any $\varepsilon > 0$ and $x, y, z \in X$, if $d(x, y) < \varepsilon$ and $d(y, z) < \varepsilon$, then $d(x, z) < 2\varepsilon$.
Then the function $\rho : X \times X \longrightarrow [0, \infty)$, defined by

$$\rho(x,y) = inf\Big\{\sum_{i=1}^{n} d(x_{i-1}, x_i);\ where\ n \in \mathbb{N}, x_0 = x\ and\ x_n = y\Big\}, \quad \big((x,y) \in X\times X\big), \tag{2.1}$$

has the following properties:

(i) *$\rho(x, z) \le \rho(x, y) + \rho(y, z)$, for all $x, y, z \in X$.*
(ii) *$\frac{d(x,y)}{4} \le \rho(x, y) \le d(x, y)$ for all $x, y \in X$. Further, ρ is symmetric (i.e. $\rho(x, y) = \rho(y, x)$) if d is.*

Now, we are ready to state one of the main results of this paper.

THEOREM 2.2. *Let (X, d) be a b-metric space and T be a self-mapping on X such that for all $x, y \in X$*

$$d\big(T(x), T(y)\big) \le a\ d\big(x, T(x)\big) + b\ d\big(y, T(y)\big) + c\ d\big(x, T(y)\big) + e\ d\big(y, T(x)\big) + f\ d(x, y), \tag{2.2}$$

where a, b, c, e, f are nonnegative, $a + b + 2(c + e) + f < 1$. Then T has a unique fixed point.

Proof. We first show that T has at most one fixed point. Let x^*, y^* be fixed points of T. We have

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\le a\ d\big(x^*, T(x^*)\big) + b\ d\big(y^*, T(y^*)\big) + c\ d(y^*, T(x^*)) + e\ d\big(x^*, T(y^*)\big) + f\ d(x^*, y^*) \\ &= (c + e + f)\ d(x^*, y^*). \end{aligned}$$

Hence $x^* = y^*$. By symmetry, it follows from (2.2) that

$$d\big(T(x), T(y)\big) \le a\ d\big(y, T(y)\big) + b\ d\big(x, T(x)\big) + c\ d\big(y, T(x)\big) + e\ d\big(x, T(y)\big) + f\ d(x, y), \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$d\big(T(x), T(y)\big) \le \frac{a + b}{2}\ [d\big(x, T(x)\big) + d\big(y, T(y)\big)] + \frac{c + e}{2}\ [d\big(x, T(y)\big) + d\big(y, T(x)\big)] + f\ d(x, y)$$

for all $x, y \in X$. Let $\alpha = \frac{a+b}{2}$, $\beta = \frac{c+e}{2}$ and put $y = T(x)$, then we have

$$\begin{aligned} d\big(T(x), T^2(x)\big) &\le \alpha[d\big(x, T(x)\big) + d\big(T(x), T^2(x)\big)] + \beta[d\big(x, T^2(x)\big) + d\big(T(x), T(x)\big)] \\ &+ f d\big(x, T(x)\big) \\ &= (\alpha + f)\ d\big(x, T(x)\big) + \alpha\ d\big(T(x), T^2(x)\big) + \beta\ d\big(x, T^2(x)\big) \\ &\le (\alpha + 2\beta + f)\ d\big(x, T(x)\big) + (\alpha + 2\beta)\ d\big(T(x), T^2(x)\big) \quad (x \in X). \end{aligned}$$

Let $r = \frac{\alpha+2\beta+f}{1-\alpha-2\beta}$. Then $0 \le r < 1$, since $a+b+2(c+e)+f < 1$. By the above inequality

$$d\big(T(x), T^2(x)\big) \le r d\big(x, T(x)\big) \quad (x \in X).$$

By induction, it follows that for all $n \in \mathbb{N}$,

$$d\big(T^n(x), T^{n+1}(x)\big) \le r^n d\big(x, T(x)\big) \quad (x \in X). \tag{2.4}$$

Take some arbitrary point $x_0 \in X$ and define

$$x_1 = T(x_0), x_2 = T(x_1), \ldots, x_n = T(x_{n-1}), \ldots.$$

By (2.4), we have

$$d(x_n, x_{n+1}) \le r^n d(x_0, x_1).$$

According to Lemma 2.1, there is a metric ρ on X such that $\frac{d(x,y)}{4} \le \rho(x,y) \le d(x,y)$ for all $x, y \in X$. For each $m > n$, we have

$$\begin{aligned} d(x_n, x_m) \le 4\rho(x_n, x_m) &\le 4\sum_{i=n}^{m-1} \rho(x_{i+1}, x_i) \\ &\le 4\sum_{i=n}^{m-1} d(x_{i+1}, x_i) \\ &\le 4d(x_1, x_0)\sum_{i=n}^{m-1} r^i \le \frac{4r^n d(x_0, x_1)}{1-r}, \end{aligned}$$

which tends to zero as n tends to infinity. Hence $\{x_n\}$ is a Cauchy sequence in complete b-metric space (X, d). Therefore, $\{x_n\}$ is convergent to some $x^* \in X$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x^*, T(x^*)) &\le 2d\big(T^{n+1}(x_0), T(x^*)\big) + 2d\big(T^{n+1}(x_0), x^*\big) \\ &\le 2a\ d(T^n(x_0), T^{n+1}(x_0)\big) + 2b\ d\big(x^*, T(x^*)\big) + 2c\ d\big(T^n(x_0), T(x^*)\big) \\ &\quad +(2e+2)\ d\big(T^{n+1}(x_0), x^*\big) + 2f\ d\big(T^n(x_0), x^*\big) \\ &\le 2a\ d(T^n(x_0), T^{n+1}(x_0)\big) + 2b\ d\big(x^*, T(x^*)\big) + 4c\ d\big(T^n(x_0), x^*\big) \\ &\quad +4c\ d\big(T(x^*), x^*\big) + (2e+2)\ d\big(T^{n+1}(x_0), x^*\big) + 2f\ d\big(T^n(x_0), x^*\big). \end{aligned}$$

By taking limit as n tends to infinity, it follows that

$$d(x^*, T(x^*)) \le (2b+4c)\ d(x^*, T(x^*)). \tag{2.5}$$

By symmetry,

$$d(x^*, T(x^*)) \le (2a+4e)\ d(x^*, T(x^*)). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned} d(x^*, T(x^*)) &\le \frac{1}{2}\big[(2b+4c)\ d(x^*, T(x^*)) + (2a+4e)\ d(x^*, T(x^*))\big] \\ &= \big(a+b+2(c+e)\big)\ d(x^*, T(x^*)). \end{aligned} \tag{2.7}$$

In view of (2.7) and the fact that $a+b+2(c+e) < 1-f$, we have $T(x^*) = x^*$.

□

In the following result, we will show that under certain condition, if $T : X \longrightarrow X$ satisfies (2.2) on a complete subset, not necessarily on the entire space, then T has a unique fixed point.

COROLLARY 2.3. *Let T be a self mapping on a b-metric space (X,d). Let Y be a complete subset of X such that $T(Y)\subseteq Y$ and for each $x,y\in Y$,*

$$d\big(T(x),T(y)\big)\le a\ d\big(x,T(x)\big)+b\ d\big(y,T(y)\big)+c\ d\big(x,T(y)\big)+e\ d\big(y,T(x)\big)+f\ d(x,y), \tag{2.8}$$

where a,b,c,e,f are nonnegative and $a+b+2(c+e)+f<1$. If for each $z,w\in X$ with $z\neq w$,

$$d\big(T(z),T(w)\big)<d\big(T(z),z\big)+d\big(T(w),w\big)+d(z,w),$$

then T has a unique fixed point on X.

Proof. By Theorem 2.2, $T|_Y:Y\longrightarrow Y$ has a fixed point $y^*\in Y$. Let $x^*\in X$ be another fixed point of T on X. Then we have

$$d(x^*,y^*)=d\big(T(x^*),T(y^*)\big)<d\big(T(x^*),x^*\big)+d\big(T(y^*),y^*\big)+d(x^*,y^*)=d(x^*,y^*).$$

This contradiction shows that $T:X\longrightarrow X$ has a unique fixed point. □

Reich's theorem [16] for b-metric spaces follows immediately from Theorem 2.2:

COROLLARY 2.4. *Let (X,d) be a complete b-metric space and $T:X\longrightarrow X$ satisfy*

$$d\big(T(x),T(y)\big)\le a\ d(x,T(x))+b\ d(y,T(y))+f\ d(x,y)\quad (x,y\in X),$$

where a,b and f are nonnegative and $a+b+f<1$. Then T has a unique fixed point.

In the following, we use our main result to show that under certain circumstances a family of self-mappings has a unique common fixed point.

COROLLARY 2.5. *Let (X,d) be a complete b-metric space and $\mathcal{F}$ be a family of self mappings on X such that for each $T,S\in\mathcal{F}$ and $x,y\in X$,*

$$d\big(T(x),S(y)\big)\le a\ d\big(x,T(x)\big)+b\ d\big(y,S(y)\big)+c\ d\big(x,S(y)\big)+e\ d\big(y,T(x)\big)+f\ d(x,y), \tag{2.9}$$

where a,b,c,e,f are nonnegative and $a+b+2(c+e)+f<1$. Then $\mathcal{F}$ has a unique common fixed point.

Proof. Assume that $T\in\mathcal{F}$. By putting $S=T$ in (2.9) and applying Theorem 2.2, we conclude T has a unique fixed point. So every element of $\mathcal{F}$ has a unique fixed point. Now we show that all elements of $\mathcal{F}$ have a common unique fixed point. Let $S,T\in\mathcal{F}$ and let x^* and y^* be fixed points of T and S respectively. Then

$$\begin{aligned} d(x^*,y^*) &= d\big(T(x^*),S(y^*)\big)\\ &\le a\ d\big(x^*,T(x^*)\big)+b\ d\big(y^*,S(y^*)\big)+c\ d\big(x^*,S(y^*)\big)+e\ d\big(y^*,T(x^*)\big)\\ +f\ d(x^*,y^*) &= (c+e+f)d(x^*,y^*).\end{aligned}$$

Since $c+e+f<1$, we have $x^*=y^*$. □

In 1995, S. Czerwick proved the following result which is a generalization one of the main results in [13, 23]:

THEOREM 2.6. [7, Theorem 3] *Let $a:(0,\infty)\longrightarrow(0,\frac{1}{2})$ be a decreasing function. Let (X,d) be a complete b-metric space and let $T:X\longrightarrow X$ satisfy*

$$d\big(T(x),T(y)\big)\le a\big(d(x,y)\big)\big[d\big(x,T(x)\big)+d\big(y,T(y)\big)\big]\quad (x\neq y;x,y\in X).$$

Then T has a unique fixed point x^* and $\lim_{n\longrightarrow\infty}d\big(T^n(x),x^*\big)=0$ for each $x\in X$.

Here we will prove Theorem 1.2 for b-metric spaces under some extra conditions. This result also generalizes Theorem 2.6.

THEOREM 2.7. *Let (X,d) be a complete b-metric space, a,b,c,e,f be monotonically decreasing functions from $(0,\infty)$ to $[0,1)$ such that $a(t)+b(t)+2(c(t)+e(t))+f(t)<1$, $2(c(t)+e(t))+4f(t)<1$ for each $t\in(0,\infty)$ and $f(t_0)>0$ for some $t_0\in(0,\infty)$. Suppose that $T:X\longrightarrow X$ satisfies*

$$d\big(T(x),T(y)\big)\le a(t)\ d\big(x,T(x)\big)+b(t)\ d\big(y,T(y)\big)+c(t)\ d\big(x,T(y)\big) \quad +e(t)\ d\big(y,T(x)\big)+f(t)\ d(x,y), \tag{2.10}$$

where $t=d(x,y)$ and $x,y\in X$ with $x\neq y$. Then T has a unique fixed point.

Proof. The proof is divided into several steps:

Step 1. *T has at most one fixed point.*
Let x^*,y^* be fixed points of T. If $x^*\neq y^*$, then

$$\begin{aligned} d(x^*,y^*) &= d(T(x^*),T(y^*))\\ &\le a\ d\big(x^*,T(x^*)\big)+b\ d\big(y^*,T(y^*)\big)+c\ d(y^*,T(x^*))+e\ d\big(x^*,T(y^*)\big)+f\ d(x^*,y^*)\\ &= (c+e+f)\ d(x^*,y^*)<d(x^*,y^*), \end{aligned}$$

where $a=a(d(x^*,y^*)),\dots,f=f(d(x^*,y^*))$. This is a contradiction. Hence $x^*=y^*$.

Step 2. *There exists a monotone decreasing function $r:(0,\infty)\longrightarrow[0,1)$ such that for $t=d\big(x,T(x)\big)\neq 0$,*

$$d\big(T^2(x),T(x)\big)\le r(t)d\big(x,T(x)\big)\quad (x\in X). \tag{2.11}$$

By symmetry, it follows from (2.10) that for $a=a(d(x,y))$, $b=b(d(x,y))$, $c=c(d(x,y))$, $e=e(d(x,y))$ and $f=f(d(x,y))$, we have

$$d\big(T(x),T(y)\big)\le a\ d\big(y,T(y)\big)+b\ d\big(x,T(x)\big)+c\ d\big(y,T(x)\big)+e\ d\big(x,T(y)\big)+f\ d(x,y), \tag{2.12}$$

where $x,y\in X$ and $x\neq y$. By (2.10) and (2.3), for the same a,b,c,e and f, we have

$$d\big(T(x),T(y)\big)\le\frac{a+b}{2}\left[d\big(x,T(x)\big)+d\big(y,T(y)\big)\right]+\frac{c+e}{2}\left[d\big(x,T(y)\big)+d\big(y,T(x)\big)\right]+f\ d(x,y) \tag{2.13}$$

for all $x,y\in X$ with $x\neq y$. Fix some $x\in X$, then for $t=d\big(x,T(x)\big)\neq 0$, $\alpha(t)=\frac{a(t)+b(t)}{2}$, $\beta(t)=\frac{c(t)+e(t)}{2}$, we have

$$\begin{aligned} d\big(T(x),T^2(x)\big) &\le \alpha(t)[d\big(x,T(x)\big)+d\big(T(x),T^2(x)\big)]+\beta(t)[d\big(x,T^2(x)\big)+d\big(T(x),T(x)\big)]\\ &+ f(t)d\big(x,T(x)\big)\\ &= (\alpha(t)+f(t))\ d\big(x,T(x)\big)+\alpha(t)\ d\big(T(x),T^2(x)\big)+\beta(t)\ d\big(x,T^2(x)\big)\\ &\le (\alpha(t)+2\beta(t)+f(t))\ d\big(x,T(x)\big)+(\alpha(t)+2\beta(t))\ d\big(T(x),T^2(x)\big)\quad (x\in X). \end{aligned}$$

Let $r(t)=\frac{\alpha(t)+2\beta(t)+f(t)}{1-\alpha(t)-2\beta(t)}$. Then r is monotonically decreasing and (2.11) holds.

Step 3. *For each $x\in X$, $\lim_{n\longrightarrow\infty}d\big(T^n(x),T^{n+1}(x)\big)=0$.*
Let $x\in X$, $n\in\mathbb{N}$ and $t=d\big(T^{n-1}(x),T^n(x)\big)$. If $T^{n-1}(x)=T^n(x)$, then the claim is proved. So we may assume that $t\neq 0$. By step 2, we have

$$d\big(T^n(x),T^{n+1}(x)\big)\le r(t)d\big(T^{n-1}(x),T^n(x)\big)<d\big(T^{n-1}(x),T^n(x)\big).$$

Therefore $\{d\big(T^{n+1}(x),T^n(x)\big)\}$ is a decreasing sequence. Let $\lim_{n\longrightarrow\infty} d\big(T^{n+1}(x),T^n(x)\big) = p$. If $p>0$, then for each $n\in\mathbb{N}$, we have $d\big(T^{n+1}(x),T^n(x)\big)\geq p$. Since r is monotone deceasing,

$$d\big(T^{n+1}(x),T^n(x)\big) < r(p)d\big(T^n(x),T^{n-1}(x)\big) \leq \cdots \leq r^n(p)d\big(x,T(x)\big).$$

Since the right hand side of the above inequality tends to zero as $n\longrightarrow\infty$, we have

$$\lim_{n\longrightarrow\infty} d\big(T^{n+1}(x),T^n(x)\big) = 0.$$

This contradiction shows that $p=0$.

Step 4. *For each $x\in X$, $\{T^n(x)\}$ converges and $x^*=\lim_{n\longrightarrow\infty}T^n(x)$ is the fixed point of T.*

Let $\alpha(t)=\frac{a(t)+b(t)}{2}$, $\beta(t)=\frac{c(t)+e(t)}{2}$, $t\in(0,\infty)$. By our assumption,

$$4\beta(t)+4f(t)=2(c(t)+e(t))+4f(t)<1 \quad (t\in(0,\infty)).$$

Therefore we can define

$$\gamma_1(t)=\tfrac{\alpha(t)+2\beta(t)+2f(t)}{1-4\beta(t)-4f(t)} \text{ and } \gamma_2(t)=\tfrac{\alpha(t)+2\beta(t)+4f(t)}{1-4\beta(t)-4f(t)} \text{ for } t\in(0,\infty).$$

Suppose that $\varepsilon>0$ and $x\in X$. Choose $n_0\in\mathbb{N}$ so that for each $n,m\geq n_0$,

$$d\big(T^n(x),T^{n-1}(x)\big) < \min\left\{\frac{\varepsilon}{8},\frac{\varepsilon}{2\gamma_2(\frac{\varepsilon}{8})}\right\}.$$

Let $n,m>n_0$ and $x\in X$ and $t=d\big(T^{n-1}(x),T^{m-1}(x)\big)$. If $T^{n-1}(x)=T^{m-1}(x)$, then $d\big(T^n(x),T^m(x)\big)=0$. So that we may assume that $t\neq 0$. By (1.4) and (2.13), we have

$$\begin{aligned}
&d\big(T^n(x),T^m(x)\big)\\
\leq\ & \alpha(t)\big[d\big(T^{n-1}(x),T^n(x)\big)+d\big(T^{m-1}(x),T^m(x)\big)\big]\\
&+\beta(t)\big[d\big(T^{n-1}(x),T^m(x)\big)+d\big(T^{m-1}(x),T^n(x)\big)\big]+f(t)\ d\big(T^{n-1}(x),T^{m-1}(x)\big)\\
\leq\ & \alpha(t)\big[d\big(T^{n-1}(x),T^n(x)\big)+d\big(T^{m-1}(x),T^m(x)\big)\big]\\
&+\beta(t)\big[2d\big(T^{n-1}(x),T^n(x)\big)+4d\big(T^n(x),T^m(x)\big)+2d\big(T^{m-1}(x),T^m(x)\big)\big]\\
&+f(t)\ \Big[2d\big(T^{n-1}(x),T^n(x)\big)+4\ d\big(T^n(x),T^m(x)\big)+4d\big(T^m(x),T^{m-1}(x)\big)\Big].
\end{aligned}$$

It follows that

$$d\big(T^n(x),T^m(x)\big)\leq\gamma_1(t)d\big(T^n(x),T^{n-1}(x)\big)+\gamma_2(t)d\big(T^m(x),T^{m-1}(x)\big).$$

If $d\big(T^{n-1}(x),T^{m-1}(x)\big)\geq\frac{\varepsilon}{8}$, then

$$\begin{aligned}
d\big(T^n(x),T^m(x)\big) &\leq \gamma_1(\frac{\varepsilon}{8})d\big(T^n(x),T^{n-1}(x)\big)+\gamma_2(\frac{\varepsilon}{8})d\big(T^m(x),T^{m-1}(x)\big)\\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \qquad (2.14)
\end{aligned}$$

Let $d\big(T^{n-1}(x),T^{m-1}(x)\big)<\frac{\varepsilon}{8}$, then

$$\begin{aligned}
d\big(T^n(x),T^m(x)\big) &\leq 2d\big(T^n(x),T^{n-1}(x)\big)+4d\big(T^{n-1}(x),T^{m-1}(x)\big)+2d\big(T^m(x),T^{m-1}(x)\big)\\
&< 2d\big(T^n(x),T^{n-1}(x)\big)+4\frac{\varepsilon}{8}+2d\big(T^m(x),T^{m-1}(x)\big) \qquad (2.15)\\
&< \frac{\varepsilon}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon}{4}=\varepsilon.
\end{aligned}$$

It follows from (2.14) and (2.15) that $\{T^n(x)\}$ is a Cauchy sequence in (X,d). Hence $x^*=\lim_{n\longrightarrow\infty}T^n(x)$ exists. So that for some $n_1\in\mathbb{N}$, $d\big(T^{n-1}(x),x^*\big)<t_0$ provided

that $n \geq n_1$. If $T^{n-1}(x) = x^*$ for infinity many n, then $T^n(x^*) = Tx^*$ for infinity many n. It follows that $T(x^*) = x^*$. Therefore, we can assume that $T^{n-1}(x) \neq x^*$ for all $n \in \mathbb{N}$. Let $t_n = d\big(T^{n-1}(x), x^*\big)$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} d\big(x^*, T(x^*)\big) &\leq 2d\big(x^*, T^n(x)\big) + 2d\big(T^n(x), T(x^*)\big) \\ &\leq 2d\big(x^*, T^n(x)\big) + 2\alpha(t_n)[d\big(x^*, T(x^*)\big) + d\big(T^n(x), T^{n-1}(x)\big)] \\ &\quad +2\beta(t_n)[2d\big(T^{n-1}(x), x^*\big) + 2d\big(x^*, T(x^*)\big) + d\big(x^*, T^n(x)\big)] \\ &\quad +2f(t_n)d\big(T^{n-1}(x), x^*\big) \quad (n \in \mathbb{N}). \end{aligned}$$

For each $n > n_1$, we have

$$2\alpha(t_n) + 4\beta(t_n) = a(t_n) + b(t_n) + 2(c(t_n) + e(t_n)) < 1 - f(t_n) \leq 1 - f(t_0).$$

Therefore

$$\begin{aligned} d\big(x^*, T(x^*)\big) &\leq \big(2\alpha(t_n) + 4\beta(t_n)\big)d\big(x^*, T(x^*)\big) + 4d\big(x^*, T^n(x)\big) \\ &\quad +2d\big(T^n(x), T^{n-1}(x)\big) + 6d\big(T^{n-1}(x), x^*\big) \\ &\leq \big[1 - f(t_0)\big]d\big(x^*, T(x^*)\big) + 4d\big(x^*, T^n(x)\big) \\ &\quad +2d\big(T^n(x), T^{n-1}(x)\big) + 6d\big(T^{n-1}(x), x^*\big). \end{aligned}$$

By taking limit as $n \longrightarrow \infty$, we see that

$$d\big(x^*, T(x^*)\big) \leq \big[1 - f(t_0)\big]d\big(x^*, T(x^*)\big).$$

Since $1 - f(t_0) < 1$, we have $T(x^*) = x^*$. □

The following result is a special case of Theorem 2.7, which is also a generalization of Theorem 2.6.

COROLLARY 2.8. *Let $\alpha, \gamma : (0, \infty) \longrightarrow [0, 1)$ be two decreasing function with $2\alpha(t) + \gamma(t) < 1$ and $\gamma(t) < \frac{1}{4}$ for all $t > 0$. Let (X, d) be a complete b-metric space and let $T : X \longrightarrow X$ satisfy*

$$d\big(T(x), T(y)\big) \leq \alpha\big(d(x, y)\big)\big[d\big(x, T(x)\big) + d\big(y, T(y)\big)\big] + \gamma(d(x, y))d(x, y) \quad (x \neq y; x, y \in X),$$

where γ is non-zero function. Then T has a unique fixed point x^* and $\lim_{n\longrightarrow\infty} d\big(T^n(x), x^*\big) = 0$ for each $x \in X$.

Proof. Put $\alpha(t) = a(t) = b(t)$, $c(t) = e(t) = 0$ and $\gamma(t) = f(t)$ for all $t > 0$ in Theorem 2.7. □

The following example shows that the above result is a genuine extension of Theorem 2.6.

EXAMPLE 2.9. Let $X = \{t_1, t_2, t_3\}$ and $d : X \times X \longrightarrow [0, \infty)$ be a symmetric function with $d(x, x) = 0$ for all $x \in X$,

$$d(t_1, t_2) = \frac{7}{4}, \; d(t_2, t_3) = 1 \text{ and } d(t_3, t_1) = 3.$$

It is easy to see that (X, d) is a complete b-metric space. Since $d(t_1, t_3) \nleq d(t_1, t_2) + d(t_2, t_3)$, d is not a metric. Define $T : X \longrightarrow X$ by

$$T(t_1) = t_2, \; T(t_2) = t_3, \; T(t_3) = t_3.$$

Let $\alpha = \frac{3}{10}$ and $\gamma = \frac{2}{10}$. Then $2\alpha + \gamma < 1$,

$$d(t_1, T(t_1)) = d(t_1, t_2) = \frac{7}{4}, \; d(t_2, T(t_2)) = d(t_2, t_3) = 1 \text{ and } d(t_3, T(t_3)) = 0.$$

Therefore

$$d(T(x),T(y)) \leq \quad \alpha\left[d(x,T(x))+d(y,T(y))\right]+\gamma d(x,y), \quad (x \neq y; x,y \in X).$$

It follows from Corollary 2.8 that T has a unique fixed point. However, Theorem 2.6 can't be used, since

$$d(T(t_1),T(t_3)) > \frac{1}{2}\left[d(t_1,T(t_1))+d(t_3,T(t_3))\right].$$

Conclusion. In this article, we establish some generalizations of Hardy-Rogers's theorem in complete b-metric space. The above consequences improve some known fixed point theorems in complete b-metric space and enable us to obtain new outcome . By our results, we can also conclude the existence of unique fixed point in some spaces which are not metric

Acknowledgements

The authors would like to thank the anonymous referee for helpful comments.

References

1. M. Abbas, H. Aydi and S. Radenovic, *Fixed point of T-Hardy-Rogers contractive mappings in partially ordered partial metric spaces*, Internat. J. Math., Article ID 313675 (2012), 11 pages.
2. H. Aydi, M. Bota, E. Karapinar and S. Mitrovic, *A fixed Point for set-valued quasi-contractions in b-metric spaces*, Fixed point theory Appl. (2012), **2012**:88.
3. S. Banach, *Sur les opérations dans les ensemble abstraits et leur application aux équations intégrales,* Fund. Mat. **3** (1922), 133–181.
4. M. Boriceanu, *Strict fixed point theorems for multivalued operators in b-metric spaces*, Int. J. Modern Math. **4**(2), (2009), 285Â–-301.
5. M. Bota, A. Molnar and C. Varga, *On EkelandÂ's variational principle in b-metric spaces*, Fixed Point Theory. **12**(2), (2011), 21-Â–28.
6. J. Caristi, *Fixed point theorems for mppings satisfying inwardness condition,* Trans. Amer. Math. Soc. **215** (1976), 241–251.
7. S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. Inform. Univ. Ostraviensis **1**, (1995), 5–11.
8. M. Edelstein, *An extension of Banach contraction principle*, Proc. Amer. Math. Soc. 12(1961), 7–10.
9. A. H. Frink, *Generalization of the G_δ-property of complete metric spaces,* Bull. Amer. Math. Soc. **43**, (1937), 133–142.
10. A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, (2003).
11. G. Gruenhage, *Generalized Metric Spaces*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, (1984), 423–501.
12. G. Hardy and T. Rogers, *A generalization of fixed point theorem of Reich,* Canad. Math. Bull. **16**(2) (1973), 201–206.
13. R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc. **60**(1968), 71–76.
14. W. A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory,* , Kluwer Academic, London (2001).
15. H. K. Nashine, Z. Kadelburg, S. N. Radenović and J. K. Kim, *Fixed point theorems under Hardy-Rogers contractive conditions on* 0*-complete ordered partial metric spaces*, Fixed Point Theory Appl. (2012), **2012**:180.
16. S. Reich, *Kannan's fixed point theorem,* Bull. Univ. Italiana, **4**(4) (1971), 1–11.
17. S. Reich, *Some remarks concerning contraction mappings,* Canad. Math. Bull. Vol. **14** (1), (1971), 121–124.
18. B. E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc. **236**, (1997), 257–290.
19. B. E. Rhoades, S. Sessa, M. S. Khan and M. D. Khan, *Some fixed point theorems for Hardy-Rogers mappings,* Internat. J. Math. **7**(1), (1984), 75–87.

20. A. Meir and E. Keeler, *A theorem on contraction mappings,* J. Math. Anal. Appl. **28** (1969), 326–329.
21. T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness,* Proc. Amer. Math. Soc. **136**(5), (2008), 1861–1869.
22. H. Yingtaweesittikul, *Suzuki type fixed point theorems for generalized muli-valued mappins in b-metric spaces,* Fixed Point Theory Appl. (2013), **2013**:215.
23. T. Zamfirescu, *Fixed point theorems in metric spaces,* Arch. Math. **23**(1), (1972), 292–298.