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ON COUPLED FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN THE INTERMEDIATE SENSE

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ABSTRACT. The study of coupled fixed points of nonlinear operators, which was introduced about three decades ago, got a boost in 2006 when Bhaskar and Lakshmikantham (2006) studied the coupled fixed points of some contractive maps in partially ordered metric spaces and applied it to solve some first order ordinary differential equations with periodic boundary problems. Since then, coupled fixed points theorems have been proved by several authors for certain contractive maps in both partially ordered and cone metric spaces. The study of coupled fixed point, previously limited to quasi-contractive maps, was recently extended to asymptotically nonexpansive mappings in uniformly convex Banach spaces by Olaoluwa, Olaleru and Chang (2013). In this paper, their results (demiclosed principle and existence result) are extended to asymptotically nonexpansive maps in the intermediate sense in a wider class of spaces. The study naturally opens up new areas of research on the study of coupled fixed points of different classes of pseudocontractive maps.

KEYWORDS : Coupled fixed point; Asymptotically nonexpansive; Uniformly convex Banach spaces.
AMS Subject Classification: 47H10, 47H09

1. INTRODUCTION

The notion of coupled fixed point was introduced by Guo and Lakshmikantham [13] in 1987. Of recent, Gnana-Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone property for contractive operators of the form $F : X \times X \longrightarrow X$ satisfying

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)],\ k<1,$$

where (X,d) is a partially ordered metric space. Their results encompassed some coupled fixed point theorems and their applications to proving the existence and

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uniqueness of the solution for a periodic boundary value problem. Ever since, many authors have established many results on coupled fixed points of quasi-contractive maps in different contexts and spaces (see e.g. [9], [19], [1], [16]).
Results on fixed points of nonexpansive mappings and pseudocontractive mappings abound in literature. The mean ergodic theorem for contractions in uniformly convex Banach spaces was proved in [3] while the authors in [4] introduced the convex approximation property of a space, proved that contractions satisfy an inequality analogue to the Zarantonello inequality (see [22]) and then studied the asymptotic behavior of contractions.
Given a nonempty subset K of a real linear normed space X, a self-mapping $T : K \longrightarrow K$ is said to be nonexpansive if the inequality $\|Tx-Ty\| \leq \|x-y\|$ holds for all $x, y \in K$. Many more general classes of mappings have been considered, including the class of asymptotically nonexpansive mappings introduced by Goebel and Kirk [12], defined by the relation $\|T^n x - T^n y\| \leq k_n\|x-y\| \ \forall n \geq 1 \ \forall x, y \in K$, where the sequence $\{k_n\} \subset [1, \infty)$ converges to 1 as $n \longrightarrow \infty$. Bruck, Kuczumow and Reich [5] introduced the definition of an asymptotically nonexpansive mapping in the intermediate sense (which is more general than an asymptotically nonexpansive map) as a continuous mapping $T : K \longrightarrow K$ such that

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in K} (\|T^n x - T^n y\| - \|x-y\|) \leq 0 \tag{1.1}$$

for any bounded subset $K \in C$. It has been proved by Kirk [15] that asymptotically nonexpansive mappings in the intermediate sense in a nonempty closed convex bounded subset of a space with characteristic of convexity $\epsilon_0(X)$ less than one, have a fixed point.

Recall that the modulus of convexity of X is the function $\delta : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x, y \in X, \ \|x\|, \|y\| \leq 1, \ \|x-y\| \geq \epsilon\right\},$$

and the number $\epsilon_0(X) = \sup\{\epsilon : \delta(\epsilon) = 0\}$ is called the characteristic of convexity of X [11]. Spaces with characteristic of convexity less than one ($\epsilon_0(X) < 1$) are known to be uniformly non-square (see [11]) hence reflexive [14]. Also if X is uniformly convex [8] if $\delta(\epsilon) > 0$ whenever $\epsilon > 0$; hence $\epsilon_0(X) = 0$. Thus spaces with characteristic of convexity less than one, are a super-class of uniformly convex spaces.

Yang et al. [21] proved the demiclosedness principle for the same class of asymptotically nonexpansive mappings in the intermediate sense using Lemma 2.2 given in [16].

Recently, Olaoluwa et al. [18] extended –for the first time– the theory of coupled fixed points to pseudo-contractive-type mappings defined on a product space (algebraic product) by defining asymptotically nonexpansive maps in the context, and studying their asymptotic behaviour, the demiclosedness property and the conditions of existence of their coupled fixed points. Our interest and main purpose is to extend their results to asymptotically nonexpansive mappings in the intermediate sense defined in a product space.

We now recall the definitions, in product spaces, of nonexpansive maps and asymptotically nonexpansive maps as introduced by Olaoluwa et al. [18] and introduce in the same context, asymptotically nonexpansive mappings in the intermediate sense.

Let K be a nonempty bounded subet of a real normed linear space X.

Definition 1.1. [18] A mapping $T : K \times K \longrightarrow K$ is said to be nonexpansive if

$$\|T(x, y) - T(u, v)\| \leq \frac{1}{2}[\|x - u\| + \|y - v\|] \ \forall x, y, u, v \in X. \tag{1.2}$$

Definition 1.2. [18] A mapping $T : K \times K \longrightarrow K$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\longrightarrow\infty} k_n = 1$ such that

$$\|T^n(x, y) - T^n(u, v)\| \leq \frac{k_n}{2}[\|x - u\| + \|y - v\|] \ \forall n \geq 1 \ \forall x, y, u, v \in X, \tag{1.3}$$

where the sequence $\{T^n\}$ is defined as follows:

$$\begin{cases} T^0(x, y) = x \\ T^{n+1}(x, y) = T(T^n(x, y), T^n(y, x)) \ n \geq 0. \end{cases} \tag{1.4}$$

The following definition is introduced as an extension of asymptotically nonexpansive mappings in the intermediate sense in product spaces:

Definition 1.3. $T : K \times K \longrightarrow K$ is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\longrightarrow\infty} \sup_{x,y\in K} (\|T^n(x, y) - T^n(u, v)\| - \|x - u\| - \|y - v\|) \leq 0 \tag{1.5}$$

Remark 1.4. The sequence $\{T^n(x, y)\}$ can be written as the sequence $\{x_n\}$ defined (see [1]) as follows:

$$\begin{cases} x_0 = x; \ y_0 = y \\ x_{n+1} = T(x_n, y_n), \ n \geq 0 \\ y_{n+1} = T(y_n, x_n), \ n \geq 0. \end{cases} \tag{1.6}$$

2. Demiclosedness principle

In [7], Chang et al. recalled the definition the definition of demi-closed maps at the origin as follows:

Definition 2.1. [7] Let X be a real Banach space and K be a closed subset of X. A mapping $T : K \longrightarrow K$ is said to be demi-closed at the origin if, for any sequence $\{x_n\}$ in K, the conditions $x_n \longrightarrow q$ weakly and $Tx_n \longrightarrow 0$ strongly, imply $Tq = 0$.

The definition of demi-closed mappings in product spaces can be proposed from the previous definition as follows:

Definition 2.2. [18] Let X be a real Banach space and K be a closed subset of K. A mapping $T : K \times K \longrightarrow K$ is said to be demi-closed at the origin if, for any sequence $\{(x_n, y_n)\}$ in $K \times K$, the conditions $x_n \longrightarrow q_1$, $y_n \longrightarrow q_2$ weakly and $F(x_n, y_n) \longrightarrow 0$, $F(y_n, x_n) \longrightarrow 0$ strongly imply $F(q_1, q_2) = F(q_2, q_1) = 0$.

In order to establish the demiclosedness principle for asymptotically nonexpansive mappings in the intermediate sense defined in a product space, it is important to estimate the difference between $T^k\left(\sum_{i=1}^n \lambda_i(x_i, y_i)\right)$ and $\sum_{i=1}^n \lambda_i T^k(x_i, y_i)$ for $\lambda \in \Delta^{n-1}$, $(x_1, y_1), \ldots, (x_n, y_n) \in K \times K$ and $k \geq 1$.
Here $\Delta^{n-1} = \{\lambda = (\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0 \text{ and } \sum_{i=1}^n \lambda_i = 1\}$. The following lemmas are useful.

Lemma 2.3. Let E be a uniformly convex Banach space and K be a nonempty closed bounded convex subset of E. For $\epsilon > 0$, there exists an integer $N_\epsilon \geq 1$ and $\delta_{2,\epsilon} > 0$ such that if $k \geq N_\epsilon$, $x_1, x_2, y_1, y_2 \in K$ and

$$\|x_1 - x_2\| + \|y_1 - y_2\| - 2\|T^k(x_1, y_1) - T^k(x_2, y_2)\| \leq \delta_{2,\epsilon},$$

then

$$\|T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) - \lambda_1 T^k(x_1, y_1) - \lambda_2 T^k(x_2, y_2)\| < \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$.

Proof. Let δ be the modulus of uniform convexity of X and define $d : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by

$$d(t) = \begin{cases} \frac{1}{2}\int_0^t \delta(s)ds, & 0 \leq t \leq 2 \\ d(2) + \frac{1}{2}\delta(2)(t-2), & t > 2. \end{cases}$$

It is well known (e.g. see [3],[16]) that d is strictly increasing, continuous, convex, satisfying $d(0) = 0$ and

$$2\lambda_1\lambda_2 d(\|u - v\|) \leq 1 - \|\lambda_1 u + \lambda_2 v\| \tag{2.1}$$

for all $\lambda = (\lambda_1, \lambda_2) \in \Delta^1$, and $u, v \in X$ such that $\|u\| \leq 1$ and $\|v\| \leq 1$.
For $\epsilon > 0$, choose $\eta_\epsilon > 0$ and $\frac{D_K}{2} d^{-1}\left(\frac{\eta_\epsilon}{D_K}\right) < \epsilon$ and put $\delta_{2,\epsilon} = \min\{\eta_\epsilon, D_K\}$. By (1.5), there exists an integer $N_\epsilon \geq 1$ (depending on K) such that if $k \geq N_\epsilon$,

$$2\|T^k(x, y) - T^k(u, v)\| - \|x - u\| - \|y - v\| < \delta_{2,\epsilon} \text{ for all } x, y, u, v \in K.$$

Let $k \geq N_\epsilon$ and let $(x_1, y_1), (x_2, y_2) \in K \times K$ with

$$\|x_1 - x_2\| + \|y_1 - y_2\| - 2\|T^k(x_1, y_1) - T^k(x_2, y_2)\| \leq \delta_{2,\epsilon}.$$

It suffices to show Lemma 2.3 in the case of $0 < \lambda_1, \lambda_2 < 1$. Put

$$u = 2\left[\frac{T^k(x_2, y_2) - T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2))}{\lambda_1(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon})}\right]$$

and

$$v = 2\left[\frac{T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2)) - T^k(x_1, y_1)}{\lambda_2(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon})}\right].$$

We have $\|u\| \leq 1$, $\|v\| \leq 1$ and

$$\lambda_1 u + \lambda_2 v = 2\left[\frac{T^k(x_2, y_2) - T^k(x_1, y_1)}{\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon}}\right]. \tag{2.2}$$

Since $u - v = 2\left[\frac{\lambda_1 T^k(x_1, y_1) + \lambda_2 T^k(x_2, y_2) - T^k(\lambda_1(x_1, y_1) + \lambda_2(x_2, y_2))}{\lambda_1\lambda_2(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon})}\right]$ and $\frac{1}{D_K}\lambda_1\lambda_2(\|x_1 - x_2\| + \|y_1 - y_2\| + \delta_{2,\epsilon}) \leq \frac{1}{D_K} \cdot \frac{1}{4}(2D_K + D_K) < 1$, we have by (2.1) and (2.2) that

$$\begin{aligned} & d\left(\frac{2}{D_K}\left\|\lambda_1 T^k(x_1,y_1)+\lambda_2 T^k(x_2,y_2)-T^k(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))\right\|\right) \\ & \quad \le \frac{1}{D_K}\lambda_1\lambda_2\left(\|x_1-x_2\|+\|y_1-y_2\|+\delta_{2,\epsilon}\right) d(\|u-v\|) \\ & \quad \le \tfrac{1}{D_K}\lambda_1\lambda_2(\|x_1-x_2\|+\|y_1-y_2\|+\delta_{2,\epsilon})\cdot \tfrac{1}{2\lambda_1\lambda_2}\left\{1-2\tfrac{\|T^k(x_2,y_2)-T^k(x_1,y_1)\|}{\|x_1-x_2\|+\|y_1-y_2\|+\delta_{2,\epsilon}}\right\} \\ & \quad = \frac{1}{2D_K}(\|x_1-x_2\|+\|y_1-y_2\|-2\|T^k(x_2,y_2)-T^k(x_1,y_1)\|+\delta_{2,\epsilon}) \\ & \quad \le \frac{2\delta_{2,\epsilon}}{2D_K}=\frac{\delta_{2,\epsilon}}{D_K}\le\frac{\eta_\epsilon}{D_K}. \end{aligned}$$

Here we have used the fact that $t\mapsto \frac{d(t)}{t}$ is strictly increasing; $t_1\le t_2 \implies \frac{d(t_1)}{t_1}\le \frac{d(t_2)}{t_2}$, with $t_1=\frac{2}{D_K}\|\lambda_1T^k(x_1,y_1)+\lambda_2T^k(x_2,y_2)-T^k(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))\|$ and $t_2=\|u-v\|$.
Consequently, from the choice of η_ϵ, we obtain

$$\|T^k(\lambda_1(x_1,y_1)+\lambda_2(x_2,y_2))-\lambda_1T^k(x_1,y_1)-\lambda_2T^k(x_2,y_2)\|\le\frac{D_K}{2}d^{-1}\left(\tfrac{\eta_\epsilon}{D_K}\right)<\epsilon. \quad \square$$

Lemma 2.4. Let E be a uniformly convex Banach space and K be a nonempty closed bounded convex subset of E. For each $\epsilon>0$ and each integer $n\ge 2$, there exists an integer $N_\epsilon\ge 1$ and $\delta_{n,\epsilon}>0$ (where N_ϵ is independent of n) such that if $k\ge N_\epsilon$, $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)\in K\times K$ and if

$$\|x_i-x_j\|+\|y_i-y_j\|-2\|T^k(x_i,y_i)-T^k(x_j,y_j)\|<\delta_{n,\epsilon}$$

for $1\le i,j\le n$, then

$$\left\|T^k\left(\sum_{i=1}^n\lambda_i(x_i,y_i)\right)-\sum_{i=1}^n\lambda_iT^k(x_i,y_i)\right\|<\epsilon$$

for all $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\Delta^{n-1}$.

Proof. Let $\epsilon>0$ and let $n\ge2$ be an arbitrary integer. Choose an integer $N_\epsilon\ge1$ in Lemma 2.3. We shall construct $\delta_{n,\epsilon}$ $(n=2,3,\dots)$ inductively. Let $\delta_{2,\epsilon}$ be as in Lemma 2.3. Suppose that all $\delta_{q,\epsilon}$ are constructed for $q=2,3,\dots,p$. Let $\epsilon'=\min\{\frac{1}{10}\delta_{p,\frac{\epsilon}{2}},\frac{\epsilon}{2}\}$ and put $\delta_{p+1,\epsilon}=\min\{\delta_{2,\epsilon'},\epsilon'\}$.
Let $\lambda\in\Delta^p$, $(x_1,y_1),\dots,(x_{p+1},y_{p+1})\in K\times K$, $k\ge N_\epsilon$ and

$$\|x_i-x_j\|+\|y_i-y_j\|-2\|T^k(x_i,y_i)-T^k(x_j,y_j)\|<\delta_{p+1,\epsilon}$$

for $1\le i,j\le p+1$.
The case $\lambda_{p+1}=1$ is trivial and so we assume $\lambda_{p+1}\ne1$. Put for $j=1,2,\dots,p$ and $i=1,2,\dots,p+1$,

$$\begin{pmatrix}u_j\\v_j\end{pmatrix}=(1-\lambda_{p+1})\begin{pmatrix}x_j\\y_j\end{pmatrix}+\lambda_{p+1}\begin{pmatrix}x_{p+1}\\y_{p+1}\end{pmatrix};\ \mu_j=\frac{\lambda_j}{1-\lambda_{p+1}},$$

$$\begin{pmatrix}u'_j\\v'_j\end{pmatrix}=(1-\lambda_{p+1})\begin{pmatrix}x'_j\\y'_j\end{pmatrix}+\lambda_{p+1}\begin{pmatrix}x'_{p+1}\\y'_{p+1}\end{pmatrix},\ \text{with}\ \begin{pmatrix}x'_i\\y'_i\end{pmatrix}=\begin{pmatrix}T(x_i,y_i)\\T(y_i,x_i)\end{pmatrix}.$$

We have:

$$\sum_{i=1}^{p+1}\lambda_i\begin{pmatrix}x_i\\y_i\end{pmatrix}=\sum_{j=1}^{p}\mu_j\begin{pmatrix}u_j\\v_j\end{pmatrix};\ \sum_{i=1}^{p+1}\lambda_i\begin{pmatrix}x'_i\\y'_i\end{pmatrix}=\sum_{j=1}^{p}\mu_j\begin{pmatrix}u'_j\\v'_j\end{pmatrix}.$$

Therefore:

$$\left\|T^k\left(\sum_{i=1}^{p+1}\lambda_i(x_i,y_i)\right)-\sum_{i=1}^{p+1}\lambda_i T^k(x_i,y_i)\right\|$$

$$=\left\|T^k\left(\sum_{j=1}^{p}\mu_j(u_j,v_j)\right)-\sum_{j=1}^{p}u'_j\right\|$$

$$\leq\left\|T^k\left(\sum_{j=1}^{p}\mu_j(u_j,v_j)\right)-\sum_{j=1}^{p}\mu_j T^k(u_j,v_j)\right\|+\sum_{j=1}^{p}\mu_j\|T^k(u_j,v_j)-u'_j\|. \tag{2.3}$$

Since $\|x_j-x_{p+1}\|+\|y_j-y_{p+1}\|-2\|T^k(x_j,y_j)-T^k(x_{p+1},y_{p+1})\|<\delta_{p+1,\epsilon}\leq\delta_{2.\epsilon'}$, we have by Lemma 2.3:

$$\begin{aligned}\|u'_j-T^k(u_j,v_j)\| &= \|(1-\lambda_{p+1})T^k(u_j,v_j)+\lambda_{p+1}T^k(u_j,v_j)\\ &\quad -T^k((1-\lambda_{p+1})(x_j,y_j)+\lambda_{p+1}(x_{p+1},y_{p+1}))\|\\ &< \epsilon',\end{aligned}$$

$$\begin{aligned}\|v'_j-T^k(v_j,u_j)\| &= \|(1-\lambda_{p+1})T^k(v_j,u_j)+\lambda_{p+1}T^k(v_j,u_j)\\ &\quad -T^k((1-\lambda_{p+1})(y_j,x_j)+\lambda_{p+1}(y_{p+1},x_{p+1}))\|\\ &< \epsilon',\end{aligned}$$

$$\|u_j-u_l\|+\|v_j-v_l\|-\|u'_j-u'_l\|-\|v'_j-v'_l\|$$

$$=(1-\lambda_{p+1})\left\{\|x_j-x_l\|+\|y_j-y_l\|-\|T^k(x_j,y_j)-T^k(x_l,y_l)\|\right.$$

$$\left.-\|T^k(y_j,x_j)-T^k(y_l,x_l)\|\right\}$$

$$=(1-\lambda_{p+1})\left\{\frac{\|x_j-x_l\|+\|y_j-y_l\|-2\|T^k(x_j,y_j)-T^k(x_l,y_l)\|}{2}\right.$$

$$\left.+\frac{\|y_j-y_l\|+\|x_j-x_l\|-2\|T^k(y_j,x_j)-T^k(y_l,x_l)\|}{2}\right\}$$

$$\leq(1-\lambda_{p+1})\left\{\frac{\delta_{p+1,\epsilon}}{2}+\frac{\delta_{p+1,\epsilon}}{2}\right\}=(1-\lambda_{p+1})\delta_{p+1,\epsilon}\leq\epsilon'$$

for all $1\leq j,l\leq p$.
Therefore we obtain by the triangle inequality:

$$\|u_j-u_l\|+\|v_j-v_l\|-\|T^k(u_j,v_j)-T^k(u_l,v_l)\|-\|T^k(v_j,u_j)-T^k(v_l,u_l)\|$$

$$\leq\ \|u_j-u_l\|+\|v_j-v_l\|-\|u'_j-u'_l\|-\|v'_j-v'_l\|$$

$$+\|u'_j-T^k(u_j,v_j)\|+\|u'_l-T^k(u_l,v_l)\|$$

$$+\|v'_j-T^k(v_j,u_j)\|+\|v'_l-T^k(v_l,u_l)\|$$

$$\leq\ 5\epsilon'\leq\tfrac{1}{2}\delta_{p,\frac{\epsilon}{2}}$$

for $1 \le j, l \le p$. Since

$$\|u_j - u_l\| + \|v_j - v_l\| - \|T^k(u_j, v_j) - T^k(u_l, v_l)\| - \|T^k(v_j, u_j) - T^k(v_l, u_l)\|$$

$$= \tfrac{1}{2}\left\{\|u_j - u_l\| + \|v_j - v_l\| - 2\|T^k(u_j, v_j) - T^k(u_l, v_l)\|\right\}$$

$$+\tfrac{1}{2}\left\{\|v_j - v_l\| + \|u_j - u_l\| - 2\|T^k(v_j, u_j) - T^k(v_l, u_l)\|\right\}$$

$$\le \tfrac{1}{2}\delta_{p,\frac{\epsilon}{2}},$$

then $\|u_j - u_l\| + \|v_j - v_l\| - 2\|T^k(u_j, v_j) - T^k(u_l, v_l)\| \le \delta_{p,\frac{\epsilon}{2}}$. Thus by inductive assumption and (2.3), the desired conclusion holds. □

The following Lemma shows that the positive number $\delta_{n,\epsilon}$ in Lemma 2.4 can be chosen independently of n.

Lemma 2.5. Let E be a uniformly convex Banach space, K be a nonempty closed bounded convex subset of E. For every $\epsilon > 0$ and every integer $n \ge 2$, there exist an integer $N_\epsilon \ge 1$ and $\delta_\epsilon > 0$ (where both N_ϵ and δ_ϵ are independent of n) such that if $k \ge N_\epsilon$, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in K \times K$ and if

$$\|x_i - x_j\| + \|y_i - y_j\| - 2\|T^k(x_i, y_i) - T^k(x_j, y_j)\| \le \delta_\epsilon$$

for $1 \le i, j \le n$, then

$$\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i, y_i)\right) - \sum_{i=1}^{n}T^k(x_i, y_i)\right\| < \epsilon$$

for all $\lambda \in \Delta^{n-1}$.

Proof. Fix $\epsilon > 0$ and an integer $n \ge 2$ arbitrarily. Denote by $N_{1,\epsilon}$ the integer $N_{\epsilon/4}$ in Lemma 2.4. By (1.5) there is an integer $N_{2,\epsilon} \ge 1$ such that if $k \ge N_{2,\epsilon}$, then we have

$$2\|T^k(x, y) - T^k(u, v)\| - \|x - u\| - \|y - v\| < \frac{\epsilon}{4} \text{ for all } x, y, u, v \in K \tag{2.4}$$

Put $N_\epsilon = \max\{N_{1,\epsilon}, N_{2,\epsilon}\}$. Let $\delta_{n,\epsilon}$ $(n = 2, 3, \ldots)$ be positive numbers determined in Lemma 2.4. Since X is uniformly convex, X is B-convex (see [4]) and since the product of B-convex spaces is also B-convex (see [10]), X^3 is B-convex, hence has the convex approximation property (C.A.P.) (see [4]) so we can choose an integer $p = p(\epsilon) \ge 1$ (independent of n) such that $coM \subset co_pM + B_{\epsilon/4} \times B_{\epsilon/4} \times B_{\epsilon/4}$ for all subsets $M \subset X^3$ whose diameters are uniformly bounded, where B_r is the open sphere centered at the origin and with r as radius, coM is the convex hall of M and

$$co_pM = \left\{\sum_{i=1}^{p} t_iX_i;\ t \in \Delta^{p-1}; X_i \in M \text{ for all } i \in \{1, \ldots, p\},\ p \text{ fixed}\right\}.$$

Put $\delta_\epsilon = \delta_{p,\frac{\epsilon}{4}}$. Let $k \ge N_\epsilon$, $(x_1, y_1), \ldots, (x_n, y_n) \in K \times K$ and

$$\|x_i - x_j\| + \|y_i - y_j\| - 2\|T^k(x_i, y_i) - T^k(x_l, y_l)\| \le \delta_\epsilon\ (1 \le i, j \le n).$$

Consider $M = \{[x_i, y_i, T^k(x_i, y_i)] \in X^3 : i = 1, 2, \cdots, n\}$. Note that there exists $r > 0$ (independent from k and n) such that $\sup_{(x,y,z)\in M}\|(x, y, z)\|_{X^3} \le r$.
Then for each $\lambda \in \Delta^{n-1}$, there exist $\mu \in \Delta^{p-1}$ and $i_1, \cdots, i_p \in \{1, \cdots, n\}$ such that

$$\left\|\sum_{i=1}^{n}\lambda_i x_i - \sum_{j=1}^{p}\mu_j x_{i_j}\right\| < \frac{\epsilon}{4},\ \left\|\sum_{i=1}^{n}\lambda_i y_i - \sum_{j=1}^{p}\mu_j y_{i_j}\right\| < \frac{\epsilon}{4},\ \text{and}$$

$$\left\|\sum_{i=1}^{n}\lambda_i T^k(x_i,y_i)-\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})\right\|<\frac{\epsilon}{4}.$$

By (2.4) and the choice of δ_ϵ we have

$$\begin{aligned}2&\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i,y_i)\right)-T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)\right\|\\&\le\left\|\sum_{i=1}^{n}\lambda_i x_i-\sum_{j=1}^{p}\mu_j x_{i_j}\right\|+\left\|\sum_{i=1}^{n}\lambda_i y_i-\sum_{j=1}^{p}\mu_j y_{i_j}\right\|+\tfrac{\epsilon}{4}\le\tfrac{3\epsilon}{4}<\epsilon\end{aligned}$$

and

$$\left\|T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)-\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})\right\|<\tfrac{\epsilon}{4}.$$

Therefore

$$\begin{aligned}&\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i,y_i)\right)-\sum_{i=1}^{n}T^k(x_i,y_i)\right\|\\&\le\left\|T^k\left(\sum_{i=1}^{n}\lambda_i(x_i,y_i)\right)-T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)\right\|\\&\quad+\left\|T^k\left(\sum_{j=1}^{p}\mu_j(x_{i_j},y_{i_j})\right)-\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})\right\|\\&\quad\left\|\sum_{j=1}^{p}\mu_j T^k(x_{i_j},y_{i_j})-\sum_{j=1}^{n}\lambda_i T^k(x_i,y_i)\right\|\\&<\epsilon.\end{aligned}$$

□

Lemma 2.5 is an extension of Lemma 1.5 of Yang et al. [21] to asymptotically nonexpansive maps in the intermediate sense defined on product spaces. From Lemma 2.5, we can now state the following theorem which is likewise an extension of their Lemma 1.6:

Theorem 2.1. (Demiclosedness Principle): Let X be a real uniformly convex Banach space and K a nonempty bounded closed convex subset of X. Let $T : K\times K \longrightarrow K$ be a mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_n\}$ and $\{y_n\}$ are sequences in K converging weakly to x^* and y^* and if

$$\begin{cases}\lim\limits_{k\longrightarrow\infty}(\limsup\limits_{n}\|x_n-T^k(x_n,y_n)\|)=0\\ \lim\limits_{k\longrightarrow\infty}(\limsup\limits_{n}\|y_n-T^k(y_n,x_n)\|)=0\end{cases}$$

then p_1-T is demiclosed at zero, i.e., for each sequences $\{x_n\},\{y_n\}\in K$, if they converge weakly to $x^*\in K$ and $y^*\in K$ respectively and $\{x_n-T(x_n,y_n)\}$ and $\{y_n-T(y_n,x_n)\}$ converge strongly to 0, then $x^*=T(x^*,y^*)$ and $y^*=T(y^*,x^*)$.

Proof. The sequences $\{x_n\}$ and $\{y_n\}$ are bounded so there exists $r>0$ such that $\{x_n\},\{y_n\}\subset C:=K\cap B_r$, where B_r is the closed ball in X with center 0 and radius r. So C is a nonempty bounded closed convex subset in K. Let us prove that $T^k(x^*,y^*)\longrightarrow x^*$ and $T^k(y^*,x^*)\longrightarrow y^*$.
For $\epsilon>0$, choose an integer $N_1(\epsilon)$ such that if $k\ge N_1(\epsilon)$, then $2\|T^k(x,y)-T^k(u,v)\|-\|x-u\|-\|y-v\|<\frac{\epsilon}{5}$ for $(x,y),(u,v)\in C\times C$ and $\limsup_n\|x_n-T^k(x_n,y_n)\|+\limsup_n\|y_n-T^k(y_n,x_n)\|<\frac{1}{4}\delta_{\frac{\epsilon}{5}}$.
Thus there exists $n_{\epsilon,k}$ such that $\|x_n-T^k(x_n,y_n)\|+\|y_n-T^k(y_n,x_n)\|<\frac{1}{4}\delta_{\epsilon/5}$ for $n\ge n_{\epsilon,k}$.
Set $\epsilon'=\min\{\frac{1}{4}\delta_{\frac{\epsilon}{5}},\frac{\epsilon}{5}\}$. Then we have $N_1(\epsilon')\ge 1$. Let $N_2(\epsilon)=\max\{N_{\frac{\epsilon}{5}},N_1(\epsilon),N_1(\epsilon')\}$ and let $j\ge N_2(\epsilon)$. Since $\{x_n\}$ and $\{y_n\}$ converge weakly to x^* and y^*, by Mazur's theorem, for each positive integer $n\ge 1$, there exist convex combinations $A_n=\sum_{i=1}^{m(n)}\lambda_i^{(n)}x_{i+n}$ and $B_n=\sum_{i=1}^{m(n)}\lambda_i^{(n)}y_{i+n}$ with $\lambda_i^{(n)}\ge 0$ and $\sum_{i=1}^{m(n)}\lambda_i^{(n)}=1$

such that $\|A_n - x^*\| \longrightarrow 0$ and $\|B_n - y^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.
Since

$$\left\{ \begin{array}{l} \|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\| \\ -\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \\ -\|T^k(y_{i+n}, x_{i+n}) - T^k(y_{j+n}, x_{j+n})\| \end{array} \right\} \leq \left\{ \begin{array}{l} \|x_{i+n} - T^k(x_{i+n}, y_{i+n})\| \\ +\|y_{i+n} - T^k(x_{i+n}, y_{i+n})\| \\ +\|x_{j+n} - T^k(x_{j+n}, y_{j+n})\| \\ +\|y_{j+n} - T^k(y_{j+n}.x_{j+n})\| \end{array} \right\} \leq \frac{1}{2}\delta_{\frac{\epsilon}{5}}$$

and

$$\left\{ \begin{array}{l} \|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\| \\ -\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \\ -\|T^k(y_{i+n}, x_{i+n}) - T^k(y_{j+n}, x_{j+n})\| \end{array} \right\} = \left\{ \begin{array}{l} \frac{1}{2}\left[\|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\|\right] \\ -\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \\ +\frac{1}{2}\left[\|y_{i+n} - y_{j+n}\| + \|x_{i+n} - x_{j+n}\|\right] \\ -\|T^k(y_{i+n}, x_{i+n}) - T^k(y_{j+n}, x_{j+n})\| \end{array} \right\} \leq \frac{1}{2}\delta_{\epsilon/5},$$

we therefore have

$$\|x_{i+n} - x_{j+n}\| + \|y_{i+n} - y_{j+n}\| - 2\|T^k(x_{i+n}, y_{i+n}) - T^k(x_{j+n}, y_{j+n})\| \leq \delta_{\epsilon/5}$$

for $1 \leq i, j \leq m(n)$; by Lemma 2.5, we have

$$\left\| T^k(A_n, B_n) - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k(x_{i+n}, y_{i+n}) \right\| < \frac{\epsilon}{5},$$

$$\left\| T^k(B_n, A_n) - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k(y_{i+n}, x_{i+n}) \right\| < \frac{\epsilon}{5}.$$

There is $L_{k,\epsilon} \geq 1$ such that $\|A_n - x^*\| + \|B_n - y^*\| < \frac{\epsilon}{5}$ for all $n \geq L_{k,\epsilon}$.
Since $x^*, y^* \in K$,

$$\begin{array}{rcl} \|T^k(x^*, y^*) - x^*\| & \leq & \|T^k(x^*, y^*) - T^k(A_n, B_n)\| \\ & & + \left\| T^k(A_n, B_n) - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T^k(x_{i+n}, y_{i_n}) \right\| \\ & & + \left\| \sum_{i=1}^{m(n)} \lambda_i^{(n)} (T^k(x_{i+n}, y_{i+n}) - x_{i+n}) \right\| + \|A_n - x^*\| \\ & < & \epsilon \end{array}$$

for $n \geq L_{k,\epsilon}$ and $k \geq N_2(\epsilon)$. Thus $\|T^k(x^*, y^*) - x^*\| < \epsilon$ for $k \geq N_2(\epsilon)$ and so $\|T^k(x^*, y^*) - x^*\| \longrightarrow 0$ as $k \longrightarrow \infty$. Similarly, $\|T^k(y^*, x^*) - y^*\| \longrightarrow 0$ as $k \longrightarrow \infty$. By the continuity of T, we have

$$\left\{ \begin{array}{l} x^* = \lim\limits_{k \longrightarrow \infty} T^{k+1}(x^*, y^*) = \lim\limits_{k \longrightarrow \infty} T(T^k(x^*, y^*), T^k(y^*, x^*)) = T(x^*, y^*) \\ y^* = \lim\limits_{k \longrightarrow \infty} T^{k+1}(y^*, x^*) = \lim\limits_{k \longrightarrow \infty} T(T^k(y^*, x^*), T^k(x^*, y^*)) = T(y^*, x^*) \end{array} \right.$$

This completes the proof. □

Theorem 2.1 extends Theorem 2.1 of Olaoluwa et al. [18] to asymptotically nonexpansive maps in the intermediate sense defined on a product space.

3. Existence of coupled fixed points

The following theorem relative to the existence of coupled fixed points of asymptotically nonexpansive maps in the intermediate sense extends the results of Kirk [15] to product spaces. The spaces considered have a characteristic of convexity less than one. Thus the result remain valid for uniformly convex Banach spaces and consequently generalize Theorem 3.1 of Olaoluwa et al. [18] on existence of coupled fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces.

Theorem 3.1. Let X be a Banach space for which $\epsilon_0 = \epsilon_0(X) < 1$ and let $K \subset X$ be nonempty, bounded, closed and convex. Suppose $T : K \times K \longrightarrow K$ is asymptotically nonexpansive in the intermediate sense. Then T has a fixed point in $K \times K$.

Proof. Let $(x, y) \in K \times K$ be fixed. Define the set $R(x, y)$ as follows:

$$R(x,y) = \left\{ \rho \in \mathbb{R} \ / \ \exists k_\rho \in \mathbb{N} \ : \ (K \times K) \bigcap \left(\bigcap_{i=k_\rho}^{\infty} B(T^i(x,y),\rho) \times B(T^i(y,x),\rho) \right) \neq \emptyset \right\}.$$

where $B(x, r)$ is the open sphere in X, of center x and radius r. K is bounded, so, if $D_K := diam K$ (diameter of K), $D_K \in R(x, y)$, hence $R(x, y) \neq \emptyset$. Let ρ^* be the g.l.b. of $R(x, y)$.
For any $\epsilon > 0$, define the sets $C_\epsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} B\left(T^i(x,y), \rho^* + \epsilon\right)\right)$ and $D_\epsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} B\left(T^i(y,x), \rho^* + \epsilon\right)\right)$. The sets C_ϵ and D_ϵ are nonempty, bounded and convex hence by the reflexivity of X the closures $\bar{C}_\epsilon$ and $\bar{D}_\epsilon$ are weakly compact and $C = \bigcap_{\epsilon>0}(\bar{C}_\epsilon \cap K) \neq \emptyset$ and $D = \bigcap_{\epsilon>0}(\bar{D}_\epsilon \cap K) \neq \emptyset$.
Let $(u, v) \in C \times D$ and let $d(u, v) = \limsup_{i \longrightarrow \infty} \|u - T^i(u, v)\| + \|v - T^i(v, u)\|$.
Suppose $\rho^*(x, y) = 0$. Then $T^n(x, y) \longrightarrow u$ and $T^n(y, x) \longrightarrow v$ as $n \longrightarrow \infty$. Let $\eta > 0$ and using (1.5), choose L such that $i \geq L$ implies

$$\sup_{(u,v),(z,t) \in K \times K} [2\|T^i(u,v) - T^i(z,t)\| - \|u - z\| - \|v - t\|] \leq \frac{1}{3}\eta.$$

Given $i \geq L$, since $T^n(x, y) \longrightarrow u$ and $T^n(y, x) \longrightarrow v$, there exists $l > i$ such that $\|T^l(x, y) - u\| + \|T^l(y, x) - v\| \leq \frac{1}{3}\eta$ and $\|T^{l-i}(x, y) - u\| + \|T^{l-i}(y, x) - v\| \leq \frac{1}{3}\eta$.

Thus if $i \geq L$,

$$\|u - T^i(u,v)\| + \|v - T^i(v,u)\|$$

$$\leq \|u - T^l(x,y)\| + \|T^l(x,y) - T^i(u,v)\| + \|v - T^l(x,y)\| + \|T^l(x,y) - T^i(u,v)\|$$

$$\begin{aligned}\leq\ & \|u - T^l(x,y)\| + \|T^i(u,v) - T^i(T^{l-i}(x,y), T^{l-i}(y,x))\| - \|u - T^{l-i}(x,y)\| \\ & + \|u - T^{l-i}(x,y)\| + \|v - T^l(y,x)\| + \|T^i(v,u) - T^i(T^{l-i}(y,x), T^{l-i}(x,y))\| \\ & - \|v - T^{l-i}(y,x)\| + \|v - T^{l-i}(y,x)\|\end{aligned}$$

$$\begin{aligned}\leq\ & \frac{2}{3}\eta \\ & + \frac{1}{2}\left[2\|T^i(T^{l-i}(x,y), T^{l-i}(y,x)) - T^i(u,v)\| - \|u - T^{l-i}(x,y)\| - \|v - T^{l-i}(y,x)\|\right] \\ & + \frac{1}{2}\left[2\|T^i(T^{l-i}(y,x), T^{l-i}(x,y)) - T^i(v,u)\| - \|u - T^{l-i}(x,y)\| - \|v - T^{l-i}(y,x)\|\right]\end{aligned}$$

$$\leq \frac{2}{3}\eta + \sup\left[2\|T^i(u,v) - T^i(z,t)\| - \|u - z\| - \|v - t\|\right]$$

$$\leq \eta.$$

This proves that $T^n(u,v) \longrightarrow u$ and $T^n(v,u) \longrightarrow v$ as $n \longrightarrow \infty$, that is, $d(u,v) = 0$. But $d(u,v) = 0$ implies $T^{N_i}(u,v) \longrightarrow u$ and $T^{N_i}(v,u) \longrightarrow v$ as $i \longrightarrow \infty$ and with the continuity of T^N this yields $T^N(u,v) = u$ and $T^N(v,u) = v$. Thus, as $i \longrightarrow \infty$,

$$\begin{cases} T(u,v) = T\left(T^{N_i}(u,v), T^{N_i}(v,u)\right) = T^{N_i+1}(u,v) \longrightarrow u \\ \\ T(v,u) = T\left(T^{N_i}(v,u), T^{N_i}(u,v)\right) = T^{N_i+1}(v,u) \longrightarrow v \end{cases}$$

so $T(u,v) = u$ and $T(v,u) = v$.

Now we assume that $\rho^*(x,y) > 0$ and $d(u,v) > 0$. In fact, we may assume this for any $x, y, u, v \in K$.

Let $\epsilon > 0$, $\epsilon \leq d(u,v)$. By definition of ρ^* there exists an integer N^* such that if $i \geq N^*$ then

$$\|u - T^i(x,y)\| + \|v - T^i(y,x)\| \leq \rho^* + \epsilon, \tag{3.1}$$

and by (1.5) there exists N^{**} such that if $i \geq N^{**}$ then

$$\sup\left[2\|T^i(u,v) - T^i(z,t)\| - \|u - z\| - \|v - t\|\right] \leq \epsilon.$$

Select j so that $j \geq N^{**}$ and so that

$$\|u - T^j(u,v) + v - T^j(v,u)\| \geq d(u,v) - \epsilon. \tag{3.2}$$

Thus if $i - j \geq N^{**}$,

$$\|T^j(u,v) - T^i(x,y)\| + \|T^j(v,u) - T^i(y,x)\|$$

$$\begin{aligned}=\ & \|T^j(u,v) - T^j(T^{i-j}(x,y), T^{i-j}(y,x))\| - \|u - T^{i-j}(x,y)\| + \|u - T^{i-j}(x,y)\| \\ & + \|T^j(v,u) - T^j(T^{i-j}(y,x), T^{i-j}(x,y))\| - \|v - T^{i-j}(y,x)\| + \|v - T^{i-j}(y,x)\|\end{aligned}$$

$$\begin{aligned}=\ & \tfrac{1}{2}\left\{2\|T^j(u,v) - T^j(T^{i-j}(x,y), T^{i-j}(y,x))\| - \|u - T^{i-j}(x,y)\| - \|v - T^{i-j}(y,x)\|\right\} \\ & + \tfrac{1}{2}\left\{2\|T^j(v,u) - T^j(T^{i-j}(y,x), T^{i-j}(x,y))\| - \|u - T^{i-j}(x,y)\| - \|v - T^{i-j}(y,x)\|\right\} \\ & + \|u - T^{i-j}(x,y)\| + \|v - T^{i-j}(y,x)\|\end{aligned}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + (\rho^* + \epsilon) = 2\epsilon + \rho^*. \tag{3.3}$$

Letting $m = \dfrac{1}{2}[u + T^j(u,v) + v + T^j(v,u)]$, by (3.3) we have:

$$\|m - T^i(x,y) - T^i(y,x)\| \leq \left(1 - \delta\left(\frac{d(u,v) - \epsilon}{\rho^* + 2\epsilon}\right)\right)(\rho^* + 2\epsilon),\ i \geq N^* + j.$$

By the minimality of ρ^* this implies that

$$\rho^* \leq \left(1 - \delta\left(\frac{d(u,v) - \epsilon}{\rho + 2\epsilon}\right)\right)(\rho^* + 2\epsilon).$$

Letting $\epsilon \longrightarrow 0$, $\rho^* \leq \left(1 - \delta\left(\frac{d(u,v)}{\rho^*}\right)\right)\rho^*$. This implies that $1 - \delta\left(\frac{d(u,v)}{\rho^*}\right) \geq 1$ and hence $\delta\left(\frac{d(u,v)}{\rho^*}\right) = 0$. It follows from the definition of ϵ_0 that $\frac{d(u,v)}{\rho^*} \leq \epsilon_0$. Hence $d(u,v) \leq \epsilon_0\rho^*(x,y)$ and letting $d(x,y) = \limsup_{i\longrightarrow\infty} \|x - T^i(x,y)\| + \|y - T^i(y,x)\|$ we have $\rho_0(x) \leq d(x,y)$ so

$$d(u,v) \leq \epsilon_0 d(x,y) \tag{3.4}$$

Also notice that $\|u - x\| + \|v - y\| \leq d(x,y) + \rho_0(x,y) \leq 2d(x,y)$.
Fix $(x_0,y_0) \in K \times K$ and define the sequence $\{(x_n,y_n)\}$ for all $n \in \mathbb{N}$ by

$$\begin{cases} x_{n+1} = u(x_n,y_n) \\ y_{n+1} = v(y_n,x_n), \end{cases}$$

where $u(x_n,y_n)$ is obtained from x_n and y_n in the same manner as $u(x,y)$ from x and y.
If for any n we have $\rho(x_n,y_n) = 0$ and $\rho(y_n,x_n) = 0$ then, as seen above, $T(x_{n+1},y_{n+1}) = x_{n+1}$ and $T(y_{n+1},x_{n+1}) = y_{n+1}$. Otherwise, by 3.4 we have $\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq 2d(x_n,y_n) \leq 2\epsilon^n d(x_0,y_0)$ and since $\epsilon_0 < 1$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Therefore there exists $(x,y) \in K \times K$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$. Also:

$$\begin{aligned}
&\|x - T^i(x,y)\| + \|y - T^i(y,x)\| \\
&\quad \leq \|x - x_n\| + \|x_n - T^i(x_n,y_n)\| + \|T^i(x_n,y_n) - T^i(x,y)\| \\
&\qquad + \|y - y_n\| + \|y_n - T^i(y_n,x_n)\| + \|T^i(y_n,x_n) - T^i(y,x)\| \\
&\quad \leq \|x - x_n\| + \|y - y_n\| + \|x_n - T^i(x_n,y_n)\| \\
&\qquad + \|y_n - T^i(y_n,x_n)\| + \|x_n - x\| + \|y_n - y\| \\
&\qquad\quad + \tfrac{1}{2}\left[2\|T^i(x_n,y_n) - T^i(x,y)\| - \|x_n - x\| - \|y_n - y\|\right] \\
&\qquad\qquad + \tfrac{1}{2}\left[2\|T^i(y_n,x_n) - T^i(y,x)\| - \|x_n - x\| - \|y_n - y\|\right]
\end{aligned}$$

Thus

$$\begin{aligned}
d(x,y) &= \limsup_{i\longrightarrow\infty} \|x - T^i(x,y)\| + \|y - T^i(y,x)\| \\
&\leq \limsup_{i\longrightarrow\infty} 2[\|x - x_n\| + \|y - y_n\|] \\
&\quad + \limsup_{i\longrightarrow\infty}[\|x_n - T^i(x_n,y_n)\| + \|y_n - T^i(y_n,x_n)\|] \\
&\quad + \limsup_{i\longrightarrow\infty}[\sup_{x,y} 2\|T^i(x,y) - T^i(u,v)\| - \|x - u\| - \|y - v\|] \\
&\leq d(x_n,y_n) + 2[\|x - x_n\| + \|y - y_n\|]
\end{aligned}$$

Since $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $d(x_n,y_n) \longrightarrow 0$ as $n \longrightarrow \infty$, this implies that $d(x,y) = 0$. But as seen before, it implies that $T(x,y) = x$ and $T(y,x) = y$. □

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