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ON THE SOLVABILITY OF GENERALIZED SET-VALUED EQUILIBRIUM PROBLEMS

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ABSTRACT. In this present article, we consider the generalized equilibria accompanied with certain multi-objective multifunctions. This class unifies numbers of problems surfaced in the area of optimization including the well-known mixed equilibrium problems. Some theorems are adopted, under the conventional assumptions of continuity, convexity, and coercivity on the objective functions, guaranteeing that these functions enjoy the existence of such equilibria. The consequences of this generalized class are of course studied and presented as well.

KEYWORDS : Generalized equilibrium, multiobjective optimization, set-valued optimization
AMS Subject Classification:

1. INTRODUCTION

The theory of optimization has always been an important subject as it evolved through its history. Amongst the developments in this area, the formulation of an equilibrium problem is one of the most prominent and promising advance which provides a unification to the classical approaches of variational inequalities, fixed point theory, saddle point theory, and several more optimization problems. Moreover, it has been a wealthy source for solving the problems frequently appeared in economics, management science, engineering, etc. The term equilibrium problem was coined by Blum and Oettli [1]. However, it was first discussed decades before by Fan [2, 3], under the influences of the minimax problems in economics. The study of equilibrium theory has then been a heavily investigated area, and it has been extended in various aspects and directions (see e.g., [4–9]).

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After the blossom of the scalar optimization theory, the vector cases were subsequently introduced. These extensions are admired for their capabilities implanted in multi-objective optimization programming. The initiation of the vector equilibria is motivated by the foundation of the vector variational inequalities introduced by Giannessi [10] in finite-dimensional Euclidean spaces. The enhanced versions where the objective functions take their values in arbitrary topological vector spaces were also inaugurated in [11]. As for the equilibrium problems, the vectorial variants were also conprehensively studied in [12–17].

On the other hand, the combinations between two kinds of optimization problems were introduced and studied. Typical important problems in this area are the mixed variational inequalities, which is notable for their distinguishing applications in engineering (see e.g., [18–26]). Due to the impact and utilities of mixed variational inequalities, several notions of mixed equilibria are then adopted in [27–29]. Although lots of numerical and computational techniques have been invented to approximate the solutions of mixed equilibrium problems (for examples, [30–37]), only a relatively small amount of qualitative results are known.

To enrich the theory of mixed equilibrium problems, we contemplate a class of generalized set-valued equilibrium problems, where various classes of the mixed equilibrium problems are determined to be embedded. This new class examines the situation where the objective functions are vectorial and multivalued in fashion. A qualitative study providing some sufficient conditions which guarantee the solvability of this class, including its significant consequences, is also conducted and employed.

The paper is organized in the following way: In section 2, we give a recollection of some background definitions and properties which are useful in our main results, and also introduce the class of generalized equilibrium problem we are interested to study. In section 3, we state and prove our main results providing the validity conditions for the problems suggested in preceding section. The consequential remarks are afterward given and studied.

2. Preliminaries

Suppose that E is a topological vector space and $Y \subset E$ being nonempty. A problem of finding a point $\widehat{x} \in Y$ with

$$f(\widehat{x}, y) \geq 0, \quad \forall y \in Y,$$

where $f : Y \times Y \to \mathbb{R}$, seems to be a fertile area for mathematicians over the past years. This problem is known today as the equilibrium problem and the point $\widehat{x}$ is referred to as an equilibrium. The very first existence theorems for solutions to this problem were introduced in [1, 3, 38] through the usages of continuity, convexity, monotonicity and compactness.

To realize the vectorial formulation, recall that a nonempty subset $C \subset E$ is said to be a cone if $\lambda C \subset C$ for $\lambda \geq 0$. If $C + C \subset C$ holds, then the cone C is called convex. By C°, we means the interiors of C. There should be no ambiguity to denote the zero element of any involved vector space by θ. A cone C is said to be pointed if $C \cap -C = \{\theta\}$ and is said to be solid if $C^\circ \neq \emptyset$.

With the perception of cones, we can define two partial ordering $\preceq$ and $\ll$ on E by

$$\begin{cases} x \preceq y \iff y - x \in C, \\ x \ll y \iff y - x \in C^\circ. \end{cases}$$

For instance, let $f : Y \times Y \to L$ a function valued in another topological vector space L with a cone $C \subset L$. A point $\widehat{x} \in Y$ is said to be a weak equilibrium (equilibrium) of f if

$$f(\widehat{x}, y) \notin -C^\circ \ (-C \setminus \{\theta\}), \quad \forall y \in Y.$$

Under many circumstances, a single action might bring more than one feasible outcomes at a time. This is where the powerful concept of multifunctions gets in. By the term multifunction, we shall refer to the function $F : A \to 2^B$ with nonempty values (it is to be understood that A, B are any nonempty sets). In this context, we shall write $F : A \rightrightarrows B$ instead. We are now consider the multi-objective multifunctions, which will be used mainly in this work: Suppose now that $F, G : Y \times Y \rightrightarrows L$. The point $\widehat{x} \in Y$ such that

$$F(y, \widehat{x}) - G(\widehat{x}, y) \not\subset C^\circ \ (C \setminus \{\theta\}), \quad \forall y \in Y \tag{GEP}$$

is called a generalized weak equilibrium (generalized equilibrum) for F and G. This class of problem contains many important special cases as one shall see as we proceed further.

On the contrary, the KKM theory has been an astonishing area as it provides a key tool in nonlinear analysis and optimization (see e.g. [7, 15, 39–41]). Recall that a multifunction $T : Y \rightrightarrows L$ is said to be a KKM if

$$\mathrm{co}(A) \subset \bigcup_{y \in A} T(y),$$

for all $A \in \langle Y \rangle$, where $\mathrm{co}(A)$ denotes the convex hull of A and $\langle A \rangle$ denotes the family of all finite subsets of Y. The celebrated lemma of Fan [2] asserts that if a KKM multifunction possesses a compact value at some point $x_0 \in Y$, then $\bigcap_{x \in Y} T(x) \neq \emptyset$.

A replacement for the above compactness of $T(x_0)$ is a favor for many nonlinear analysts. In [42], the following coercivity conditions were introduced:

Definition 2.1 ([42]). Suppose that $T : Y \rightrightarrows L$ a multifunction. A family $\{(C_i, K_i)\}_{i \in I}$ is said to be coercing for T if the following properties are satisfied:

- (C1) for each $i \in I$, $C_i \subset K$ for some compact convex set $K \subset Y$ and $K_i \subset L$ is compact;
- (C2) for each $i, j \in I$, there exists $k \in I$ such that
$$C_i \cup C_j \subset C_k;$$
- (C3) for each $i \in I$, there exists $k \in I$ with
$$\bigcap_{x \in C_k} T(x) \subset K_i.$$

The following KKM principle was subsequently proposed, which shows the above coercing conditions successfully overcome the necessity of the compactness of $T(x_0)$.

Lemma 2.2 ([42]). Let E be a topological vector space, $K \subset E$ be nonempty and convex, and $X \subset K$ be nonempty. Suppose that $T : X \rightrightarrows K$ is a KKM multifunction with compactly closed values (w.r.t. Y) at each $x \in X$. If T admits a coercing family, then $\bigcap_{x \in X} T(x) \neq \emptyset$.

Most existence results for any general class of an equilibrium imposed some kinds of continuity, monotonicity, convexity, and coercivity upon the objective (multi)functions. The rest of this section is thus devoted to recall some background definitions and properties which will be applied to the multifunctions F and G in (GEP).

Definition 2.3. A multifunction $T : Y \rightrightarrows L$ is said to be C-lower semi-continuous if for any point $y \in Y$ and any neighbourhood V of $T(y)$, there exists a neighbourhood $U \subset Y$ of y such that $T(U) \subset V + C$. Moreover, T is said to be C-upper semi-continuous if $-T$ is C-lower semi-continuous.

It is proved in [43] that the following conditions characterize each others:

(i) T is C-lower semi-continuous;

(ii) for each $e \in L$, the set

$$T^{-}(e + C^{\circ}) \overset{\textbf{def}}{=} \{x \in E\, ;\, T(x) \cap (e + C^{\circ}) \neq \emptyset\}$$

is open;

(iii) for each $x \in E$ and each $e \in C^{\circ}$, there exists a neighbourhood U of x such that

$$T(U) \subset T(x) - e + C^{\circ}.$$

Definition 2.4. A multifunction $F : Y \times Y \rightrightarrows E$ is said to be C-monotone if

$$F(x, y) + F(y, x) \subset -C, \quad \text{for all } x, y \in Y.$$

Definition 2.5. Suppose that K is nonempty and convex, a multifunction $F : K \rightrightarrows L$ is said to be C-convex if

$$F\left(\sum_{i=1}^{n} \lambda_i x_i\right) \subset \sum_{i=1}^{n} \lambda_i F(x_i) - C.$$

where for each $i \in \{1, 2, \cdots, n\}$, $x_i \in Y$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. In addition, F is C-concave if $-F$ is C-convex.

3. Main results

Here, we shall consider the conditions under which the solvability is possible for the problem (GEP). The section is organized into the sequence of problems known throughout the literature in terms the generalized vector equilibria. Note that some problems can be seen explicitly, while some are not.

3.1. Generalized equilibria.

Theorem 3.1. Let E, L be two topological vector spaces, $K \subset E$ a nonempty closed convex set and $C \subset L$ a pointed closed convex cone. Suppose that $F, G : K \times K \rightrightarrows L$ are two multifunctions with the following properties:

(H1) F is C-monotone;

(H2) $\theta \in F(x, x) \cap G(x, x)$ for all $x \in K$;

(H3) $F(x, \cdot)$ is C-lower semi-continuous and $F(\cdot, y)$ is C-concave;

(H4) $G(\cdot, y)$ is C-upper semi-continuous and $G(x, \cdot)$ is C-convex;

(H5) there exists a collection $\{(C_i, K_i)\}_{i \in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with $\{x \in K\, ;\, F(y, x) - G(x, y) \not\subset C^{\circ},\ \forall y \in C_k\} \subset K_i$.

Then, F and G possess at least one generalized weak equilibrium.

As for the proof, we shall consider this theorem through the following sequence of lemmas.

Lemma 3.2. The multifunction

$$H(y) \overset{\textbf{def}}{=} \{x \in K\, ;\, F(y, x) - G(x, y) \not\subset C^{\circ}\}$$

has closed values at each $y \in K$.

Proof. Suppose that $y \in K$ and $(x_n) \subset H(y)$ with $x_n \longrightarrow x$. Assume that $x \notin H(y)$. Thus, it follows from (H3) and (H4) that

$$\begin{aligned} F(y, x_n) - G(x_n, y) &\subset F(y, x) - G(x, y) - 2d + C^\circ \\ &\subset -2d + C^\circ, \end{aligned}$$

for all $d \in C^\circ$. For each $m \in \mathbb{N}$, $\frac{1}{m}d \in C^\circ$. Hence, $F(y, x_n) - G(x_n, y) \subset \bigcap_{m\in\mathbb{N}}(-\frac{2}{m}d + C^\circ) = C^\circ$. This yields a contradiction. □ □

Lemma 3.3. H is a KKM multifunction.

Proof. Let $A \overset{\textbf{def}}{=} \{y_j\,;\, j \in J\} \in \langle K\rangle$ and $z \in \mathrm{co}(A)$. Thus, z can be expressed by $z = \sum_{j\in J} \lambda_j y_j$ with $\lambda_j \geq 0$ and $\sum_{j\in J} \lambda_j = 1$. Assume that H is not KKM so that $z \notin \bigcup_{j\in J} H(y_j)$. It means that

$$\bigcap_{j\in J}(F(y_j, z) - G(z, y_j)) \subset C^\circ.$$

From (H3) and (H4), we may deduce that

$$\begin{cases} \theta \in F(z, z) \subset \sum_{k\in J} \lambda_k F(y_k, z) + C, \\ \theta \in G(z, z) \subset \sum_{k\in J} \lambda_k G(z, y_k) - C. \end{cases}$$

We subsequently have

$$\theta \in F(z, z) - G(z, z) \subset \sum_{k\in J}[F(y_k, z) - G(z, y_k)] + C \subset C^\circ + C \subset C^\circ,$$

which leads to a contradiction (otherwise the cone cannot be pointed). □ □

Lemma 3.4. For each $i \in I$, we can find $k \in I$ with

$$\bigcap_{y\in C_k} H(y) \subset K_i.$$

Proof. The desired result follows immediately from (H5). □ □

With the lemmas above, we may obtain a simple proof of Theorem 3.1.

of Theorem 3.1. Since E is Hausdorff, Lemma 3.2 implies that H has compactly closed values for each $y \in K$. Now, from Lemmas 3.3, 3.4 and 2.2, resp., we have

$$\bigcap_{y\in K} H(y) \neq \emptyset.$$

Take any $\widehat{x} \in \bigcap_{y\in K} H(y)$, it follows directly that $\widehat{x}$ is a generalized weak equilibrium for F and G. □ □

As initial consequences, we might consider the following corollaries.

Corollary 3.5. Let E, L be two topological vector spaces, $K \subset E$ a nonempty closed convex set and $C \subset L$ a pointed closed convex cone. Suppose that $G : K \times K \rightrightarrows L$ is a multifunction with the following properties:

1. (1) $\theta \in G(x, x)$ for all $x \in K$;
2. (2) $G(\cdot, y)$ is C-upper semi-continuous and $G(x, \cdot)$ is C-convex;
3. (3) there exists a collection $\{(C_i, K_i)\}_{i\in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with

$$\{x \in K\,;\, G(x, y) \not\subset -C^\circ,\ \forall y \in C_k\} \subset K_i.$$

Then, there exists a point $\widehat{x} \in K$ with

$$G(\widehat{x}, y) \not\subset -C^\circ, \quad \text{for all } y \in Y.$$

Proof. Consider Theorem 3.1 as $F = \theta$. □ □

Corollary 3.6. Let E, L be two topological vector spaces, $K \subset E$ a nonempty closed convex set and $C \subset L$ a pointed closed convex cone. Suppose that the multifunction $H : K \times K \rightrightarrows L$ possesses the following properties:

(1) $H(x, \cdot)$ is C-upper semi-continuous and C-concave;
(2) and $H(\cdot, y)$ is C-lower semi-continuous and C-convex;
(3) there exists a collection $\{(C_i, K_i)\}_{i\in I}$ satisfying (C1), (C2) and for each $i \in I$, there exists $k \in I$ with

$$\{x \in K\,;\, H(y, x) - H(x, y) \not\subset -C^\circ,\, \forall y \in C_k\} \subset K_i.$$

Then, there exists a point $\widehat{x} \in K$ with

$$H(y, \widehat{x}) - H(\widehat{x}, y) \not\subset -C^\circ, \quad \text{for all } y \in Y.$$

Proof. Define a multifunction

$$G(x, y) \overset{\textbf{def}}{=} H(y, x) - H(x, y).$$

Then, $\theta \in G(x, x)$ for all $x \in K$, $G(\cdot, y)$ is C-upper semi-continuous and $G(x, \cdot)$ is C-convex. Apply Corollary 3.5 to complete the proof. □ □

3.2. Strong solutions. According to Theorem 3.1, we shall give a supplementary assumption to assure the existence of a generalized weak equilibrium.

Corollary 3.7. In addition to Theorem 3.1, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then F and G possess at least one 2-equilibrium.

Proof. With this assumption, we can replace the cone C in Theorem 3.1 with $\widetilde{C}$ and still obtain the result that

$$F(y, x) - G(x, y) \not\subset \widetilde{C}^\circ.$$

Since $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, we have

$$F(y, x) - G(x, y) \not\subset C \setminus \{\theta\}. \quad □$$

□

Corollary 3.8. In addition to Theorem 3.5, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then F and G possesses at least one generalized equilibrium.

Proof. As in the previous corollary, set $F = \theta$. □ □

3.3. Saddle points. We have mentioned in the earlier section the problem of finding a saddle point. For instance, let E_1, E_2, L be three topological vector spaces and K_1, K_2 be two nonempty closed convex subsets of E_1 and E_2, respectively. Suppose that $C \subset L$ is a pointed closed convex cone. Now, consider the multifunction $F : K_1 \times K_2 \rightrightarrows L$. A point $(\overline{x}_1, \overline{x}_2) \in K_1 \times K_2$ is called a weak saddle point (strong saddle point) if

$$F(y_1, \overline{x}_2) - F(\overline{x}_1, y_2) \not\subset -C^\circ(-C \setminus \{\theta\}), \quad \text{for all } (y_1, y_2) \in K_1 \times K_2.$$

Theorem 3.9. Let E_1, E_2, L be three topological vector spaces, K_1, K_2 two nonempty closed convex subsets of E_1, E_2, resp., $C \subset L$ a pointed closed convex cone. Suppose that $F : K_1 \times K_2 \rightrightarrows L$ is a multifunction with the following properties:

(i) $F(x,\cdot)$ is C-upper semi-continuous and C-concave;
(ii) $F(\cdot,y)$ is C-lower semi-continuous and C-convex;
(iii) there exists a collection $\{(C_i,K_i)\}_{i\in I}$ satisfying (C1), (C2) and for each $i\in I$, there exists $k\in I$ with

$$\{x\in K\,;\,F(y_1,x_2)-F(x_1,y_2)\not\subset -C^\circ,\,\forall (y_1,y_2)\in C_k\}\subset K_i.$$

Then, F possesses at least one (K_1,K_2)-weak saddle point.

Proof. Set $K\overset{\textbf{def}}{=}K_1\times K_2$. We may see that K is closed and convex. Consider the multifunction

$$G(x,y)\overset{\textbf{def}}{=}F(y_1,x_2)-F(x_1,y_2).$$

It is easy to verify that $\theta\in G(x,x)$ for all $x\in K_1\times K_2$, $G(x,\cdot)$ is C-convex and $G(\cdot,y)$ is C-upper semi-continuous. By Corollary 3.5, G has a weak equilibrium $\widehat{x}=(\widehat{x}_1,\widehat{x}_2)\in K$ which is in turn a weak equilibrium of G and is in turn a (K_1,K_2)-weak saddle point of F. □ □

Corollary 3.10. *In addition to Theorem 3.9, if there exists a pointed closed convex cone $\widetilde{C}\subset L$ with $C\setminus\{\theta\}\subset\widetilde{C}^\circ$, then F possesses at least one saddle point.*

Proof. Combine the proofs of Corollary 3.8 and Theorem 3.9. □ □

3.4. Non-cooperative game equilibrium.

Let $I=\{1,2,\cdots,n\}$ ($n\in\mathbb{N}$) denotes the set of players. To each player $i\in I$, we assign a set K_i of strategies with K_i being nonempty, closed and convex in some topological vector space E_i. Suppose that L is a topological vector space with a pointed closed convex cone C. A loss multifunction for each player i is the function $F_i: K=\prod_{i\in I}K_i\rightrightarrows L$. For $x=(x_i)_{i\in I}\in K$, we write $x^{-i}|y_i\overset{\textbf{def}}{=}(x_1,x_2,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_n)$, where $y_i\in K_i$.

A point $\widehat{x}=(\widehat{x}_i)_{i\in I}\in K$ is said to be a weak non-cooperative game equilibrium (strong non-cooperative game equilibrium) if for each $i\in I$,

$$F_i(\widehat{x}^{-i}|y_i)-F_i(\widehat{x})\not\subset -C^\circ,\quad \text{for all } y=(y_i)_{i\in I}\in K.$$

Theorem 3.11. *Let $(E_i)_{i\in I}$ be a sequence of topological vector spaces, (K_i) a sequence of nonempty closed convex sets with $K_i\subset E_i$. Suppose that for each $i\in I$, $F_i: K=\prod_{i\in I}K_i\rightrightarrows L$, where L is a topological vector space with a pointed closed convex cone C. Also assume that the following properties hold for each $i\in I$:*

(N1) F_i is C-continuous, i.e., F_i is both C-upper and C-lower semi-continuous;
(N2) at each $i\in I$, $F_i(x^{-i}|\cdot)$ is C-convex;
(N3) there exists a collection $\{(C_j,Q_j)\}_{j\in J}$, where for each $j\in J$, $C_j,Q_j\subset E\overset{\textbf{def}}{=}\prod_{i\in I}E_i$, satisfying (C1), (C2) and for each $j\in J$, there exists $k\in J$ with

$$\{(x_i)_{i\in I}\in K\,;\,\sum_{i\in I}[F_i(x^{-i}|y_i)-F_i(x)]\not\subset -C^\circ,\,\forall (y_i)_{i\in I}\in C_k\}\subset Q_j.$$

Then, the sequence $(F_i)_{i\in I}$ has at least one weak non-cooperative game equilibrium.

Proof. Define a multifunction $G: K\times K\rightrightarrows L$ by

$$G(x,y)\overset{\textbf{def}}{=}\sum_{i\in I}[F_i(x^{-i}|y_i)-F_i(x)].$$

It is clear that $\theta\in G(x,x)$ for all $x\in K$, $G(x,\cdot)$ is C-convex and $G(\cdot,y)$ is C-lower semi-continuous. Applying Corollary 3.5, we obtain the existence of a weak equilibrium of H, which is in turn a weak equilibrium of G. That is, there exists a point $\widehat{x} = (\widehat{x}_i)_{i\in I} \in K$ with

$$\sum_{i\in I}[F_i(\widehat{x}^{-i}|y_i) - F_i(\widehat{x})] \not\subset -C^\circ, \quad \forall y = (y_i)_{i\in I} \in K.$$

For each $\ell \in I$, we may take $y \in K$ such that $y_i = \widehat{x}_i$ for all $i \in I \setminus \{\ell\}$ into account and conclude that

$$F_\ell(\widehat{x}^{-\ell}|y_\ell) - F_\ell(\widehat{x}) \not\subset -C^\circ, \quad \forall y_\ell \in K_\ell. \qquad \square$$

$\square$

Corollary 3.12. In addition to Theorem 3.11, if there exists a pointed closed convex cone $\widetilde{C} \subset L$ with $C \setminus \{\theta\} \subset \widetilde{C}^\circ$, then the sequence $(F_i)_{i\in I}$ has at least one non-cooperative game equilibrium.

Proof. Combine the proof of Corollary 3.8 and Theorem 3.11. $\square$ $\square$

3.5. Mixed equilibrium problems. There are several classes of different mixed equilibrium problems and generalized mixed equilibrium problems. However, in this paper, we shall consider only on the major ones.

According to the problem (GEP), we may consider this as a mixture of two multifunctions F and G, with $F(x, y) = -\widetilde{F}(y, x)$ for some $\widetilde{F} : Y \times Y \rightrightarrows L$ such that $\theta \in \widetilde{F}(x, x)$ for all $x \in Y$. Then, (GEP) can be rewritten as the problem of finding $\widehat{x} \in Y$ such that

$$F(\widehat{x}, y) + G(\widehat{x}, y) \not\subset -C^\circ, \quad \forall y \in Y.$$

In other words, the problem (GEP) conveys the mixed equilibrium problemsas it includes the two set-valued equilibrium problems corresponded to $\widetilde{F}$ and G, respectively. Thus, if $\widetilde{F}$ is $(-C)$-monotone, $\widetilde{F}(\cdot, y)$ is C-upper semi-continuous, and $\widetilde{F}(x, \cdot)$ is C-convex, then it follows from Theorem 3 that this mixed equilibrium problem has a solution.

The next theorem overcome the situation when the function $\widetilde{F}$ is not valid.

Theorem 3.13. In addition to Theorem 3.1, assume further that

- (M1) $F(x, \cdot)$ is C-convex;
- (M2) for each fixed $x, y \in K$, the multifunction
$$t \in [0, 1] \mapsto F(ty + (1 - t)x, y)$$
is C-upper semi-continuous at $t = 0$;
- (M3) there is a solution $\widehat{x} \in K$ of the problem (GEP) that admits an absolute exclusion, i.e.,
$$F(y, \widehat{x}) - G(\widehat{x}, y) \subset L \setminus C^\circ, \quad \forall y \in K.$$

Then, $\widehat{x}$ also solves the following problem:

$$F(\widehat{x}, y) + G(\widehat{x}, y) \not\subset -C^\circ, \quad \text{for all } y \in K.$$

Proof. We first define two multifunctions $F', G' : K \times K \rightrightarrows L$ such that

$$F'(x, y) \overset{\textbf{def}}{=} \begin{cases} F(x, y), & \text{if } x \neq y, \\ \{\theta\}, & \text{otherwise,} \end{cases} \quad \text{and} \quad G'(x, y) \overset{\textbf{def}}{=} \begin{cases} G(x, y), & \text{if } x \neq y, \\ \{\theta\}, & \text{otherwise.} \end{cases}$$

It is clear that F' and G' preserves the C-convexity, C-semi continuity, and C-monotonicity of F and G, respectively. For $t \in [0, 1]$ and $y \in K$, we write

$$x_t \overset{\textbf{def}}{=} ty + (1 - t)\widehat{x}.$$

From the C-convexity of $F'(x,\cdot)$ and $G'(x,\cdot)$, we have

$$\begin{cases} \theta \in tF'(x_t,y)+(1-t)F'(x_t,\widehat{x})-C, \\ \theta \in (1-t)tG'(\widehat{x},y)-(1-t)G'(\widehat{x},x_t)-C. \end{cases}$$

Consequently, we have

$$\theta \in tF'(x_t,y)+(1-t)[F'(x_t,\widehat{x})-G'(\widehat{x},x_t)]+(1-t)tG'(\widehat{x},y)-C.$$

We now claim that $tF'(x_t,y)+(1-t)tG'(\widehat{x},y)\not\subset -C^\circ$. Otherwise, if we suppose to the contrary, it follows that for some $\xi\in tF'(x_t,y)+(1-t)tG'(\widehat{x},y)$ and $\xi'\in (1-t)[F'(x_t,\widehat{x})-G'(\widehat{x},x_t)]$, we have

$$\theta\in\xi+\xi'-C\subset\xi'-C^\circ-C\subset\xi'-C^\circ.$$

Hence, it is the case that $\xi'\in C^\circ$, which contradicts (M3). So we have proved our claim. For $t\neq 0$, we further obtain that

$$tF'(x_t,y)+(1-t)tG'(\widehat{x},y)\not\subset -C^\circ. \tag{3.1}$$

Define a multifunction $H:[0,1]\rightrightarrows L$ by

$$H(t)\stackrel{\mathbf{def}}{=}F'(x_t,y)+(1-t)G'(\widehat{x},y),\quad \forall t\in[0,1].$$

We may see from (M2) that H is C-upper semi-continuous at $t=0$.

Let us now verify that $H(0)\not\subset -C^\circ$, since if it is so, the combination with the fact that $F'(\widehat{x},y)+G'(\widehat{x},y)\subset F(\widehat{x},y)+G(\widehat{x},y)$ will eventually implies our desired result. Assume to the contrary that $H(0)\subset -C^\circ$. Thus, we may find an open set N with $H(0)\subset N\subset -C^\circ$, which immediately give

$$N-C\subset -C^\circ-C\subset -C^\circ.$$

Since H is C-upper semi-continuous, we can find an open set $P\subset\mathbb{R}$ such that

$$H(P\cap[0,1])\subset N-C\subset -C^\circ.$$

Taking any $t\in P\cap(0,1]$, we have from the above inclusion that $H(t)\subset -C^\circ$. This contradicts with (3.1), and so this proves the theorem. □ □

Conclusion

We close this paper with recalling that a new class of generalized equilibrium problems is formulated. It turns out that many esteemed problems in optimization are included and unified. Explicit and implicit consequences are also deduced and studied. Ultimately and most importantly, our results enlarge the validity support of the approximation models for several generalized equilibrium problems.

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