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COINCIDENCE POINT AND COMMON FIXED POINT THEOREMS IN CONE b-METRIC SPACES

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ABSTRACT. The main purpose of this paper is to obtain sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings in cone b-metric spaces. Our results extend and generalize several well known comparable results in the existing literature. Finally, some examples are provided to illustrate our results.

KEYWORDS: Cone b-metric space, point of coincidence, weakly compatible mappings, common fixed point.
AMS Subject Classification: 54H25, 47H10.

1. Introduction

Over the past two decades a considerable amount of research work for the development of fixed point theory have executed by several mathematicians. There has been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space initiated by Bakhtin[2]. In[6], Huang and Zhang introduced the concept of cone metric spaces as a generalization of metric spaces and proved some important fixed point theorems in such spaces. After that a series of articles have been dedicated to the improvement of fixed point theory. In most of those articles, the authors used normality property of cones in their results. Recently, Hussain and Shah[8] introduced the concept of cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They studied some topological properties and improved some recent results about KKM mappings in the setting of a cone b-metric space. In this work, we shall establish sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings without the assumption of normality in cone b-metric spaces. The results generalize and improve some recent results in the literature. Furthermore, we support our results by examples.

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2. Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature. Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

$$(i) \quad P \text{ is closed, nonempty and } P \neq \{\theta\};$$
$$(ii) \quad a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \ \Rightarrow\ ax + by \in P;$$
$$(iii) \quad P \cap (-P) = \{\theta\}.$$

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$(equivalently, $y \succeq x$) if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq k\,\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. Throughout this paper, we suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is a partial ordering on E with respect to P.

Definition 2.1. [6] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 2.2. [8] Let X be a nonempty set and E a real Banach space with cone P. A vector valued function $d : X \times X \to E$ is said to be a cone b-metric function on X with the coefficient $s \geq 1$ if the following conditions are satisfied:

(i) $\theta \preceq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = \theta$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \preceq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a cone b-metric space.

Observe that if $s = 1$, then the ordinary triangle inequality in a cone metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of cone b-metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b-metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 2.3. [8] Let $X = \{-1, 0, 1\}$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \to P$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = \theta$, $x \in X$ and $d(-1, 0) = (3, 3)$, $d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, d) is a cone b-metric space, but not a cone metric space since the triangle inequality is not satisfied. Indeed, we have that

$$d(-1, 1) + d(1, 0) = (1, 1) + (1, 1) = (2, 2) \prec (3, 3) = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.4. [9] Let $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\} \subseteq E$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x, y) = (\mid x - y \mid^p, \alpha \mid x - y \mid^p)$, where $\alpha \geq 0$ and $p > 1$ are two constants. Then (X, d) is a cone b-metric space with $s = 2^{p-1}$, but not a cone metric space.

Definition 2.5. [8] Let (X, d) be a cone b-metric space, $x \in X$ and (x_n) be a sequence in X. Then

(i): (x_n) converges to x whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$;

(ii): (x_n) is a Cauchy sequence whenever, for every $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$;

(iii): (X, d) is a complete cone b-metric space if every Cauchy sequence is convergent.

Remark 2.6. [8] Let (X, d) be a cone b-metric space over the ordered real Banach space E with a cone P. Then the following properties are often used:

(i): If $a \preceq b$ and $b \ll c$, then $a \ll c$.

(ii): If $a \ll b$ and $b \ll c$, then $a \ll c$.

(iii): If $\theta \preceq u \ll c$ for each $c \in int(P)$, then $u = \theta$.

(iv): If $c \in int(P)$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

(v): Let $\theta \ll c$. If $\theta \preceq d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then eventually $d(x_n, x) \ll c$, where (x_n), x are a sequence and a given point in X.

(vi): If $\theta \preceq a_n \preceq b_n$ and $a_n \to a$, $b_n \to b$, then $a \preceq b$, for each cone P.

(vii): If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

(viii): $\alpha\, int(P) \subseteq int(P)$ for $\alpha > 0$.

(ix): For each $\delta > 0$ and $x \in int(P)$ there is $0 < \gamma < 1$ such that $\parallel \gamma x \parallel < \delta$.

(x): For each $\theta \ll c_1$ and $c_2 \in P$, there is an element $\theta \ll d$ such that $c_1 \ll d$, $c_2 \ll d$.

(xi): For each $\theta \ll c_1$ and $\theta \ll c_2$, there is an element $\theta \ll e$ such that $e \ll c_1$, $e \ll c_2$.

Definition 2.7. [1] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.8. [11] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx) \text{ whenever } Sx = Tx.$$

Proposition 2.9. [1] *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Definition 2.10. Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. A mapping $T : X \to X$ is called expansive if there exists a real constant $k > s$ such that

$$d(Tx, Ty) \succeq k\, d(x, y)$$

for all $x, y \in X$.

3. TOPOLOGY IN CONE b-METRIC SPACES

In this section our concern is to introduce some topological aspects in cone b-metric spaces. This will facilate the initiation of open and closed sets, limit points of sets and other allied notions in the setting of cone b-metric spaces.

Definition 3.1. [8] Let (X, d) be a cone b-metric space and $B \subseteq X$.

- **(i):** $b \in B$ is called an interior point of B whenever there is $\theta \ll p$ such that $B_0(b,p) \subseteq B$, where $B_0(b,p) := \{y \in X : d(y,b) \ll p\}$.
- **(ii):** An element $x \in X$ is called a limit point of B whenever for every $\theta \ll e$, $B_0(x,e) \setminus (B \setminus \{x\}) \neq \emptyset$. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
- **(iii):** A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A, that is, for any $a \in A$, there exists $c \in int\, P$ such that the open ball $B_0(a,c) \subseteq A$.
- **(iv):** A subset $B \subseteq X$ is called bounded whenever there exist $\theta \ll c$ and $x_0 \in X$ such that $d(b, x_0) \ll c$ for all $b \in B$.
- **(v):** A subset $B \subseteq X$ is called compact whenever every open cover of B has a finite subcover.

Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Then the family of sets $\{B(x,c) : x \in X,\ \theta \ll c\}$ where $B(x,c) = \{y \in X : d(y,x) \ll c\}$ is a sub basis for a topology on X. This topology is denoted by τ_{cb}. It is to be noted that τ_{cb} is a Hausdorff topology. Suppose for each c with $\theta \ll c$, we have $B(x,c) \cap B(y,c) \neq \emptyset$. So, there exists $z \in X$ such that $d(z,x) \ll \frac{c}{2s}$ and $d(z,y) \ll \frac{c}{2s}$. Hence,

$$d(x,y) \preceq s(d(x,z) + d(z,y)) \ll c.$$

This implies that $d(x,y) = \theta$, that is, $x = y$.

Proposition 3.2. [8] *Let (X, d) be a cone b-metric space and τ_{cb} be the topology defined above. Then for any nonempty subset $A \subseteq X$ we have*
(i) A is closed if and only if for any sequence (x_n) in A which converges to x, we have $x \in A$;
(ii) If we define $\bar{A}$ to be the intersection of all closed subsets of X which contains A, then for any $x \in \bar{A}$ and for any $c \in int\, P$, we have $B_0(x,c) \cap A \neq \emptyset$.

Theorem 3.3. [8] *Let (X, d) be a cone b-metric space and τ_{cb} be the topology defined above. Then for any nonempty subset $A \subseteq X$, the following properties are equivalent:*
(i) A is compact.
(ii) For any sequence (x_n) in A, there exists a subsequence (x_{n_k}) of (x_n) which converges, and $\lim\limits_{n\to\infty} x_n \in A$.

4. Main Results

In this section, we prove some point of coincidence and common fixed point results in cone b-metric spaces.

Theorem 4.1. *Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings f, $g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq \lambda\, d(gx, gy) \tag{4.1}$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s})$ is a constant. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and choose $x_1 \in X$ such that $fx_0 = gx_1$. This is possible since $f(X) \subseteq g(X)$. Continuing this process, we can construct a sequence (x_n) in X such that $fx_n = gx_{n+1},\ n = 0, 1, 2, \cdots$.
By using (4.1), we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &\preceq \lambda d(gx_{n+1}, gx_n) \\ &= \lambda d(fx_n, fx_{n-1}) \\ &\preceq \lambda^2 d(gx_n, gx_{n-1}) \\ &= \lambda^2 d(fx_{n-1}, fx_{n-2}) \\ &\ . \\ &\ . \\ &\ . \\ &\preceq \lambda^n d(fx_1, fx_0). \end{aligned} \tag{4.2}$$

For any $m, n \in \mathbb{N}$ with $m > n$, we have by using condition (4.2) that

$$\begin{aligned} d(fx_n, fx_m) &\preceq s\left[d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_m)\right] \\ &\preceq sd(fx_n, fx_{n+1}) + s^2 d(fx_{n+1}, fx_{n+2}) + \cdots \\ &\quad + s^{m-n-1}\left[d(fx_{m-2}, fx_{m-1}) + d(fx_{m-1}, fx_m)\right] \\ &\preceq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n-1}\lambda^{m-1}\right] d(fx_0, fx_1) \\ &\preceq \left[s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n-1}\lambda^{m-2} + s^{m-n}\lambda^{m-1}\right] d(fx_0, fx_1) \\ &= s\lambda^n \left[1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-2} + (s\lambda)^{m-n-1}\right] d(fx_0, fx_1) \\ &\preceq \frac{s\lambda^n}{1 - s\lambda}\, d(fx_0, fx_1). \end{aligned} \tag{4.3}$$

It is to be noted that $\frac{s\lambda^n}{1-s\lambda}\, d(fx_0, fx_1) \to \theta$ as $n \to \infty$. Let $\theta \ll c$ be given. Then we can find $m_0 \in \mathbb{N}$ such that

$$\frac{s\lambda^n}{1 - s\lambda}\, d(fx_0, fx_1) \ll c,$$

for each $n > m_0$.

Therefore, it follows from (4.3) that

$$d(fx_n, fx_m) \preceq \frac{s\lambda^n}{1 - s\lambda}\, d(fx_0, fx_1) \ll c$$

for all $m > n > m_0$.

So (fx_n) is a Cauchy sequence in $f(X)$. Suppose that $f(X)$ is a complete subspace of X. Then there exists $y \in f(X) \subseteq g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. In case, $g(X)$ is complete, this holds also with $y \in g(X)$. Let $u \in X$ be such that $gu = y$. For $\theta \ll c$, one can choose a natural number $n_0 \in \mathbb{N}$ such that $d(y, fx_n) \ll \frac{c}{s(\lambda+1)}$ and $d(gx_n, gu) \ll \frac{c}{s(\lambda+1)}$ for all $n > n_0$.

Now,

$$\begin{aligned} d(y, fu) &\preceq s[d(y, fx_n) + d(fx_n, fu)] \\ &\preceq s[d(y, fx_n) + \lambda\, d(gx_n, gu)] \\ &\ll c,\ \textit{for all } n > n_0, \end{aligned}$$

which gives that $d(y, fu) = \theta$, i.e., $fu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.
For uniqueness, let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$d(v, y) = d(fx, fu) \preceq \lambda\, d(gx, gu) = \lambda\, d(v, y).$$

By Remark 2.6(vii), we have $d(v, y) = \theta$ i.e., $v = y$.
Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. □

The following Corollary is the Theorem 2.1[9].

Corollary 4.2. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq \lambda\, d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s})$ is a constant. Then f has a unique fixed point in X. Furthermore, the iterative sequence $(f^n x)$ converges to the fixed point.

Proof. The proof follows from Theorem 4.1 by taking $g = I$, the identity mapping on X.

□

Corollary 4.3. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $g : X \to X$ is onto and satisfies*

$$d(gx, gy) \succeq k\, d(x, y)$$

for all $x, y \in X$, where $k > s$ is a constant. Then g has a unique fixed point in X.

Proof. Taking $f = I$ in Theorem 4.1, we obtain the desired result.

□

Remark 4.4. Corollary 4.3 gives a sufficient condition for the existence of unique fixed point of an expansive mapping in cone b-metric spaces.

Theorem 4.5. *Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\ g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq a\, d(fx, gx) + b\, d(fy, gy) \tag{4.4}$$

for all $x, y \in X$, where $a,\ b \geq 0$ with $a + sb < 1$. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. As in Theorem 4.1, we can construct a sequence (x_n) in X such that $fx_n = gx_{n+1},\ n = 0,\ 1,\ 2,\ \cdots$.
Using (4.4), we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &\preceq a\, d(fx_{n+1}, gx_{n+1}) + b\, d(fx_n, gx_n) \\ &= a\, d(fx_{n+1}, fx_n) + b\, d(fx_n, fx_{n-1}) \end{aligned}$$

which implies that

$$d(fx_{n+1}, fx_n) \preceq k\, d(fx_n, fx_{n-1}) \tag{4.5}$$

where $k = \frac{b}{1-a}$. It is easy to see that $0 \leq k < \frac{1}{s}$.
By repeated application of (4.5), we obtain

$$d(fx_{n+1}, fx_n) \preceq kd(fx_n, fx_{n-1}) \preceq k^2 d(fx_{n-1}, fx_{n-2}) \preceq \cdots \preceq k^n d(fx_1, fx_0).$$

By an argument similar to that used in Theorem 4.1, it follows that (fx_n) is a Cauchy sequence in $f(X)$. If $f(X)$ is a complete subspace of X, then there exists $y \in f(X) \subseteq g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. In case, $g(X)$ is complete, this holds also with $y \in g(X)$. Let $u \in X$ be such that $gu = y$. For $\theta \ll c$, one can choose a natural number $n_0 \in \mathbb{N}$ such that $d(y, fx_n) \ll \frac{1-bs}{2(s+as^2)}c$ and $d(gx_n, gu) \ll \frac{1-bs}{2as^2}c$ for all $n > n_0$.

Now,

$$\begin{aligned} d(y, fu) &\preceq s[d(y, fx_n) + d(fx_n, fu)] \\ &\preceq s[d(y, fx_n) + a\, d(fx_n, gx_n) + b\, d(fu, gu)] \\ &\preceq s[d(y, fx_n) + as\, d(fx_n, y) + as\, d(y, gx_n) + b\, d(fu, y)]. \end{aligned}$$

So it must be the case that

$$(1 - bs)d(y, fu) \preceq (s + as^2)d(y, fx_n) + as^2 d(y, gx_n). \tag{4.6}$$

Therefore, we obtain from condition (4.6) that

$$\begin{aligned} d(y, fu) &\preceq \frac{s + as^2}{1 - bs} d(y, fx_n) + \frac{as^2}{1 - bs} d(gu, gx_n) \\ &\ll c \text{ for all } n > n_0. \end{aligned} \tag{4.7}$$

This implies that $d(y, fu) = \theta$, i.e., $fu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.

For uniqueness, let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$d(v, y) = d(fx, fu) \preceq a\, d(fx, gx) + b\, d(fu, gu) = \theta.$$

By Remark 2.6(vii), we have $d(v, y) = \theta$ i.e., $v = y$.
Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. □

Corollary 4.6. *Let (X, d) be a complete cone b-metric space with the coefficient $s \geq 1$. Suppose the mapping $f : X \to X$ satisfies the contractive condition*

$$d(fx, fy) \preceq a\, d(fx, x) + b\, d(fy, y)$$

for all $x, y \in X$, where $a,\ b \geq 0$ with $a + sb < 1$. Then f has a unique fixed point in X.

Proof. Proof follows from Theorem 4.5 by taking $g = I$.

□

Theorem 4.7. *Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$. Suppose the mappings $f,\ g : X \to X$ satisfy the contractive condition*

$$d(fx, fy) \preceq a\, d(fx, gy) + b\, d(fy, gx) \tag{4.8}$$

for all $x, y \in X$, where $a,\ b \geq 0$ with $max\{a,\ b\} < \frac{1}{s^2+s}$. If $f(X) \subseteq g(X)$ and $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a unique point of *coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary. As in Theorem 4.1, we can construct a sequence (x_n) in X such that $fx_n = gx_{n+1},\ n = 0, 1, 2, \cdots$.
Using (4.8), we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &\preceq a\, d(fx_{n+1}, gx_n) + b\, d(fx_n, gx_{n+1}) \\ &= a\, d(fx_{n+1}, fx_{n-1}) \\ &\preceq as[d(fx_{n+1}, fx_n) + d(fx_n, fx_{n-1})]. \end{aligned}$$

This implies that

$$d(fx_{n+1}, fx_n) \preceq \frac{as}{1-as}\, d(fx_n, fx_{n-1}). \tag{4.9}$$

Therefore, we obtain from condition (4.9) that

$$d(fx_{n+1}, fx_n) \preceq k\, d(fx_n, fx_{n-1}) \tag{4.10}$$

where $k = \frac{as}{1-as}$. It is easy to see that $0 \leq k < \frac{1}{s}$.
By repeated application of (4.10), we obtain

$$d(fx_{n+1}, fx_n) \preceq kd(fx_n, fx_{n-1}) \preceq k^2 d(fx_{n-1}, fx_{n-2}) \preceq \cdots \preceq k^n d(fx_1, fx_0).$$

By an argument similar to that used in Theorem 4.1, it follows that (fx_n) is a Cauchy sequence in $f(X)$. If $f(X)$ is a complete subspace of X, then there exists $y \in f(X) \subseteq g(X)$ such that $fx_n \to y$ and also $gx_n \to y$. In case, $g(X)$ is complete, this holds also with $y \in g(X)$. Let $u \in X$ be such that $gu = y$. For $\theta \ll c$, one can choose a natural number $n_0 \in \mathbb{N}$ such that $d(y, fx_n) \ll \frac{1-bs^2}{2(s+as)}c$ and $d(gx_n, gu) \ll \frac{1-bs^2}{2bs^2}c$ for all $n > n_0$.

Now,

$$\begin{aligned} d(y, fu) &\preceq s[d(y, fx_n) + d(fx_n, fu)] \\ &\preceq s[d(y, fx_n) + a\, d(fx_n, gu) + b\, d(fu, gx_n)] \\ &\preceq s[d(y, fx_n) + a\, d(fx_n, y) + bs\, d(y, gx_n) + bs\, d(fu, y)]. \end{aligned}$$

So it must be the case that

$$(1 - bs^2)d(y, fu) \preceq (s + as)d(y, fx_n) + bs^2 d(y, gx_n). \tag{4.11}$$

Therefore, we obtain from condition (4.11) that

$$\begin{aligned} d(y, fu) &\preceq \frac{s+as}{1-bs^2} d(y, fx_n) + \frac{bs^2}{1-bs^2} d(gu, gx_n) \\ &\ll c \textit{ for all } n > n_0. \end{aligned}$$

This implies that $d(y, fu) = \theta$, i.e., $fu = y$ and hence $fu = gu = y$. Therefore, y is a point of coincidence of f and g.

For uniqueness, let v be another point of coincidence of f and g. So $fx = gx = v$ for some $x \in X$. Then

$$\begin{aligned} d(v, y) = d(fx, fu) &\preceq a\, d(fx, gu) + b\, d(fu, gx) \\ &= a\, d(v, y) + b\, d(y, v) \\ &= (a + b)\, d(v, y). \end{aligned}$$

Since $a+b<1$, by Remark 2.6(vii), we have $d(v,y)=\theta$ i.e., $v=y$.
Therefore, f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.9, f and g have a unique common fixed point in X. □

Corollary 4.8. *Let (X,d) be a complete cone b-metric space with the coefficient $s\geq 1$. Suppose the mapping $f:X\to X$ satisfies the contractive condition*

$$d(fx,fy)\preceq a\,d(fx,y)+b\,d(fy,x)$$

for all $x,y\in X$, where $a,\ b\geq 0$ with $max\{a,\ b\}<\frac{1}{s^2+s}$. Then f has a unique fixed point in X.

Proof. Proof follows from Theorem 4.7 by taking $g=I$. □

We conclude with some examples.

Example 4.9. Let $E=\mathbb{R}^2$, the Euclidean plane and $P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$ a cone in E. Let $X=[0,1]$ and $p>1$ be a constant. We define $d:X\times X\to E$ as

$$d(x,y)=(\mid x-y\mid^p,\mid x-y\mid^p)$$

for all $x,y\in X$. Then (X,d) is a cone b-metric space with the coefficient $s=2^{p-1}$. Let us define $f,g:X\to X$ as

$$fx=\frac{x}{4}-\frac{x^2}{8},\ for\ all\ x\in X$$

and

$$gx=\frac{x}{2},\ for\ all\ x\in X.$$

Then, for every $x,y\in X$ one has

$$\begin{aligned}
d(fx,fy) &= (\mid fx-fy\mid^p,\mid fx-fy\mid^p)\\
&= \left(\mid \frac{1}{4}(x-y)-\frac{1}{8}(x-y)(x+y)\mid^p,\mid \frac{1}{4}(x-y)-\frac{1}{8}(x-y)(x+y)\mid^p\right)\\
&= \left(\mid \frac{x}{2}-\frac{y}{2}\mid^p.\mid \frac{1}{2}-\frac{1}{4}(x+y)\mid^p,\mid \frac{x}{2}-\frac{y}{2}\mid^p.\mid \frac{1}{2}-\frac{1}{4}(x+y)\mid^p\right)\\
&\preceq \frac{1}{2^p}\left(\mid \frac{x}{2}-\frac{y}{2}\mid^p,\mid \frac{x}{2}-\frac{y}{2}\mid^p\right)\\
&= \frac{1}{2^p}\,d(gx,gy).
\end{aligned}$$

Thus, we have all the conditions of Theorem 4.1 and $0\in X$ is the unique common fixed point of f and g.

Example 4.10. Let $E=\mathbb{R}^2$, the Euclidean plane and $P=\{(x,y)\in\mathbb{R}^2 : x,y\geq 0\}$ a cone in E. Let $X=[0,1]$ and $p>1$ be a constant. We define $d:X\times X\to E$ as

$$d(x,y)=(\mid x-y\mid^p,\mid x-y\mid^p)$$

for all $x,y\in X$. Then (X,d) is a cone b-metric space with the coefficient $s=2^{p-1}$. Let us define $f,g:X\to X$ as

$$\begin{aligned}
fx &= \frac{x}{16},\ for\ all\ x\in[0,\frac{1}{2})\\
&= \frac{x}{12},\ for\ all\ x\in[\frac{1}{2},1]
\end{aligned}$$

and

$$gx = \frac{x}{2},\ for\ all\ x \in X.$$

Now we verify that for every $x, y \in X$ one has

$$d(fx, fy) \preceq a\, d(fx, gx) + b\, d(fy, gy)$$

where $a,\ b \geq 0$ with $a + sb < 1$.

Case-I If $x, y \in [0, \frac{1}{2})$, then

$$\begin{aligned} d(fx, fy) &= (|\ fx - fy\ |^p, |\ fx - fy\ |^p) \\ &= \frac{1}{16^p}\ (|\ x - y\ |^p, |\ x - y\ |^p) \\ &\preceq \frac{1}{16^p}\ ((|\ x\ | + |\ y\ |)^p, (|\ x\ | + |\ y\ |)^p) \\ &\preceq \frac{2^p}{16^p}\ (x^p + y^p, x^p + y^p) \\ &= \frac{2^p}{7^p} . \frac{7^p}{16^p}\ (x^p + y^p, x^p + y^p)\,. \end{aligned}$$

Also,

$$\begin{aligned} d(fx, gx) + d(fy, gy) &= (|\ fx - gx\ |^p, |\ fx - gx\ |^p) + (|\ fy - gy\ |^p, |\ fy - gy\ |^p) \\ &= \left(|\ \frac{x}{16} - \frac{x}{2}\ |^p, |\ \frac{x}{16} - \frac{x}{2}\ |^p\right) + \left(|\ \frac{y}{16} - \frac{y}{2}\ |^p, |\ \frac{y}{16} - \frac{y}{2}\ |^p\right) \\ &= \frac{7^p}{16^p}\ (x^p + y^p, x^p + y^p)\,. \end{aligned}$$

Therefore,

$$d(fx, fy) \preceq \frac{2^p}{7^p}\,[d(fx, gx) + d(fy, gy)] \preceq \frac{2^p}{5^p}\,[d(fx, gx) + d(fy, gy)].$$

Case-II If $x, y \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} d(fx, fy) &\preceq \frac{2^p}{12^p}\,(x^p + y^p, x^p + y^p) \\ &= \frac{2^p}{5^p} . \frac{5^p}{12^p}\ (x^p + y^p, x^p + y^p)\,. \end{aligned}$$

and,

$$d(fx, gx) + d(fy, gy) = \frac{5^p}{12^p}\ (x^p + y^p, x^p + y^p)\,.$$

Therefore,

$$d(fx, fy) \preceq \frac{2^p}{5^p}\,[d(fx, gx) + d(fy, gy)].$$

Case-III If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1]$, then

$$\begin{aligned} d(fx, fy) &= (|\ fx - fy\ |^p, |\ fx - fy\ |^p) \\ &= \left(|\ \frac{x}{16} - \frac{y}{12}\ |^p, |\ \frac{x}{16} - \frac{y}{12}\ |^p\right) \\ &\preceq 2^p\ \left((\frac{x}{16})^p + (\frac{y}{12})^p, (\frac{x}{16})^p + (\frac{y}{12})^p\right) \\ &= \frac{2^p}{5^p}\left((\frac{5x}{16})^p + (\frac{5y}{12})^p, (\frac{5x}{16})^p + (\frac{5y}{12})^p\right) \\ &\preceq \frac{2^p}{5^p}\left((\frac{7x}{16})^p + (\frac{5y}{12})^p, (\frac{7x}{16})^p + (\frac{5y}{12})^p\right). \end{aligned}$$

Also,

$$\begin{aligned} d(fx,gx)+d(fy,gy) &= (\mid fx-gx\mid^p,\mid fx-gx\mid^p)+(\mid fy-gy\mid^p,\mid fy-gy\mid^p)\\ &= \Big(\mid \frac{x}{16}-\frac{x}{2}\mid^p,\mid \frac{x}{16}-\frac{x}{2}\mid^p\Big)+\Big(\mid \frac{y}{12}-\frac{y}{2}\mid^p,\mid \frac{y}{12}-\frac{y}{2}\mid^p\Big)\\ &= \Big((\frac{7x}{16})^p+(\frac{5y}{12})^p,(\frac{7x}{16})^p+(\frac{5y}{12})^p\Big). \end{aligned}$$

Therefore,

$$d(fx,fy)\preceq \frac{2^p}{5^p}\,[d(fx,gx)+d(fy,gy)].$$

Thus, we have

$$d(fx,fy)\preceq \frac{2^p}{5^p}\,[d(fx,gx)+d(fy,gy)]$$

for all $x,y\in X$, where $a+sb=(1+s)\frac{2^p}{5^p}\leq 2s.\frac{2^p}{5^p}=2^p.\frac{2^p}{5^p}=\frac{4^p}{5^p}<1$ since $s=2^{p-1}$. We see that $f(X)\subseteq g(X)$, $g(X)$ is complete, f and g are weakly compatible. Therefore, all the conditions of Theorem 4.5 are satisfied and $0\in X$ is the unique common fixed point of f and g.

Example 4.11. Let $X=\{1,2,3\}$, $E=\mathbb{R}^2$, $P=\{(x,y):x\geq 0,y\geq 0\}$. Define $d:X\times X\to P$ by $d(x,y)=d(y,x)$ for all $x,y\in X$, $d(x,x)=\theta$, $x\in X$ and $d(1,2)=(8,8)$, $d(2,3)=d(1,3)=(2,2)$. We observe that

$$d(1,2)=(8,8)\npreceq d(1,3)+d(3,2)=(2,2)+(2,2)=(4,4).$$

This shows that the triangle inequality does not hold true and so (X,d) is not a cone metric space. It is easy to verify that (X,d) is a cone b-metric space with the coefficient $s=2$. Let us define $f,g:X\to X$ as

$$fx=3,\ for\ all\ x\in X$$

and

$$\begin{aligned} gx &= 3,\ for\ x\in\{1,3\}\\ &= 1,\ for\ x=2. \end{aligned}$$

Then for every $x,y\in X$ one has

$$d(fx,fy)\preceq a\,d(fx,gy)+b\,d(fy,gx)$$

for all $a,\ b\geq 0$.

We see that $f(X)\subseteq g(X)$, $f(X)$ is complete, f and g are weakly compatible. Therefore, all the conditions of Theorem 4.7 are satisfied and $3\in X$ is the unique common fixed point of f and g.

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References

1. M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341 (2008), 416 – 420.
2. I.A.Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal.,Gos. Ped. Inst. Unianowsk 30 (1989), 26 – 37.
3. I. Beg and M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, Fixed Point Theory and Applications 2006 (2006), Article ID 74503, 7 pages.

4. S.Czerwik, Nonlinear set valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fiz. Univ. Modena 46 (1998), 263 – 276.
5. Z.M.Fadail and A.G.B.Ahmad, Coupled coincidence point and common coupled fixed point results in cone b-metric spaces, Fixed Point Theory and Applications 2013 (2013), 2013:177.
6. L.-G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468 – 1476.
7. O.Hadzic, Common fixed point theorem for family of mappings in complete metric spaces, Math. Japon 29 (1984), 127 – 134.
8. N.Hussain and M.H.Shah, KKM mappings in cone b-metric spaces, comput. Math. Appl. 62 (2011), 1677 – 1684.
9. H.Huang and S.Xu, Fixed point theorems of contractive mappings in cone b-metric spaces and applications, Fixed Point Theory and Applications 2013 (2013), 2013:112.
10. D.Ilić, V.Rakočević, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341 (2008), 876 – 882.
11. G. Jungck, Common fixed points for noncontinuous nonself maps on non-metric spaces, Far East J. Math. Sci. 4 (1996), 199 – 215.
12. J. O. Olaleru, Some generalizations of fixed point theorems in cone metric spaces, Fixed Point Theory and Applications 2009 (2009), Article ID 657914.
13. H.K.Pathak, Weak commuting mappings and fixed points, Ind. J. Pure Appl. Math. 17 (1986), 201 – 211.
14. B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Analysis: Theory, Methods and Applications 47 (2001), 2683 – 2693.
15. S.Rezapour, R.Hamlbarani, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", J. Math. Anal. Appl. 345 (2008), 719 – 724.
16. S.Rezapour, M.Derafshpour and R.Hamlbarani, A review on topological properties of cone metric spaces, in Proceedings of the Conference on Analysis, Topology and Applications (ATA'08), Vrnjacka Banja, Serbia, May-June 2008.
17. S.P.Singh and B.A.Meade, On common fixed point theorem, Bull. Aus. Math. Soc. 16 (1977), 49 – 53.
18. P. Vetro, Common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo 56 (2007), 464 – 468.