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VISCOSITY ITERATIVE SCHEME FOR SPLIT FEASIBILITY PROBLEMS

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ABSTRACT. In this paper, we intend to solve a split feasibility problem by viscosity iterative algorithm. The bounded perturbation resilience of the method is examined in Hibert spaces. As tools, averaged mappings and resolvents of maximal monotone operators are the specialized procedure to simplify the proofs of the main results. Under mild conditions, we prove that our algorithms converge to a solution of the split feasibility problem. Moreover, we show the convergence and result of the algorithms by a numerical example.

KEYWORDS: Viscosity iterative algorithm, Split feasibility problem, Maximal monotone operator.
AMS Subject Classification: 47H09, 47H10.

1. Introduction

Let C and Q be nonempty closed convex subsets in real Hibert space H_1 and H_2,respectively. Let P_C be the metric projection from H_1 onto C and P_Q be the metric projection from H_2 onto Q. The problem to find

$$u^* \in C \quad with \quad Au^* \in Q \tag{1.1}$$

where A is a bounded linear operator from H_1 to H_2, if such u^* exist, this problem is called the split feasibility problem(see [1]). If problem (1.1) has a solution (say that $C \cap A^{-1}Q$ is nonempty). $u^* \in C \cap A^{-1}Q$ is equivalent to

$$u^* = P_C(I - \lambda A^*(I - P_Q)A)u^*, \tag{1.2}$$

where $\lambda > 0$ and A^* is the adjoint operator of A.

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The SFP was first introduced by Censor and Elfving [2] in 1994. They used their multidistance method to obtain iterative algorithms for solving the SFP. After that, Byrne [3] proposed his CQ algorithm which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(I - \lambda A^*(I - P_Q)A)x_n, \quad \forall n \geq 0. \tag{1.3}$$

Let $B : H_1 \to 2^{H_1}$ be a mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for all $\lambda > 0$. Let $T : H_2 \to H_2$ be nonexpansive mapping.

In 2015, Takahashi et al. [4] proposed the following algorithm:

$$x_{n+1} = J^B_{\gamma_n}(x_n - \lambda_n A^*(I - T)Ax_n), \tag{1.4}$$

where $\{\lambda_n\}$ is a sequence in $[0, 1]$ and they proved that the sequence $\{u_n\}$ converges weakly to a point $u^* \in B^{-1}0 \cap A^{-1}Fix(T)$ in the framework of Hilbert spaces. That is this problems is to find a point $u^* \in H_1$ such that

$$0 \in Bu^* \quad and \quad Au^* \in Fix(T). \tag{1.5}$$

The set of all solution (1.5) denoted by $\Gamma = B^{-1}0 \cap A^{-1}Fix(T)$.
there are many authers have studied the SFP and its extensions by means of fixed-point methods and weak-strong convergence theorems of solutions have been established in Hilbert or Banach spaces (see [5, 6, 7, 8]).

Let F be an algorithm operator. Let $\{x_n\}$ be a sequence, generated by $x_{n+1} = Fx_n$, and let $\{y_n\}$ be a sequence, generated by $y_{n+1} = F(y_n + \beta_n v_n)$, where $\{\beta_n\}$ is a sequence of nonnegative real numbers and $\{v_n\}$ is a sequence in H such that

$$\sum_{n=0}^{\infty} \beta_n < \infty \quad and \quad \|v_n\| \leq M, \quad \forall n \geq 0. \tag{1.6}$$

An algorthmic operator F is call bounded perturbation resilient if the following is ture: if the sequence $\{x_n\}$ is convergent, then $\{y_n\}$ is also convergent (see [9]).

In 2017, Xu [10] presented the bounded perturbation resilience and superiorization techniques for the projected scaled gradient(PSG).The iterative method is defined as following:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n P_C(x_n - \lambda_n D(x_n)\nabla h(x_n) + e(x_n)), \quad \forall n \geq 0 \tag{1.7}$$

where $\{\lambda_n\}$, $\{\alpha_n\}$ are a sequence in $[0, 1]$, h is a continuous differentiable and convex function, and $D(x_n)$ is a diagonal scaling matrix. The weak convergence was proved in [10].

In 2018, Guo and Chi [11] proposed the following proximal gradient algorithm with perturbations:

$$x_{n+1} = t_n f(x_n) + (1 - t_n)prox_{\lambda_n g}(1 - \lambda_n \nabla h)x_n + e(x_n), \tag{1.8}$$

where $\{\lambda_n\}$, $\{t_n\}$ are a sequence in $[0, 1]$ and f is a contractive, for solving nonsmooth composite convex optimization problem. They obtained strong convergence and bounded resilience of the above method.

In 2019, Duan and Zheng [12] presented a viscosity approximation method for solving problem (1.5):

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)J^B_{\lambda_n}(x_n - \tau_n A^*(1 - T)Ax_n + e(x_n)), \tag{1.9}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\}$ are a sequence in $[0,1]$ and they gave the bounded perturbation of (1.9) yields a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ x_{n+1} &= \alpha_n f(y_n) + (1-\alpha_n) J^B_{\lambda_n}(y_n - \tau_n A^*(I-T)Ay_n + e(y_n)), \end{aligned} \tag{1.10}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\},\ \{\beta_n\}$ are a sequence in $[0,1]$

In this paper, we extend work in [12] and purpose the following process for solving problem (1.5) :

$$x_{n+1} = \alpha_n f(x_n + e(x_n)) + (1-\alpha_n) J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n + e(x_n)), \tag{1.11}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\}$, are a sequence in $[0,1]$, f is contractive, and we give a sequence $\{x_n\}$ generated by the iterative process:

$$\begin{aligned} y_n &= x_n + \beta_n v_n \\ x_{n+1} &= \alpha_n f(y_n + e(y_n)) + (1-\alpha_n) J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)), \end{aligned} \tag{1.12}$$

where $\{\lambda_n\},\ \{\tau_n\},\ \{\alpha_n\},\ \{\beta_n\}$ are a sequence in $[0,1]$ and f is contractive.

After that we prove the convergence point of the iterative method which is also the unique solution of some variational inequality problem. A numerical example is also given to demonstrate the effectiveness of our iterative schemes.

2. Preliminaries

Let $\{x_n\}$ be a sequence in a real Hilbert space H. First, We give notations:

- Denote $\{x_n\}$ converging weakly to x by $x_n \rightharpoonup x$ and $\{x_n\}$ converging strongly to x by $x_n \to x$.
- Denote the set of fixed points of mapping T by $Fix(T) = \{x \in H : Tx = x\}$
- Denotetheweak ω-limit set of $\{x_n\}$ by $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$.

Definition 2.1. A mapping $F : H \to H$ is said to be

(i) Lipschizian if there exist a positive constant L such that

$$||Fx - Fy|| \le L||x - y||, \quad \forall x, y \in H.$$

In particular, if $L = 1$, we say that F is nonexpansive, namely,

$$||Fx - Fy|| \le L||x - y||, \quad \forall x, y \in H,$$

if $L \in [0, 1)$, we say that F is contractive.

(ii) α-averaged mapping (α-av for short) if

$$F = (1-\alpha)I + \alpha T,$$

where $\alpha \in [0, 1)$ and $T : H \to H$ is nonexpansive.

Definition 2.2. A mapping $B : H \to H$ is said to be

(i) monotone if

$$\langle Bx - By, x - y\rangle \le 0, \quad \forall x, y \in H.$$

(ii) $\eta-$strongly monotone if there exists a posive constant eta such that

$$\langle Bx - By, x - y\rangle \ge \eta||x - y||^2, \quad \forall x, y \in H.$$

(iii) α-inverse strongly monotone (for short α-ism) if there exist a positive constant α such that

$$\langle Bx - By, x - y\rangle \geq \alpha||Bx - By||^2, \quad \forall x, y \in H.$$

In paticular, if $\alpha = 1$, we say that B is firmly nonexpansive, namely,

$$\langle Bx - By, x - y\rangle \geq ||Bx - By||^2, \quad \forall x, y \in H.$$

Definition 2.3. Let $B : H \to H$ be amonotone mapping. Then B is maximal monotone if there exists no monotone operater $A : H \to 2^H$ such that $graA$ properly contains $graB$,i.e. for every $(x, u) \in H \times H$,

$$(x, u) \in graB \Leftrightarrow \forall(y, v) \in graB, \langle x - y, u - v\rangle \geq 0.$$

Lemma 2.4. *Let H be a real Hillbert space. Theere holds the following inequality*

$$||x + y||^2 \leq ||x||^2 + 2\langle x + y, y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5. *Let $f : H \to H$ be a $k \in (0, 1)$ and let $T : H \to H$ be a nonexpansive mapping.Then*

(i) *$I - f$ is (1-k)-strongly monotone:*

$$\langle(I - f)x - (I - f)y, x - y\rangle \geq (1 - \rho)||x - y||^2, \quad \forall x, y \in H.$$

(ii) *$I - T$ is monotone:*

$$\langle(I - T)x - (I - T)y, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

Proposition 2.6. [4] *Assume that H_1 and H_2 are Hilbert space. Let $B : H_1 \to 2_1^H$ be a maximal monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then*

(i) *$A^*(I - T)A$ is $\frac{1}{2||A||^2}$-ism.*

(ii) *For $0 < \tau < \frac{1}{2||A||^2}$,*
$I - \tau A^*(I - T)A$ is $\tau||A||^2$- averaged and $J_\lambda^B(I - \tau A^*(I - T)A)$ is $\frac{1+\tau||A||^2}{2}$ averaged.

Lemma 2.7. [13] *Let B be a maximal monotone operator. Let $J_\gamma^B = (I + \gamma B)^{-1}$ and $J_\lambda^B = (I + \lambda B)^{-1}$, where $\gamma > 0$ and $\lambda > 0$ are two real numbers, be the resolvent operators of B. Then*

$$J_\gamma^B x = J_\lambda^B(\frac{\lambda}{\gamma}x + (1 - \frac{\lambda}{\gamma})J_\gamma^B x), \quad \forall x \in H.$$

Lemma 2.8. [14] *Let H be a real Hillbert space, and let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq 0$.if $\{x_n\}$ is a sequence in H weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y,then $(I - T) = y$; in particulary, if $y = 0$, then $x \in Fix(T)$.*

Lemma 2.9. [15] *Assume $\{\sigma_n\}$ is a sequence of nonnegative real numbers such that*

$$\sigma_{n+1} \leq (1 - \rho_n)\sigma_n + \rho_n\delta_n, \quad n \geq 0,$$
$$\sigma_{n+1} \leq \sigma_n - \varphi_n + \phi_n, \quad n \geq 0,$$

where $\{\rho_n\}$ is a sequence in $(0, 1)$, $\{\varphi_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\phi_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^\infty \rho_n = \infty$;
(ii) $\lim_{n\to\infty} \phi_n = 0$;
(iii) $\lim_{k\to\infty} \varphi_{n_k} = 0 \Rightarrow \limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence $(n_k) \subset (n)$.*

Then $\lim_{n\to\infty} \sigma_n = 0$.

Lemma 2.10. [4] *Let* H_1 *and* H_2 *be Hilbert space. Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone mapping and let* $J_\lambda^B = (I + \lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$. *Let* $T : H_2 \to H_2$ *be a nonexpansive mapping and let* $T : H_1 \to H_2$ *be a bounded linear operater. Suppose that* $B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. *Let* $\lambda, \tau > 0$. *Then the following equality holds:*

$$Fix(J_\lambda^B(I - \tau A^*(I - T)A)) = (A^*(I - T)A + B)^{-1}0 = B^{-1}0 \cap A^{-1}Fix(T)$$

3. Main Results

In [1] proposed the viscosity approximation method:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)S(x_n), \quad \forall n \geq 0,$$

which converges strongly to a fixed point u^* of the nonexpansive mapping S. In [5] further proved that $u^* \in Fix(S)$ is also the unique solution of the following variational inequality problem:

$$\langle (I - f)u^*, \hat{u} - u^* \rangle \geq 0, \quad \forall \hat{u} \in Fix(S), \tag{3.1}$$

where $f : H \to H$ is a k-contraction.

In this section, we present a viscosity iterative algorithm for solving problem (1.5). Rewrite iteration (1.11) as

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n + e(x_n)) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &= \alpha_n f(x_n) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n) + \widehat{e}_n, \quad \forall n \geq 0, \end{aligned}$$

where

$$\begin{aligned} \widehat{e}_n = {} & \alpha_n(f(x_n + e(x_n)) - f(x_n)) + (1 - \alpha_n)(J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ & - J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n)). \end{aligned}$$

Since $J_{\gamma_n}^B$ is nonexpansive and f is contractive, it is easy to get

$$\begin{aligned} \|\hat{e}_n\| &\leq \alpha_n\|f(x_n + e(x_n)) - f(x_n)\| + (1 - \alpha_n)\|J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \\ &\quad - J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n)\| \\ &\leq \alpha_n k\|e(x_n)\| + (1 - \alpha_n)\|e(x_n)\| \\ &= (\alpha_n k + 1 - \alpha_n)\|e(x_n)\| \\ &\leq \|e(x_n)\|. \end{aligned}$$

Theorem 3.1. *Let* H_1, H_2 *be two real Hilbert spaces and let* $A : H_1 \to H_2$ *be a bounded linear operator with* $L = \|A^*A\|$, *where* A^* *is the adjoint of* A. *Suppose that* $B : H_1 \to 2^{H_1}$ *is a maximal monotone operator and* $T : H_2 \to H_2$ *is a nonexpansive mapping. Assume that* $\Gamma = B^{-1}0 \cap A^{-1}Fix(T) \neq \emptyset$. *Let* f *be a* k*-contractive on* H_1 *with* $0 \leq k < 1$. *Choose* $x_0 \in H_1$ *arbitrarily and define a sequence* $\{x_n\}$ *in the following manner:*

$$x_{n+1} = \alpha_n f(x_n + e(x_n)) + (1 - \alpha_n)J_{\gamma_n}^B(x_n - \lambda_n A^*(I - T)Ax_n + e(x_n)) \tag{3.2}$$

if the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^\infty \|e(x_n)\| < \infty$.

Then $\{x_n\}$ converges strongly to $u^* \in \Gamma$, which is also the unique solution of variational inequality problem (3.1).

Proof. Let $V_{\lambda_n} = J^B_{\gamma_n}((I - \lambda_n A^*(I-T)A)$. From Proposition 2.6, it follows that $J^B_{\gamma_n}((I - \lambda_n A^*(I-T)A)$ is $\frac{1+\lambda_n L}{2} - av$ as $0 < \lambda_n < \frac{1}{L}$.
Step 1. show that $\{x_n\}$ is bounded. For any $u^* \in \Gamma$, we have

$$
\begin{aligned}
&\|x_{n+1} - u^*\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n + \widehat{e}_n - u^*\| \\
&= \|\alpha_n f(x_n) + V_{\lambda_n}x_n - \alpha_n V_{\lambda_n}x_n + \widehat{e}_n - u^*\| \\
&= \|(\alpha_n f(x_n) - \alpha_n u^*) + (V_{\lambda_n}x_n - u^*) - (\alpha_n V_{\lambda_n}x_n - \alpha_n u^*) + \widehat{e}_n\| \\
&= \|\alpha_n (f(x_n) - u^*) + (1-\alpha_n)(V_{\lambda_n}x_n - u^*) + \widehat{e}_n\| \\
&\le \alpha_n \|f(x_n) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - u^*)\| + \|\widehat{e}_n\| \\
&= \alpha_n \|f(x_n) - f(u^*) + f(u^*) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - u^*)\| + \|\widehat{e}_n\| \\
&\le \alpha_n \|f(x_n) - f(u^*)\| + \alpha_n\|f(u^*) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - u^*)\| + \|\widehat{e}_n\| \\
&= \alpha_n \|f(x_n) - f(u^*)\| + \alpha_n\|f(u^*) - u^*\| + (1-\alpha_n)\|(V_{\lambda_n}x_n - V_{\lambda_n}u^*)\| + \|\widehat{e}_n\| \\
&\le \alpha_n k\|x_n - u^*\| + \alpha_n\|f(u^*) - u^*\| + (1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\| \\
&= (1 - \alpha_n + \alpha_n k)\|x_n - u^*\| + \alpha_n(\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}) \\
&= (1 - \alpha_n(1-k))\|x_n - u^*\| + \alpha_n(1-k)\left[\frac{\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}}{1-k}\right].
\end{aligned}
$$

From condition (i), (iv) and $\alpha_n > 0$, we get $\left\{\frac{\|\widehat{e}_n\|}{\alpha_n}\right\}$ is bounded.Thus there exists $M_1 > 0$ such that $\sup\left\{\|f(u^*) - u^*\| + \frac{\|\widehat{e}_n\|}{\alpha_n}\right\} \le M_1$, for all $n \ge 0$. By Mathematical Induction, we get $\|x_n - u^*\| \le \max\left\{\|x_0 - u^*\|, \frac{M_1}{1-k}\right\}$, which implies that the sequence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{V_{\lambda_n}x_n\}$ and $\{A^*(I-T)Ax_n\}$.
Step 2. Show that for any sequence$\{n_k\} \subset \{n\}$,

$$\lim_{n\to\infty} \|x_{n_k} - V_{\lambda_{n_k}}x_{n_k}\| = 0.$$

Fixing $u^* \in \Gamma$, we have

$$
\begin{aligned}
&\|x_{n+1} - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n + \widehat{e}_n - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&= \|\alpha_n (f(x_n) - u^*) + (1-\alpha_n)(V_{\lambda_n}x_n - u^*)\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&\le \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + 2\|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&= \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\|\alpha_n(f(x_n) - u^*) + (1-\alpha_n)(V_{\lambda_n}x_n - u^*)\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&\leq \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + 2\Big[\|\alpha_n(f(x_n) - u^*)\| + \|(1-\alpha_n)(V_{\lambda_n}x_n - u^*)\|\Big]\|\widehat{e}_n\| + \|\widehat{e}_n\|^2 \\
&= \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + \Big[2\alpha_n\|(f(x_n) - u^*)\| + 2(1-\alpha_n)\|V_{\lambda_n}x_n - u^*\| + \|\widehat{e}_n\|\Big]\|\widehat{e}_n\| \\
&\leq \alpha_n^2\|f(x_n) - u^*\|^2 + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle \\
&\quad + \Big(2\alpha_n\|f(x_n) - u^*\| + 2(1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\|\Big)\|\widehat{e}_n\| \\
&\leq 2\alpha_n^2\Big(\|f(x_n) - f(u^*)\|^2 + \|f(u^*) - u^*\|^2\Big) + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x_n) - u^*, V_{\lambda_n}x_n - u^*\rangle + M_2\|\widehat{e}_n\| \\
&\leq 2\alpha_n^2\Big(\|f(x_n) - f(u^*)\|^2 + \|f(u^*) - u^*\|^2\Big) + (1-\alpha_n)^2\|V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\Big(\|f(x_n) - f(u^*)\|\|x_n - u^*\| + \langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle\Big) + M_2\|\widehat{e}_n\| \\
&\leq 2\alpha_n^2 k\|x_n - u^*\|^2 + 2\alpha_n^2\|f(u^*) - u^*\|^2 + (1-\alpha_n^2)\|x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)k\|x_n - u^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle + M_2\|\widehat{e}_n\| \\
&= \Big(2\alpha_n^2 k + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)k\Big)\|x_n - u^*\|^2 + 2\alpha_n^2\|f(u^*) - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle + M_2\|\widehat{e}_n\| \\
&= (1 - \alpha_n(2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k))\|x_n - u^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle + 2\alpha_n^2\|f(u^*) - u^*\|^2 + M_2\|\widehat{e}_n\|, \qquad (3.3)
\end{aligned}
$$

where

$$M_2 = \sup_{n\in\mathbb{N}}\Big\{2\alpha_n\|f(x_n) - u^*\| + 2(1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\|\Big\}.$$

Note that

$$V_{\lambda_n} = J_{r_n}^B(I - \lambda_n A^*(I-T)A) = (1-w_n)I + w_n U_n, \qquad (3.4)$$

such that $w_n = \frac{1+\lambda_n L}{2}$, and U_n is nonexpansive. By condition (iii), we get

$$\frac{1}{2} < \liminf_{n\to\infty} w_n \leq \limsup_{n\to\infty} w_n < 1$$

Since $u^* \in \Gamma$, then $V_{\lambda_n}u^* = u^*$. Furthermore, we have $(1-w_n)u^* + w_n U_n u^* = u^*$. It is clear that $U_n u^* = u^*$.

$$
\begin{aligned}
&\|x_{n+1} - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n + \widehat{e}_n - u^*\|^2 \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\langle \alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*, \widehat{e}_n\rangle + \|\widehat{e}_n\|^2 \\
&\leq \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + 2\|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|\|\widehat{e}_n\| + \|\widehat{e}_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 \\
&\quad + \Big(2\alpha_n\|f(x_n) - u^*\| + 2(1-\alpha_n)\|x_n - u^*\| + \|\widehat{e}_n\|\Big)\|\widehat{e}_n\| \\
&\le \|\alpha_n f(x_n) + (1-\alpha_n)V_{\lambda_n}x_n - u^*\|^2 + M_2\|\widehat{e}_n\| \\
&= \|V_{\lambda_n}x_n - u^* + \alpha_n(f(x_n) - V_{\lambda_n}x_n)\|^2 + M_2\|\widehat{e}_n\| \\
&= \|V_{\lambda_n}x_n - u^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|(1-w_n)x_n + w_nU_nx_n - u^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|x_n - w_nx_n + w_nU_nx_n - (1-w_n)u^* - w_nU_nu^*\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= \|(1-w_n)(x_n - u^*) + w_n(U_nx_n - U_nu^*)\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&= (1-w_n)\|x_n - u^*\|^2 + w_n\|U_nx_n - U_nu^*\|^2 - w_n(1-w_n)\|U_nx_n - x_n\|^2 \\
&\quad + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\| \\
&\le \|x_n - u^*\|^2 - w_n(1-w_n)\|U_nx_n - x_n\|^2 + \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 \\
&\quad + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\|. \qquad (3.5)
\end{aligned}
$$

Furthermore, we set

$$
\begin{aligned}
&\sigma_n = \|x_n - u^*\|^2, \quad \rho_n = \alpha_n(2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k), \\
&\delta_n = \frac{1}{2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k}\Big[2\alpha_n\|f(u^*) - u^*\|^2 + M_2\frac{\|\widehat{e}_n\|}{\alpha_n} \\
&\qquad + 2(1-\alpha_n)\langle f(u^*) - u^*, V_{\lambda_n}x_n - u^*\rangle\Big], \\
&\varphi_n = w_n(1-w_n)\|U_nx_n - x_n\|^2, and \\
&\phi_n = \alpha_n^2\|f(x_n) - V_{\lambda_n}x_n\|^2 + 2\alpha_n\langle V_{\lambda_n}x_n - u^*, f(x_n) - V_{\lambda_n}x_n\rangle + M_2\|\widehat{e}_n\|.
\end{aligned}
$$

Note that

$$
\rho_n \to 0, \quad \sum_{n=0}^{\infty}\rho_n = \infty \quad (\lim_{n\to\infty}(2 - \alpha_n(1+2k^2) - 2(1-\alpha_n)k) = 2(1-k) > 0)
$$

and $\phi_n \to 0 \quad (\alpha_n \to 0)$. By lemma 2.9, we have $\varphi_{n_k} \to 0 \quad (k \to \infty)$ implies that $\lim_{k\to\infty}\sup\delta_{n_k} \le 0$ for any subsequence $\{n_k\} \subset \{n\}$. Indeed, $\varphi_{n_k} \to 0 (k \to \infty)$ implies that $\|U_{n_k}x_{n_k} - x_{n_k}\| \to 0 (k \to \infty)$ due to condition (iii). From (3.3) we have

$$
\|x_{n_k} - V_{\lambda_{n_k}}x_{n_k}\| = w_{n_k}\|x_{n_k} - U_{n_k}x_{n_k}\| \to 0. \qquad (3.6)
$$

Step 3. Show that

$$
\omega_w\{x_{n_k}\} \subset \Gamma \qquad (3.7)
$$

where $\omega_w\{x_{n_k}\}$ is the set of all weak cluster points of $\{x_{n_k}\}$.
Let $\widehat{u} \in \omega_w\{x_{n_k}\}$ and $x_{n_{k_j}}$ is a subsequence of x_{n_k} weakly converging to $\widehat{u}$. We use $\{x_{n_k}\}$ to denote $x_{n_{k_j}}$ and we assume that $\lambda_{n_k} \to \lambda$.Then $0 < \lambda < \frac{1}{L}$. In the same

way, we take a subsequence $\{\gamma_{n_k}\}$ of γ_n by condition (ii) and assume that $\gamma_{n_k} \to \gamma$. Let $V_\lambda = J_\gamma^B(I - \lambda A^*(I-T))A$, we see that V is nonexpansive. Set

$$t_k = x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k}; \quad z_k = x_{n_k} - \lambda_n A^*(I-T)Ax_{n_k}.$$

By the resolvent identity, we conclude that

$$\begin{aligned}
&\|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \\
&= \|J^B_{\gamma_{n_k}}(x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k}) - J^B_\gamma(x_{n_k} - \lambda A^*(I-T)Ax_{n_k})\| \\
&= \|J^B_{\gamma_{n_k}}(t_k) - J^B_\gamma(z_k)\| \\
&= \|J^B_\gamma(\frac{\gamma}{\gamma_{n_k}} t_k + (1 - \frac{\gamma}{\gamma_{n_k}})J^B_{\gamma_{n_k}}) - J^B_{\gamma_n}(z_k)\| \\
&\le \|\frac{\gamma}{\gamma_{n_k}} t_k + (1 - \frac{\gamma}{\gamma_{n_k}})J^B_{\gamma_{n_k}}) - z_k\| \\
&= \|\frac{\gamma}{\gamma_{n_k}} t_k - \frac{\gamma}{\gamma_{n_k}} z_k + (1 - \frac{\gamma}{\gamma_{n_k}})J^B_{\gamma_{n_k}}) - z_k + \frac{\gamma}{\gamma_{n_k}} z_k\| \\
&= \|\frac{\gamma}{\gamma_{n_k}}(t_k - z_k) + (1 - \frac{\gamma}{\gamma_{n_k}})(J^B_{\gamma_{n_k}} t_k - z_k\| \\
&\le \frac{\gamma}{\gamma_{n_k}}\|t_k - z_k\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}\|x_{n_k} - \lambda_{n_k} A^*(I-T)Ax_{n_k} - x_{n_k} + \lambda A^*(I-T)Ax_{n_k}\| \\
&\quad + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}\| - (\lambda_{n_k} - \lambda)A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}| - (\lambda_{n_k} - \lambda)|\|A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\| \\
&= \frac{\gamma}{\gamma_{n_k}}|(\lambda_{n_k} - \lambda)|\|A^*(I-T)Ax_{n_k}\| + (1 - \frac{\gamma}{\gamma_{n_k}})\|J^B_{\gamma_{n_k}} t_k - z_k\|. \qquad (3.8)
\end{aligned}$$

Since $\gamma_{n_k} \to \gamma$ and $\lambda_{n_k} \to \lambda$ as $k \to \infty$, then $\|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \to 0$. As a result, we get

$$\|x_{n_k} - V_\lambda x_{n_k}\| \le \|x_{n_k} - V_{\lambda_{n_k}} x_{n_k}\| + \|V_{\lambda_{n_k}} x_{n_k} - V_\lambda x_{n_k}\| \to 0 \qquad (3.9)$$

From lemma 2.8, we have $\omega_w\{x_{n_k}\} \subset Fix(V_\lambda)$. It follows from lemma 2.10 that $\omega_w\{x_{n_k}\} \subset S$. We also have

$$\begin{aligned}
\limsup_{k\to\infty}\langle f(u^*) - u^*, V_{\lambda_{n_k}} x_{n_k} - u^*\rangle &= \limsup_{k\to\infty}\langle f(u^*) - u^*, x_{n_k} - u^*\rangle \\
&\quad + \limsup_{k\to\infty}\langle f(u^*) - u^*, V_{\lambda_{n_k}} x_{n_k} - x_{n_k}\rangle \qquad (3.10)
\end{aligned}$$

and

$$\limsup_{k\to\infty}\langle f(u^*) - u^*, x_{n_k} - u^*\rangle = \langle f(u^*) - u^*, \widehat{u} - u^*\rangle, \forall \widehat{u} \in \Gamma. \qquad (3.11)$$

It is easy to get from (3.10) tend to zero. Since u^* is the unique solution of variational inequality problem (3.1), we get

$$\langle f(u^*) - u^*, \widehat{u} - u^*\rangle \le 0.$$

Hence

$$\limsup_{k\to\infty} \delta_{n_k} \le 0.$$

The bounded perturbation of (3.2) by the following iterative method:

$$\begin{cases} y_n = x_n + \beta_n v_n, \\ x_{n+1} = \alpha_n f(y_n + e(y_n)) + (1-\alpha_n) J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)), \end{cases} \tag{3.12}$$

where $\{\lambda_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are a sequence in $[0,1]$ and f is contractive.

□

Theorem 3.2. *Let $\{\beta_n\}$ and $\{v_n\}$ be satisfied by condition (1.6). Let H_1, H_2 be two real Hilbert spaces and let A be a bounded linear operator with $L = \|A^*A\|$, where A^* is the adjoint of A. Suppose that $\Gamma = B^{-1}0 \cap A^{-1}Fix(T) \neq 0$. Let f be k-contrative mapping on H, with $0 \le k < 1$. Choose $x_0 \in H_1$arbitrarily and define the sequence $\{x_n\}$ by (3.12).If the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \infty$;
(iii) $0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < \frac{1}{L}$;
(iv) $\sum_{n=0}^{\infty} \|e(y_n)\| < \infty$.

Then $\{x_n\}$ converges strongly to u^*, where u^* is a solution of problem (1.5),which is also the unique solution of variational inequality problem (3.1)

Proof. we can rewrite (3.12) as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n) + \widehat{e}_n, \tag{3.13}$$

where

$$\begin{aligned} \widehat{e}_n = {} & \alpha_n(f(y_n + e(y_n)) - f(x_n)) + (1-\alpha_n)(J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)) \\ & - J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n)), \end{aligned} \tag{3.14}$$

Since $A^*(I-T)A$ is $\frac{1}{2L}$-ism, then it is $2L$-Lipschitz. Thus,

$$\begin{aligned} & \|\widehat{e}_n\| \\ & \le \alpha_n \|f(y_n + e(y_n)) - f(x_n)\| \qquad (3.15) \\ & \quad + (1-\alpha_n)\|J^B_{\gamma_n}(y_n - \lambda_n A^*(I-T)Ay_n + e(y_n)) - J^B_{\gamma_n}(x_n - \lambda_n A^*(I-T)Ax_n)\| \\ & \le \alpha_n k \|y_n + e(y_n) - x_n\| \\ & \quad + (1-\alpha_n)\|y_n - \lambda_n A^*(I-T)Ay_n + e(y_n) - (x_n - \lambda_n A^*(I-T)Ax_n)\| \\ & = \alpha_n k \|y_n - x_n + e(y_n)\| \\ & \quad + (1-\alpha_n)\|y_n - x_n - \lambda_n(A^*(I-T)Ay_n - A^*(I-T)Ax_n) + e(y_n)\| \\ & \le \alpha_n k \|y_n - x_n\| + \alpha_n k \|e(y_n)\| \\ & \quad + (1-\alpha_n)\Bigg(\|y_n - x_n\| + \lambda_n \|A^*(I-T)Ay_n - A^*(I-T)Ax_n\| + e(y_n)\|\Bigg) \\ & \le \alpha_n k \|y_n - x_n\| + \alpha_n k \|e(y_n)\| \\ & \quad + (1-\alpha_n)\Bigg(\|y_n - x_n\| + 2\lambda_n L \|y_n - x_n\| + \|e(y_n)\|\Bigg) \\ & = (\alpha_n k + 1 + 2\lambda_n L - \alpha_n - 2\alpha_n \lambda_n L)\|y_n - x_n\| + (\alpha_n k + 1 + \alpha_n)\|e(y_n)\| \\ & = (\alpha_n k + (1-\alpha_n)(1 + 2\lambda_n L))\|y_n - x_n\| + (1 + (1+k)\alpha_n)\|e(y_n)\| \\ & \le (\alpha_n k + (1-\alpha_n)(1 + 2\lambda_n L))\beta_n \|V_n\| + (1 + (1-k)\alpha_n)\|e(y_n)\| \qquad (3.16) \end{aligned}$$

From (1.6) and

$$\sum_{n=1}^{\infty} \|e(y_n)\| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \|\widehat{e}_n\| < \infty$$

Thus, we find from theorem 3.1 that algorithm (3.2) is bounded perturbation resilient.

□

4. NUMERICAL RESULTS

In this section, we consider the following numerical examples to present the effectiveness, realization and convergence of Theorem 3.1.

Example 4.1. Let $H_1 = H_2 = \mathbb{R}^2$. Define $h(x) = \frac{1}{14}x$. Take $B : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ asfollows:

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}, \quad T : y = (y(1), y(2)^T \longmapsto (y(1), y(2)+\sin(y(2)))^T \text{ and } \quad e(x) = \begin{pmatrix} \frac{1}{n^4} \\ \frac{1}{n^4} \end{pmatrix}.$$

Observe that B is a positive linear operator. Then it is maximal monotone. T is $\frac{1}{2}$-av and the set of fixed points $Fix(T) = \{y \,|\, (y(1), 0)^T\}$ is nonempty. Then it is nonexpansive. Hence, we obtain the resolvent mapping $J_\gamma^B = (I + \gamma B)^{-1}$. It follows that

$$J_\gamma^B = \frac{1}{(3\gamma+1)(9\gamma+1)} \begin{pmatrix} 9\gamma+1 & 0 \\ 0 & 3\gamma+1 \end{pmatrix}$$

. Generatea 2×2 random matrix A, and compute the Lipschitz constant $L = \|A^T A\|$, where A^T represents the transpose of A. Take $\gamma_n = 0.9$, $\lambda_n = \lambda = \frac{1}{100L}$ and $\alpha_n = \frac{1}{4n+5}$.

According to the iterative process of Theorem 3.1, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \frac{1}{4n+5}\left(\frac{1}{14}(x_n + e(x_n)) + (1 - \frac{1}{4n+5}\right)J_{\gamma_n}^B(x_n - \lambda_n A^T(I-T)Ax_n + e(x_n)).$$

As $n \to \infty$, we have $\{x_n\} \to u^*$. Taking random initial guess x_0 and the stopping criteria is $\|x_{n+1} - x_n\| < \epsilon$, we obtain the numerical experiment results in Table 1.

TABLE 1. $x_0 = rand(2,1)$

ϵ	$\lambda_n = \frac{1}{100L}$	n	Time	x_n	$\|x_{n+1} - x_n\|$
10^{-6}	0.006745	20	0.010811	(0.000003, 0.000001)	8.068095×10^{-7}
10^{-7}	0.022563	30	0.013420	(0.000001, 0.000000)	8.854955×10^{-8}
10^{-8}	0.011875	46	0.009761	(0.000000, 0.000000)	9.290480×10^{-8}

Next, we consider the algorithm with bounded perturbation resilience. Choose the bounded sequence $\{v_n\}$ and the summable nonnegative real sequence $\{\beta_n\}$ as follows:

$$v_n = \begin{cases} -\dfrac{d_n}{\|d_n\|}, & \text{if } 0 \neq d_n \in B(x_n), \\ 0, & \text{if } 0 \in B(x_n), \end{cases}$$

where $B(x_n) = (3x_n(1),\, 9x_n(2))^T,\ x_n(i),\ i = 1,2$ denote the ith element of x_n, and $\beta_n = c^n$, for some $c \in (0,1)$. Setting $c = 0.9$, the numerical results can be seen in Table 2.

TABLE 2. $x_0 = rand(2,1)$

ϵ	$\lambda_n = \frac{1}{100L}$	n	Time	x_n	$\|x_{n+1} - x_n\|$
10^{-6}	0.006923	33	0.015636	(0.000000, 0.000000)	6.782486×10^{-7}
10^{-7}	0.006444	60	0.015266	(0.000000, 0.000000)	8.078633×10^{-8}
10^{-8}	0.005079	83	0.015765	(0.000000, 0.000000)	7.836764×10^{-9}

5. Conclusion

We have introduced a viscosity iterative scheme and obtained the strong convergence. We also consider the bounded perturbation resilience of the proposed method and get theoretical convergence results.

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