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MODIFY REGULARIZATION METHOD VIA PROXIMAL POINT ALGORITHMS FOR ZEROS OF SUM ACCRETIVE OPERATORS OF FIXED POINT AND INVERSE PROBLEMS

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ABSTRACT. In this paper, we investigate the regularization method via a proximal point algorithm for solving treating the sum of two accretive operators for a fixed point set and inverse problems. Strong convergence theorems are established in the framework of Banach spaces. Furthermore, we also apply our result to variational inequality and equilibrium problems.

KEYWORDS: Regularization method, proximal point algorithm, zero points, accretive operators, inverse problems.
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1. Introduction

Many important problems have reformulation which require finding common zero points of nonlinear operators, for instance, inverse problems, variational inequality, optimization problems and fixed point problems. In this paper,we use $A^{-1}(0)$ to denote the set of zeros point of A. A well-known method for solving zero points of maximal monotone operators is the *proximal point algorithm* (PPA). First, Martinet [1] introduced the PPA in a Hilbert space H, that is, for starting $x_0 \in H$, a sequence $\{x_n\}$ generated by

$$x_{n+1} = J_{r_n}^A(x_n)\ \forall n \in \mathbb{N}, \tag{1.1}$$

where A is maximal monotone operators, $J_{r_n}^A = (I+r_nA)^{-1}$ is the resolvent operator of A and $\{r_n\} \subset (0,\infty)$ is a regularization sequence. An iterative (1.1) is equivalent

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to

$$x_n \in x_{n+1} + r_n A x_{n+1} \ \forall n \in \mathbb{N}.$$

If $\phi(x) : H \to \mathbb{R} \cup \{\infty\}$ is a proper convex and lower semicontinuous function, then $J^A_{r_n}$ reduces to

$$x_{n+1} = \arg\min \left\{ \phi(y) + \frac{1}{2r_n} \|x_n - y\|^2, \ y \in H \right\} \ \forall n \in \mathbb{N}. \tag{1.2}$$

Later, Rockafellar [2] studied the proximal point algorithm in framework of a Hilbert space and he proved that if $\liminf_{n \longrightarrow \infty} r_n > 0$ and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to a solution of a zero point of A. Rockafellar [2] has given a more practical method which is an inexact variant of the method:

$$x_{n+1} = J^A_{r_n} x_n + e_n, \ \forall n \in \mathbb{N}, \tag{1.3}$$

where $\{e_n\}$ is an error sequence. It was shown that if $e_n \longrightarrow 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$, with $0 \in A(z)$.

In 2011, Sahu and Yao [3] also extended PPA for the zero of an accretive operator in a Banach space which has a uniformly Gâteaux differentiable norm by combining the prox-Tikhonov method and the viscosity approximation method. They introduced the iterative method to define the sequence $\{x_n\}$ as follows:

$$x_{n+1} = J^A_{r_n}((1 - \alpha_n)x_n + \alpha_n f(x_n)), \ \forall n \in \mathbb{N}, \tag{1.4}$$

$$z_{n+1} = J^A_{r_n}((1 - \alpha_n)z_n + \alpha_n f(z_n) + e_n), \ \forall n \in \mathbb{N}, \tag{1.5}$$

where A is an accretive operator such that $A^{-1}(0) \neq \emptyset$ and f is a contractive mapping on C and $\{e_n\}$ is an error sequence. Strong convergent were established in both algorithms. This is a source of idea about resolvent operator can be approximated by contractions.

In the same year, PPA extended to the case of sum of two monotone operators A and B by use the technique of forward-backward splitting methods. Manaka and Takahashi [4] introduced the following iterative scheme in a Hilbert space:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S J^A_{\lambda_n}(I - \lambda_n B)x_n, \ \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a positive sequence, $S : C \to C$ is a nonexpansive mapping, A is a maximal monotone operator, B is an inverse strongly monotone mapping and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. They prove that a sequence $\{x_n\}$ converges weakly to some point $z \in Fix(S) \cap (A+B)^{-1}(0)$ provided that the control sequence satisfies some conditions. From [4], then we concern with the problem for finding a common element of $Fix(S) \cap (A + B)^{-1}(0)$.

In 2012, López et al. [5] use the technique of forward-backward splitting methods for accretive operators in Banach spaces. They considered the following algorithms with errors:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n J^A_{r_n}(x_n - r_n(Bx_n + a_n)) + b_n \tag{1.6}$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J^A_{r_n}(x_n - r_n(Bx_n + a_n)) + b_n, \tag{1.7}$$

where $u \in E$, $\{a_n\}$, $\{b_n\} \subset E$ and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. An operator A is a maximal accretive operator and B is an inverse strongly accretive. They prove that a sequence $\{x_n\}$ in equation (1.6) and (1.7) is weakly and strongly convergence, respectively.

In 2014, Cho et al. [6] introduced the following iterative scheme in a Hilbert space:

$$\begin{cases} x_1 \in C, \\ z_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ y_n = J^A_{r_n}(z_n - r_n B z_n + e_n) \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n) S y_n), \ for\ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are a sequences in $(0,\ 1)$, $\{r_n\}$ is a positive sequence, $A : C \to H$ is an inverse strongly monotone mapping, B is a maximal monotone operator, and $J^A_{\lambda_n} = (I + \lambda_n A)^{-1}$ is the resolvent of A. Let $S : C \to C$ is a strictly pseudo-contractive mapping with $k \in [0, 1)$, and $f : C \to C$ be a contractive mapping. They prove that a sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A+B)^{-1}(0)$ if the control sequence satisfies some restrictions.

Motivated by [3, 4, 5, 6], then we are interested in the problems for finding a common element of fixed point of nonexpansive S and element of the (quasi) variational inclusion problem as follow:

$$Find\ x \in C\ such\ that\ x \in Fix(S) \cap (A+B)^{-1}(0), \tag{1.8}$$

where A be single-valued nonlinear mapping and B be a multi-valued mapping.

The purpose of this paper is to introduce an iterative algorithm which is modify regularization method and use technique of forward-backward splitting methods for finding a common element of the set solution of nonexpansive S and the set solution of fixed point of the variational inclusion problems, where A is an m-accretive operator and B is an inverse-strongly accretive operator in the framework of Banach space with a uniformly convex and 2-uniformly smooth.

2. Preliminaries

Let E be a Banach space and let E^* be its dual. Let $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. For all $x \in E$ and $x^* \in E^*$, the value of x^* at x be denoted by $\langle x, x^* \rangle$. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\}$, for all $x \in E$. A *single-value* normalized duality mapping is denoted by j, which means a mapping $j : E \longrightarrow E^*$ such that, for all $u \in E$, $j(u) \in E^*$ satisfying the following:

$$\langle u, j(u) \rangle = \|u\| \|j(u)\|, \ \|j(u)\| = \|u\|.$$

If $E = H$ is a Hilbert space, then $J = I$, where I is identity mapping. If E is *smooth Banach space*, then J is single-valued j.

A Banach space E is called an *Opial's space* if for each sequence $\{x_n\}_{n=0}^{\infty}$ in E such that $\{x_n\}$ converges weakly to some x in E, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

hold for all $y \in E$ with $y \neq x$. In fact, for any normed linear space X admit the weakly sequentially continuous duality mapping implies X is Opial space. So, a Banach space with a weakly sequentially continuous duality mapping has the Opial's property; see [7].

The modulus of convexity of E is the function $\delta_E : (0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\epsilon) := \inf\{1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1; \|x - y\| \geq \epsilon\}.$$

E is said to be *uniformly convex* if and only if $\delta(\epsilon) > 0$, for each $\epsilon \in (0, 2]$. It known that a uniformly convex Banach space is reflexive and strictly convex.

Let $S(E)$ be the unit sphere defined by $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm $\|\cdot\|$ of E is said to be *Gâteaux differentiable norm*, if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S(E)$. In this case, space E is called *smooth.* A spaces E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.1) exist uniformly for all $x \in S(E)$. The norm of E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for all $x, y \in S(E)$. It is known that if the norm of E is smooth, then the duality mapping J is single-valued and norm to *weak** uniformly continuous on each bounded subset of E.

On the other hand, the *modulus of smoothness* of E is the function $\rho : [0,\infty) \to [0,\infty)$ defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x, y \in S(E), \|x\| = 1, \|y\| \le t\}.$$

A Banach space E be an *smooth* if $\rho_E(t) > 0$ for all $t > 0$. A Banach space E be an *uniformly smooth* if and only if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$. A Banach space E is said to be *q-uniformly smooth*, if for $1 < q \le 2$ be a fixed real number, there exists a constant $c > 0$ such that $\rho(t) \le ct^q$ for all $t > 0$. It known that every q-uniformly smooth space is smooth. In the case $\rho(t) \le ct^2$ for $t > 0$, these is 2-uniformly smooth. The examples of uniformly convex and 2-uniformly smooth Banach space are L_p, l_p or Sobolev spaces W^p_m, where $p \ge 2$. It is well known that, Hilbert spaces are 2-uniformly convex and 2-uniformly smooth. We known that if E is a reflexive Banach space, then every bounded sequence in E has a weakly convergent subsequence. Note that all uniformly convex and 2-uniformly smooth Banach space is reflexive.

Next, we recall the definitions of some operators.

(i) Let $f : C \longrightarrow C$ be an operator. Then f is called *k-contraction* if there exists a coefficient k $(0 < k < 1)$ such that

$$\|fx - fy\| \le k\|x - y\|, \quad \forall x, y \in C.$$

(ii) Let $S : C \longrightarrow C$ be an operator. Then s is called *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

(iii) Let $B : C \longrightarrow E$ be an operator. Then B is called *α-inverse-strongly accretive* if there exists a constant $\alpha > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Bx - By, j(x - y)\rangle \ge \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

(iv) A set-valued operator $A : D(A) \subseteq E \longrightarrow 2^E$ is called *accretive* if there exists $j(x - y) \in J(x - y)$ such that $u \in A(x)$, and $v \in A(y)$,

$$\langle u - v, j(x - y)\rangle \geqslant 0, \quad \forall x, y \in D(A).$$

(v) A set-valued operator $A : D(A) \subseteq E \longrightarrow 2^E$ is called *m-accretive* if A is accretive and $R(I + rA) = E$ for some $r > 0$, where I is identity mapping.

Let C and D are nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is said to be *sunny* if $Q(x + t(x - Q(x))) = Q(x)$ whenever $x + t(x - Q(x)) \in C$ for all $x \in C$ and $t \ge 0$.

A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

Lemma 2.1. [8] *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*
(i) Q is sunny and nonexpansive;
(ii) $\|Qx - Qy\|^2 \le \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$;
(iii) $\|(x - y) - (Qx - Qy)\|^2 \le \|x - y\|^2 - \|Qx - Qy\|^2$
(iv) $\langle x - Qx, J(y - Qx)\rangle \le 0, \forall x \in E, y \in C$.

Lemma 2.2. [9] *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let S be a nonexpansive mapping of C into itself with $Fix(S) \neq \emptyset$. Then, the set $Fix(S)$ is a sunny nonexpansive retract of C.*

It well known that if $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from E onto C, that is $Q_C = P_C$. Let C be a nonempty closed convex subset of E.

In the sequel for the proof of our main results, we also need the following lemmas.

Lemma 2.3. [10] *Let E be a Banach space and J be a normal duality mapping. Then there exits $j(x + y) \in J(x + y)$ for any given $x, y \in E$. Then*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle, \ j(x + y) \in J(x + y), \tag{2.2}$$

for any given $x, y \in E$.

Lemma 2.4. [5] *Let E be a real Banach space and let C be a nonempty closed and convex subset of E. Let $B : C \longrightarrow E$ be a single valued operator and α-inverse strongly accretive operator and let A is an m-accretive operator in E with $D(A) \supset C$ and $D(B) \supset C$.Then*

$$Fix\big(J_r^A(I - rB)\big) = (A + B)^{-1}(0).$$

where $J_r^A = (I + rA)^{-1}$ be a resolvent of A for $r > 0$.

Lemma 2.5. [11](*The Resolvent Identity*) *Let E be a Banach space and let A be an m-accretive operator.For $r > 0$, $s > 0$ and $x \in E$, then*

$$J_r^A x = J_s^A\left(\frac{s}{r}x + \left(1 - \frac{s}{r}\right)J_r^A x\right).$$

Lemma 2.6. [12] *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E with the 2-uniformly smooth constant K. Let the mapping $B : C \longrightarrow E$ be a α-inverse strongly accretive operator. Then, we have*

$$\|(I - rB)x - (I - rB)y\|^2 \le \|x - y\|^2 - 2r(\alpha - K^2 r)\|Bx - By\|^2, \tag{2.3}$$

where I is identity mapping. In particular, if $r \in (0, \frac{\alpha}{K^2})$, then $(I - rB)$ is a nonexpansive.

Lemma 2.7. [13] (*Demiclosed principle*) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and $S : C \longrightarrow E$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. Then $I - S$ is demiclosed at zero, i.e., $x_n \rightharpoonup x$ and $x_n - Sx_n \longrightarrow 0$ implies $x = Sx$.

Lemma 2.8. [14] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.9. [15] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq (1-t_n)a_n + t_n b_n + c_n, \forall n \geq 0,$$

where $\{t_n\}$ *is a number sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} t_n = 0$ *and* $\Sigma_{n\to\infty}^{\infty} t_n = \infty$*,* $\{b_n\}$ *is a sequence such that* $\limsup_{n\to\infty} b_n \leq 0$ *and* $\{c_n\}$ *is a positive number sequence such that* $\Sigma_{n=0}^{\infty} c_n < \infty$*. Then,* $\lim_{n\to\infty} a_n = 0$*.*

3. Main results

Before prove our main result, we need the following lemma:

Lemma 3.1. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty closed convex subset of* E*. Let* $A : D(A) \subseteq C \longrightarrow 2^E$ *be a m-accretive operator and* $B : C \longrightarrow E$ *be an* α*-inverse strongly accretive operator. Let* $S : C \longrightarrow C$ *be a nonexpansive mapping and let* $f : C \longrightarrow C$ *be a contraction mapping with the constant* $k \in (0,1)$*. Let* $J_{r_n}^A = (I + r_n A)^{-1}$ *be a resolvent of* A *for* $r_n > 0$ *such that* $Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$*. If defined operator* $W_n : C \to C$ *by* $W_n := SJ_{r_n}^A((I - r_n B)[\alpha_n fx + (1-\alpha_n)x] + e_n)$ *for all* $x \in C$*, where* $\alpha_n \in (0,1)$*,* $r_n > 0$*. Then* W_n *is a contraction operator and has a unique fixed point.*

Proof. Since S, $J_{r_n}^A$, and $(I - r_n B)$ are nonexpansive. Then we known that W_n is nonexpansive. Since f be a contraction mapping with coefficient $k \in (0,1)$. We have

$$\begin{aligned}
\|W_n x - W_n y\| &= \|SJ_{r_n}^A((I - r_n B)[\alpha_n f(x) + (1-\alpha_n)x] + e_n) \\
&\quad - SJ_{r_n}^A((I - r_n B)[\alpha_n f(y) + (1-\alpha_n)y] + e_n)\| \\
&\leq \|((I - r_n B)[\alpha_n f(x) + (1-\alpha_n)x] + e_n) \\
&\quad - ((I - r_n B)[\alpha_n f(y) + (1-\alpha_n)y] + e_n)\| \\
&\leq \|[\alpha_n f(x) + (1-\alpha_n)x] - [\alpha_n f(y) + (1-\alpha_n)y]\| \\
&\leq \|(\alpha_n f(x) + (1-\alpha_n)x) - (\alpha_n f(y) + (1-\alpha_n y)\| \\
&= \|\alpha_n(f(x) - f(y)) + (1-\alpha_n)(x-y)\| \\
&\leq \alpha_n\|f(x) - f(y)\| + (1-\alpha_n)\|x-y\| \\
&\leq \alpha_n k\|x-y\| + (1-\alpha_n)\|x-y\| \\
&= (\alpha_n k + (1-\alpha_n))\|x-y\|.
\end{aligned}$$

Since $0 < (\alpha_n k + (1-\alpha_n)) < 1$, it follows that W_n is a contraction mapping of C into it self. By Banach contraction principle, then there exist a unique fixed point, i.e., we say $\bar{x} = W_n\bar{x}$. Moreover, by use lemma 2.2, then the set $Fix(W_n)$ is sunny nonexpansive retraction of C. Hence there exist a unique fixed point $\bar{x} \in Fix(W_n) = Fix(S) \cap (A+B)^{-1}(0) := \Omega$, namely $Q_\Omega f(\bar{x}) = \bar{x} = W_n\bar{x}$. □

Theorem 3.2. *Let* E *be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let* C *be a nonempty closed convex subset of* E*. Let* $A : D(A) \subseteq C \longrightarrow 2^E$ *be a m-accretive operator and* $B : C \longrightarrow E$ *be an* α*-inverse strongly accretive operator. Let* $S : C \longrightarrow C$ *be a nonexpansive mapping and let* $f : C \longrightarrow C$ *be a contraction mapping with the constant* $k \in (0,1)$*. Let* $J_{r_n}^A = (I + r_n A)^{-1}$ *be a resolvent of* A *for* $r_n > 0$*. Assume that* $Fix(S) \cap (A+B)^{-1}(0) \neq \emptyset$*.*

For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by the following:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S J_{r_n}^A (y_n - r_n B y_n + e_n), \ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$*;*
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$*;*
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A+B)^{-1}(0)$, where $\bar{x} = Q_\Omega f(\bar{})$ and $Q_F f$ is a sunny nonexpansive retraction from E onto Ω.

Proof. **Step 1** We want to show that $\{x_n\}$ is bounded. Fixed $p \in Fix(S) \cap (A + B)^{-1}(0) \neq \emptyset$. So, we have $p \in Fix(S)$ and $p \in (A+B)^{-1}(0) = Fix(J_{r_n}^A(I - r_n B))$ (see Lemma 2.4). Observe that, we consider

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - p\| \\ &\leq \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n \left(\|f(x_n) - f(p)\| + \|f(p) - p\|\right) + (1-\alpha_n)\|x_n - p\| \\ &\leq \alpha_n k \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= [\alpha_n k + (1-\alpha_n)]\|x_n - p\| + \alpha_n \|f(p) - p\| \\ &= [1 - \alpha_n(1-k)]\|x_n - p\| + \alpha_n \|f(p) - p\|. \end{aligned} \tag{3.2}$$

We set $z_n := S J_{r_n}^A (y_n - r_n B y_n + e_{n+1})$. Since $J_{r_n}^A$ and $I - r_n B$ are nonexpansive, and from (3.2), it follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n) z_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &= \beta_n \|x_n - p\| + (1-\beta_n)\|S J_{r_n}^B (y_n - r_n A y_n + e_n) - Sp\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|J_{r_n}^A (y_n - r_n B y_n + e_n) - p\| \\ &= \beta_n \|x_n - p\| + (1-\beta_n)\|J_{r_n}^A (y_n - r_n B y_n + e_n) - J_{r_n}^A (I - r_n B)p\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|(y_n - r_n B y_n + e_n) - (I - r_n B)p\| \\ &= \beta_n \|x_n - p\| + (1-\beta_n)\|(I - r_n B)y_n - (I - r_n B)p + e_n\| \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\left(\|(I - r_n B)y_n - (I - r_n B)p\| + \|e_n\|\right) \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)\left[\|y_n - p\| + \|e_n\|\right] \\ &\leq \beta_n \|x_n - p\| + (1-\beta_n)[(1 - \alpha_n(1-k))\|x_n - p\| \\ &\quad + \alpha_n \|f(p) - p\|] + (1-\beta_n)\|e_n\| \\ &= \beta_n \|x_n - p\| + [(1-\beta_n) - \alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n \|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [\beta_n + (1-\beta_n) - \alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n \|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [1 - (1-\beta_n)\alpha_n(1-k)]\|x_n - p\| \\ &\quad + (1-\beta_n)\alpha_n \|f(p) - p\| + (1-\beta_n)\|e_n\| \\ &= [1 - \lambda_n(1-k)]\|x_n - p\| + \lambda_n \|f(p) - p\| + \|e_n\|, \end{aligned}$$

where $\lambda_n := (1-\beta_n)\alpha_n$. Then, it follows that

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \|e_n\| \\
&\leq \max\left\{\|x_{n-1}-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \|e_{n-1}\| + \|e_n\| \\
&\leq \max\left\{\|x_{n-2}-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \|e_{n-2}\| + \|e_{n-1}\| + \|e_n\| \\
&\vdots \\
&\leq \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-k}\right\} + \sum_{i=0}^{n}\|e_i\| < \infty.
\end{aligned}$$

It follows by mathematical induction, we conclude that

$$\|x_{n+1}-p\| \leq \max\left\{\|x_0-p\|, (1-k)^{-1}\|f(p)-p\|\right\} + \sum_{i=0}^{n}\|e_i\|, \ \forall n \geq 0.$$

By condition (d), this implies that $\{x_n\}$ is bounded.

From $y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n$, we obtain

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - p\| \\
&\leq \alpha_n\|f(x_n)-p\| + (1-\alpha_n)\|x_n-p\|.
\end{aligned} \tag{3.3}$$

From (3.3) and since $\{x_n\}$ is bounded, so $\{y_n\}$ and $\{z_n\}$ are bounded too.

Step 2 We want to show that $\lim_{n\longrightarrow\infty}\|x_{n+1}-x_n\| = 0$. By lemma 2.8, we set $v_n := y_n - r_nAy_n + e_n$, then $z_n := SJ^B_{r_n}v_n$, it follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|SJ^A_{r_{n+1}}v_{n+1} - SJ^A_{r_n}v_n\| \\
&\leq \|J^A_{r_{n+1}}v_{n+1} - J^A_{r_n}v_n\| \\
&\leq \|J^A_{r_{n+1}}v_{n+1} - J^A_{r_{n+1}}v_n\| + \|J^A_{r_{n+1}}v_n - J^A_{r_n}v_n\| \\
&\leq \|v_{n+1}-v_n\| + \|J^A_{r_{n+1}}v_n - J^A_{r_n}v_n\|.
\end{aligned} \tag{3.4}$$

Next, we compute $\|v_{n+1}-v_n\|$ that

$$\begin{aligned}
\|v_{n+1}-v_n\| &= \|(y_{n+1} - r_{n+1}By_{n+1} + e_{n+1}) - (y_n - r_nBy_n + e_n)\| \\
&= \|(I-r_nB)y_{n+1} - (I-r_nB)y_n + (r_n - r_{n+1})By_{n+1} + e_{n+1} - e_n\| \\
&\leq \|(I-r_nB)y_{n+1} - (I-r_nB)y_n\| + \mid r_n - r_{n+1} \mid \|By_{n+1}\| + \|e_{n+1}-e_n\| \\
&\leq \|y_{n+1}-y_n\| + \mid r_n - r_{n+1} \mid \|By_{n+1}\| + \|e_{n+1}\| + \|e_n\|.
\end{aligned} \tag{3.5}$$

Next, we compute $\|y_{n+1}-y_n\|$ that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|(\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1})(x_{n+1})) - (\alpha_n f(x_n) + (1-\alpha_n)x_n)\| \\
&= \|\alpha_{n+1}f(x_{n+1}) - \alpha_n f(x_{n+1}) + \alpha_n f(x_{n+1}) - \alpha_n f(x_n) + (1-\alpha_{n+1})x_n \\
&\quad -(1-\alpha_{n+1})x_n - (1-\alpha_n)x_n\| \\
&= \|(\alpha_{n+1}-\alpha_n)f(x_{n+1}) + \alpha_n(f(x_{n+1}) - f(x_n)) + (1-\alpha_{n+1})(x_{n+1}-x_n) \\
&\quad +x_n((1-\alpha_{n+1}) - (1-\alpha_n))\| \\
&\leq |\alpha_{n+1}-\alpha_n|\|f(x_{n+1}) - x_n\| + \alpha_n\|f(x_{n+1}) - f(x_n)\| + (1-\alpha_{n+1})\|x_{n+1}-x_n\| \\
&= (1-\alpha_{n+1})\|x_{n+1}-x_n\| + h_n
\end{aligned}$$

$$\leq \quad \|x_{n+1} - x_n\| + h_n, \tag{3.6}$$

where $h_n = |\alpha_{n+1} - \alpha_n| \|f(x_{n+1}) - x_n\| + \alpha_n \|f(x_{n+1}) - f(x_n)\|$.

That is

$$\|v_{n+1} - v_n\| \quad \leq \quad \|x_{n+1} - x_n\| + h_n + g_n, \tag{3.7}$$

where $g_n = | r_n - r_{n+1} | \|By_{n+1}\| + \|e_{n+1}\| + \|e_n\|$.

Next, we compute $\|J^A_{r_{n+1}} v_n - J^A_{r_n} v_n\|$ by the resolvent identity (see Lemma 2.5) that

$$\begin{aligned}
\|J^A_{r_{n+1}} v_n - J^A_{r_n} v_n\| &= \|J^A_{r_n}\left(\frac{r_n}{r_{n+1}} v_n + (1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n\right) - J^A_{r_n} v_n\| \\
&\leq \|\left(\frac{r_n}{r_{n+1}} v_n + (1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n\right) - v_n\| \\
&= \|(\frac{r_n}{r_{n+1}} - 1) v_n + (1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n\| \\
&= \|(1 - \frac{r_n}{r_{n+1}}) J^A_{r_{n+1}} v_n - (1 - \frac{r_n}{r_{n+1}}) v_n\| \\
&= \|\frac{r_{n+1} - r_n}{r_{n+1}} (J^A_{r_{n+1}} v_n - v_n)\| \\
&\leq | \frac{r_{n+1} - r_n}{r_{n+1}} | \|J^A_{r_{n+1}} v_n - v_n\|.
\end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\|z_{n+1} - z_n\| \quad \leq \quad \|x_{n+1} - x_n\| + h_n + g_n + | \frac{r_{n+1} - r_n}{r_{n+1}} | \|J^A_{r_{n+1}} v_n - v_n\|.$$

In view of the condition (a), (c), and (d), it follows that

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \quad \leq \quad 0.$$

We take lim sup, it follows that

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \quad \leq \quad 0.$$

By lemma 2.8, we conclude that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0 \tag{3.9}$$

that is $\lim_{n\to\infty} \|SJ^A_{r_n}(v_n) - x_n\| = 0$. From (4.1), we observe that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\beta_n x_n + (1 - \beta_n) z_n - x_n\| \\
&\leq (1 - \beta_n) \|z_n - x_n\|.
\end{aligned}$$

By (3.9), then we conclude that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

Step 3 To show that $\lim_{n\to\infty} \|By_n - Bp\| = 0$, $\lim_{n\to\infty} \|J^A_{r_n}(v_n) - y_n\| = 0$ and $\lim_{n\to\infty} \|SJ^A_{r_n}(v_n) - J^A_{r_n}(v_n)\| = 0$.

Step 3.1 First, we observe that $\lim_{n\to\infty} \|By_n - Bp\| = 0$. Notice that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n x_n + (1 - \beta_n) SJ^A_{r_n} v_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|SJ^A_{r_n} v_n - p\|^2 \\
&= \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|v_n - (I - r_n B) p\|^2 \\
&= \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|(y_n - r_n B y_n + e_n) - (I - r_n B) p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) [\|(I - r_n B) y_n - (I - r_n B) p\|^2
\end{aligned}$$

$$\begin{aligned}
&+2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|] \\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\|y_n-p\|^2-2r_n(\alpha-K^2r_n)\|By_n-Bp\|^2\right) \\
&+2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|.
\end{aligned}$$

Set $\bar{g_n} := (1-\beta_n)2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)p\|$, we get

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \qquad (3.11)\\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\left(\|y_n-p\|^2-2r_n(\alpha-K^2r_n)\|By_n-Bp\|^2\right)+\bar{g_n} \\
=\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2-2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2+\bar{g_n} \\
=\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2 \\
& -2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2+\bar{g_n}.
\end{aligned}$$

Set $\bar{h_n} := 2r_n(\alpha-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2$, we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2-\bar{h_n}+\bar{g_n} \\
\leq\ & \beta_n\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+(1-\beta_n)(1-\alpha_n)\|x_n-p\|^2 \\
& -\bar{h_n}+\bar{g_n} \\
=\ & (1-\alpha_n(1-\beta_n))\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2-\bar{h_n}+\bar{g_n}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& 2r_n(\alpha_n-K^2r_n)(1-\beta_n)\|By_n-Bp\|^2 \\
\leq\ & (1-\alpha_n(1-\beta_n))\|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
\leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|(x_n-p)-(x_{n+1}-p)\| \\
& +(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n} \\
=\ & \|(x_n-p)+(x_{n+1}-p)\|\|x_n-x_{n+1}\|+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2+\bar{g_n}.
\end{aligned}$$

In view of the condition (a), (c), (d), and from (3.10), we conclude that $\lim_{n\longrightarrow\infty}\|By_n-Bp\|^2=0$. This implies

$$\lim_{n\longrightarrow\infty}\|By_n-Bp\|=0. \qquad (3.12)$$

Step 3.2 Second, we will show that $\lim_{n\to\infty}\|J^A_{r_n}(v_n)-y_n\|=0$, we observe that

$$\begin{aligned}
& \|J^A_{r_n}(v_n)-p\|^2 \\
\leq\ & \|J^A_{r_n}(v_n)-p\|\|(y_n-r_nBy_n+e_n)-(p-r_nBp)\| \\
=\ & \frac{1}{2}\{\|J^A_{r_n}(v_n)-p\|^2+\|(y_n-r_nBy_n+e_n)-(p-r_nBp)\|^2 \\
& -\|\left(J^A_{r_n}(v_n)-p\right)-(y_n-r_nBy_n+e_n)-(p-r_nBp)\|^2\} \\
=\ & \frac{1}{2}\{\|J^A_{r_n}(v_n)-p\|^2+\|(I-r_nB)y_n-(I-r_nB)p+e_n\|^2 \\
& -\|J^A_{r_n}(v_n)-y_n-r_nBy_n-e_n+r_nBp\|^2\} \\
=\ & \frac{1}{2}\{\|J^A_{r_n}(v_n)-p\|^2+\|(I-r_nB)y_n-(I-r_nB)p\|^2+\bar{g_n} \\
& -\|\left(J^A_{r_n}(v_n)-y_n-e_n\right)-r_n(By_n-Bp)\|^2\} \\
\leq\ & \frac{1}{2}\big\{\|J^A_{r_n}(v_n)-p\|^2+\|y_n-p\|^2+\bar{g_n} \\
& -(\|J^A_{r_n}(v_n)-y_n-e_n\|^2-2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\| \\
& +\|r_nBy_n-r_nBp\|^2)\big\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{2}\big\{\|J^A_{r_n}(v_n)-p\|^2+\|y_n-p\|^2+\bar{g_n}-\|J^A_{r_n}(v_n)-y_n-e_n\|^2 \\
& +2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|-\|r_nBy_n-r_nBp\|^2\big\}. \qquad (3.13)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \|J^A_{r_n}(v_n)-p\|^2 \\
\le \quad & \|y_n-p\|^2+\bar{g_n}-\|J^A_{r_n}(v_n)-y_n-e_n\|^2 \\
& +2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|-\|r_nBy_n-r_nBp\|^2 \\
= \quad & \|\alpha_nf(x_n)+(1-\alpha_n)x_n-p\|^2-\|J^A_{r_n}(v_n)-y_n-e_n\|^2 \\
& -\|r_nBy_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n} \\
\le \quad & \alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\|J^A_{r_n}(v_n)-y_n+e_n\|^2 \\
& -r_n\|By_n-Bp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}. \qquad (3.14)
\end{aligned}
$$

From (3.14), this implies that

$$
\begin{aligned}
& \|x_{n+1}-p\|^2 \\
\le \quad & \beta_n\|x_n-p\|^2+(1-\beta_n)\|SJ^A_{r_n}(v_n)-p\|^2 \\
\le \quad & \beta_n\|x_n-p\|^2+(1-\beta_n)\|J^A_{r_n}(v_n)-p\|^2 \\
\le \quad & \beta_n\|x_n-p\|^2+(1-\beta_n)\big\{\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2 \\
& -\|J^A_{r_n}(v_n)-y_n+e_n\|^2-r_n\|By_n-Bp\|^2 \\
& +2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}\big\} \\
= \quad & (1-\alpha_n)\|x_n-p\|^2+(1-\beta_n)\alpha_n\|f(x_n)-p\|^2 \\
& -(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2-(1-\beta_n)r_n^2\|By_n-r_nBp\|^2 \\
& +(1-\beta_n)2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+(1-\beta_n)\bar{g_n} \\
\le \quad & \|x_n-p\|^2+\alpha_n\|f(x_n)-p\|^2-(1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2 \\
& -r_n^2\|By_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& (1-\beta_n)\|J^A_{r_n}(v_n)-y_n+e_n\|^2 \\
\le \quad & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2-r_n^2\|By_n-r_nBp\|^2 \\
& +2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}. \\
= \quad & \|x_n-p\|^2-\|x_{n+1}-p\|^2+s_n \\
= \quad & \|(x_n-p)+(x_{n+1}-p)\|\|(x_n-p)-(x_{n+1}-p)\|+s_n \\
= \quad & \|(x_n-p)+(x_{n+1}-p)\|\|x_n-x_{n+1}\|+s_n, \qquad (3.15)
\end{aligned}
$$

where we set $s_n:=\alpha_n\|f(x_n)-p\|^2-r_n^2\|By_n-r_nBp\|^2+2r_n\|By_n-Bp\|\|J^A_{r_n}(v_n)-y_n-e_n\|+\bar{g_n}$.
From (3.15), in view of the condition (a), (c), (d), and equation (3.10), we conclude that

$$
\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n-e_n\|=0.
$$

This in turn implies that

$$
\lim_{n\longrightarrow\infty}\|J^A_{r_n}(v_n)-y_n\|=0. \qquad (3.16)
$$

Step 3.3 Lastly, we will show that $\lim_{n\to\infty} \|SJ_{r_n}^A(v_n) - J_{r_n}^A(v_n)\| = 0$, we see that

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)x_n - x_n\| \\ &= \alpha_n\|f(x_n) - x_n\|.\end{aligned}$$

By condition (a), then

$$\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0. \tag{3.17}$$

Next, from (3.16) and equation (3.17), then we see that

$$\|J_{r_n}^A(v_n) - y_n\| \leq \|J_{r_n}^A(v_n) - y_n\| + \|y_n - x_n\|.$$

That is

$$\lim_{n\longrightarrow\infty} \|J_{r_n}^A(v_n) - x_n\| = 0. \tag{3.18}$$

From equation (3.9) and (3.18), then we see that

$$\|SJ_{r_n}^A(v_n) - J_{r_n}^A(v_n)\| \leq \|SJ_{r_n}^A(v_n) - x_n\| + \|x_n - J_{r_n}^A(v_n)\|.$$

That is

$$\lim_{n\longrightarrow\infty} \|SJ_{r_n}^A(v_n) - J_{r_n}^A(v_n)\| = 0. \tag{3.19}$$

Step 4 Since E is a uniformly convex and 2-uniformly smooth Banach space, then E is reflexive Banach space. By reflexive Banach space and from $\{x_n\}$, $\{y_n\}$ are bounded, then it has a weakly convergence subsequence. We may assume that $x_{n_i} \rightharpoonup \hat{x}$. In view of $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$, then there exist a subsequence of $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to $\hat{x}$. we can say that $\{y_{n_i}\}$ also converges weakly to $\hat{x}$, i.e, $y_{n_i} \rightharpoonup \hat{x}$, without loss of generality. To show that $\hat{x} \in Fix(S) \cap (A+B)^{-1}(0) = \Omega$.

(i) First, we want to show that $\hat{x} \in Fix(S)$. Now, we have $y_{n_i} \rightharpoonup \hat{x}$. Since we known that $\{J_{r_n}^A(v_n)\}$ is bounded and form $\lim_{n\longrightarrow\infty} \|J_{r_n}^A(v_n) - y_n\| = 0$, then we say that $\{J_{r_{n_i}}^A(v_{n_i})\} \rightharpoonup \hat{x}$.

From (3.19), we have $\lim_{n\longrightarrow\infty} \|SJ_{r_{n_i}}^A(v_{n_i}) - J_{r_{n_i}}^A(v_{n_i})\| = 0$. By demiclosed principle, this implies $S\hat{x} = \hat{x}$, namely we prove that $\hat{x} \in Fix(S)$. **(ii)** Next, to show that $J_r^A(I - rB)\hat{x} = \hat{x}$. Since a Banach space with weakly continuous duality mapping has the Opial's condition, see [7]. Suppose $\hat{x} \neq J_r^A(I - rB)\hat{x}$. By the Opial's condition and condition (c), (d), then we have

$$\begin{aligned}&\liminf_{i\to\infty} \|y_{n_i} - \hat{x}\| \\ < \;& \liminf_{i\to\infty} \|y_{n_i} - J_r^A(I - rB)\hat{x}\| \\ \leq \;& \liminf_{i\to\infty} \{\|y_{n_i} - J_{r_{n_i}}^A(v_{n_i})\| + \|J_{r_{n_i}}^A(v_{n_i}) - J_{r_n}^A(I - r_nB)\hat{x}\|\} \\ = \;& \liminf_{i\to\infty} \{\|y_{n_i} - J_{r_{n_i}}^A(v_{n_i})\| + \|J_r^A(v_{n_i}) - J_r^A(I - rB)\hat{x}\|\} \\ \leq \;& \liminf_{i\to\infty} \{\|y_{n_i} - J_{r_{n_i}}^A(v_{n_i})\| + \|v_{n_i} - (I - rB)\hat{x}\|\} \\ = \;& \liminf_{i\to\infty} \{\|y_{n_i} - J_{r_{n_i}}^A(v_{n_i})\| + \|(I - rB)y_{n_i} - (I - rB)\hat{x}\| + \|e_{n_i}\|\} \\ \leq \;& \liminf_{i\to\infty} \{\|y_{n_i} - J_{r_{n_i}}^A(v_{n_i})\| + \|y_{n_i} - \hat{x}\| + \|e_{n_i}\|\}.\end{aligned}$$

By (3.16) and condition (d), hence

$$\liminf_{i\to\infty} \|y_{n_i} - \bar{x}\| \quad < \quad \liminf_{i\to\infty} \|y_{n_i} - \hat{x}\|.$$

This is contradiction. Therefore, $J_r^A(I-rB)\hat{x}=\hat{x}$.
This complete the proof that $\hat{x}\in Fix(S)\cap(A+B)^{-1}(0)=\Omega$.

Step 5 We defined operator $W_n:C\to C$ by $W_n:=SJ_{r_n}^A((I-r_nB)[\alpha_nfx+(1-\alpha_n)x]+e_n)$ for all $x\in C$, where $\alpha_n\in(0,1)$, $r_n>0$. From lemma 3.1 an operators W_n is a contraction operator and has a unique fixed point. Moreover, by use lemma 2.2, we known that $\bar{x}\in Fix(W_n)=Fix(S)\cap(A+B)^{-1}(0):=\Omega$, namely $Q_\Omega f(\bar{x})=\bar{x}=W_n\bar{x}$. (Now, $\hat{x}=\bar{x}$ too)

Next, we will show that $\limsup_{n\longrightarrow\infty}\langle f(\bar{x})-\bar{x},j(y_n-\bar{x})\rangle\le 0$, where $\lim_{t\longrightarrow 0}x_t=\bar{x}=Q_\Omega f(\bar{x})$ and x_t solves equation $x_t=SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t), \forall t\in(0,1)$.

(i) We want to show that $\lim_{n\longrightarrow\infty}\|W_nx_n-y_n\|=0$. Consider

$$\begin{aligned}\|W_nx_n-y_n\| &\le \|SJ_{r_n}^A((I-r_nB)[\alpha_nf(x_n)+(1-\alpha_n)x_n]+e_n)-x_n\|+\|x_n-y_n\|\\ &= \|z_n-x_n\|+\|x_n-y_n\|. \end{aligned} \tag{3.20}$$

From (3.9) and (3.17), then

$$\lim_{n\longrightarrow\infty}\|W_nx_n-y_n\|=0. \tag{3.21}$$

(ii) We want to show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},j(y_n-\bar{x})\rangle\le 0$. We compute

$$\begin{aligned} & \|x_t-y_n\|^2\\ =\ & \|SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-y_n\|^2\\ =\ & \langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-W_nx_n+W_nx_n-y_n,j(x_t-y_n)\rangle\\ =\ & \langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-W_nx_n,j(x_t-y_n)\rangle\\ & +\langle W_nx_n-y_n,j(x_t-y_n)\rangle\\ =\ & \langle SJ_{r_n}^A(I-r_nB)(tf(x_t)+(1-t)x_t)-SJ_{r_n}^A((I-r_nB)y_n+e_n),j(x_t-y_n)\rangle\\ & +\langle W_nx_n-y_n,j(x_t-y_n)\rangle\\ \le\ & \langle (I-r_nB)(tf(x_t)+(1-t)x_t)-(I-r_nB)y_n-e_n,j(x_t-y_n)\rangle\\ & +\|W_nx_n-y_n\|\|x_t-y_n\|\\ =\ & \langle (I-r_nB)(tf(x_t)+(1-t)x_t)-(I-r_nB)y_n,j(x_t-y_n)\rangle+\langle e_n,j(x_t-y_n)\rangle\\ & +\|W_nx_n-y_n\|\|x_t-y_n\|\\ \le\ & \langle (tf(x_t)+(1-t)x_t)-x_t+x_t-y_n,j(x_t-y_n)\rangle+\|e_n\|\|x_t-y_n\|\\ & +\|W_nx_n-y_n\|\|x_t-y_n\|\\ \le\ & \langle t(f(x_t)-x_t),j(x_t-y_n)\rangle+\langle x_t-y_n,j(x_t-y_n)\rangle+\|e_n\|\|x_t-y_n\|\\ & +\|W_nx_n-y_n\|\|x_t-y_n\|\\ \le\ & t\langle f(x_t)-x_t,j(x_t-y_n)\rangle+\|x_t-y_n\|^2+\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|\\ \le\ & -t\langle f(x_t)-x_t,j(y_n-x_t)\rangle+\|x_t-y_n\|^2+\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\| \end{aligned} \tag{3.22}$$

It follows that

$$t\langle f(x_t)-x_t,j(y_n-x_t)\rangle \le \|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|.$$

Then

$$\langle f(x_t)-x_t,j(y_n-x_t)\rangle \le \frac{1}{t}\{\|e_n\|\|x_t-y_n\|+\|W_nx_n-y_n\|\|x_t-y_n\|\}.$$

By virtue of (3.21) and condition (d), we found that

$$\limsup_{n\to\infty}\langle f(x_t)-x_t,j(y_n-x_t)\rangle\le 0. \tag{3.23}$$

Since $x_t \to \bar{x}$, as $t \to 0$ and the fact that j is norm-to- weak* uniformly continuous on bounded subset of E, we obtain

$$\begin{aligned}
& |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle - \langle f(x_t) - x_t, j(y_n - x_t)\rangle| \\
\leq\ & |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle - \langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\rangle| \\
& + |\langle f(\bar{x}) - \bar{x}, j(y_n - x_t)\rangle - \langle f(x_t) - x_t, j(y_n - x_t)\rangle| \\
\leq\ & |\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x}) - j(y_n - x_t)\rangle| + |\langle f(\bar{x}) - \bar{x} - f(x_t) + x_t, j(y_n - x_t)\rangle| \\
\leq\ & \|f(\bar{x}) - \bar{x}\| \|j(y_n - \bar{x}) - j(y_n - x_t)\| + \|f(\bar{x}) - \bar{x} - f(x_t) + x_t\| \|y_n - x_t\| \\
& \longrightarrow 0, \ as\ t \longrightarrow 0.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exist $\delta > 0$ such that $\forall t \in (0, \delta)$ the following inequality holds:

$$\langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \quad \leq \quad \langle f(x_t) - x_t, j(y_n - x_t)\rangle + \epsilon.$$

Taking $\limsup_{n\to\infty}$ in the above inequality, we find that

$$\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \quad \leq \quad \limsup_{n\to\infty} \langle f(x_t) - x_t, j(y_n - x_t)\rangle + \epsilon.$$

Since ϵ is arbitrary and (3.23), we obtain that

$$\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \quad \leq \quad 0. \tag{3.24}$$

Step 6 Next, we prove that $\{x_n\}$ converges strongly to $\bar{x} = Q_\Omega f(\bar{x})$ by using the lemma 2.3 and lemma 2.9. We note that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|\beta_n x_n + (1 - \beta_n) S J^A_{r_n}(v_n) - \bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|S J^A_{r_n}(v_n) - \bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|S J^A_{r_n}(v_n) - S\bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|J^A_{r_n}(v_n) - \bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|J^A_{r_n}(v_n) - J^A_{r_n}(I - r_n B)\bar{x}\|^2 \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|v_n - (I - r_n B)\bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|(y_n - r_n B y_n + e_n) - (I - r_n B)\bar{x}\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \|(I - r_n B) y_n - (I - r_n B)\bar{x} + e_n\|^2 \\
&= \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) [\|(I - r_n A) y_n - (I - r_n A)\bar{x}\|^2 \\
&\quad + 2\langle e_n, j((I - r_n B) y_n - (I - r_n B)\bar{x} + e_n)\rangle \\
&\leq \beta_n \|x_n - \bar{x}\|^2 + (1 - \beta_n) \left[\|y_n - \bar{x}\|^2 + 2\|e_n\| \|(I - r_n B) y_n - (I - r_n B)\bar{x} + e_n\|\right].
\end{aligned} \tag{3.25}$$

Consider

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &= \langle \alpha_n f(x_n) + (1 - \alpha_n) x_n - \bar{x}, j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - \bar{x}) + (1 - \alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - f(\bar{x})) + \alpha_n (f(\bar{x}) - \bar{x}) + (1 - \alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle \\
&= \langle \alpha_n (f(x_n) - f(\bar{x})) + (1 - \alpha_n)(x_n - \bar{x}), j(y_n - \bar{x})\rangle + \langle \alpha_n (f(\bar{x}) - \bar{x}), j(y_n - \bar{x})\rangle \\
&\leq \|\alpha_n (f(x_n) - f(\bar{x})) + (1 - \alpha_n)(x_n - \bar{x})\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&\leq [\alpha_n k \|x_n - \bar{x}\| + (1 - \alpha_n)\|x_n - \bar{x}\|] \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&= [1 - \alpha_n (1 - k)] \|x_n - \bar{x}\| \|y_n - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle \\
&= (1 - \alpha_n (1 - k)) \frac{\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2}{2} + \alpha_n \langle f(\bar{x}) - \bar{x}, j(y_n - \bar{x})\rangle
\end{aligned}$$

$$= \frac{1-\alpha_n(1-k)}{2}\left(\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2\right)+\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle.$$

It follows that

$$\begin{aligned} & 2\|y_n-\bar{x}\|^2 \qquad (3.26)\\ \leq\ & (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+(1-\alpha_n(1-k))\|y_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \\ \leq\ & (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle. \end{aligned}$$

Therefore, we obtain

$$\|y_n-\bar{x}\|^2 \leq (1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle. \quad (3.27)$$

Replace (3.27) in (3.25) that

$$\begin{aligned} & \|x_{n+1}-\bar{x}\|^2 \\ \leq\ & \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)\left[(1-\alpha_n(1-k))\|x_n-\bar{x}\|^2+2\alpha_n\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle\right] \\ & +(1-\beta_n)2\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\| \\ =\ & (1-\alpha_n(1-k)(1-\beta_n))\|x_n-\bar{x}\|^2+2\alpha_n(1-\beta_n)\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \\ & +2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\| \\ =\ & (1-\lambda_n)\|x_n-\bar{x}\|^2+\frac{2\lambda_n}{(1-k)}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle+c_n, \end{aligned}$$

where $c_n := 2(1-\beta_n)\|e_n\|\|(I-r_nB)y_n-(I-r_nB)\bar{x}+e_n\|$, and $\lambda_n = \alpha_n(1-k)(1-\beta_n)$.

If we set $b_n = \frac{2}{(1-k)}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle$ and we have $\limsup_{n\longrightarrow\infty}\langle f(\bar{x})-\bar{x}, j(y_n-\bar{x})\rangle \leq 0$, then we see that $\limsup_{n\to\infty} b_n \leq 0$, and also that $\sum_{n=0}^{\infty} c_n < \infty$.

By lemma 2.8 and condition (a), (b), and (d), we conclude that $\|x_n-\bar{x}\|^2 \longrightarrow 0$, as $n \longrightarrow \infty$. This implies

$$\lim_{n\longrightarrow\infty}\|x_n-\bar{x}\| = 0,$$

i.e., x_n converges strongly to $\bar{x}$. □

Next, we will utilize theorem 3.2 to study some strong convergence theorem in L_p with $2 \leq p < \infty$. Since L_p, where $p \geq 2$ are uniformly convex and 2-uniformly smooth Banach space with $K = p-1$, then we consider $E = L_p$ and we derive that following theorem:

Theorem 3.3. *Let C be a nonempty closed convex subset of an L_p for $2 \leq p < \infty$. Let A, B, S, f, $J_{r_n}^A$ be the same as in theorem 3.2. Let $\{\alpha_n\}$, $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{(p-1)^2})$ and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions (a),(b) and (d) in theorem 3.2 and conditions (c) $\lim_{n\longrightarrow\infty} r_n = r$, and $r \in (0, \frac{\alpha}{(p-1)^2})$. Then the sequence $\{x_n\}$ is defined by (4.1) converges strongly to a point $\bar{x} \in Fix(S) \cap (A+B)^{-1}(0)$.*

Consider a mapping $S \equiv I$ in theorem 3.2, we can obtain the following corollary direct.

Corollary 3.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E. Let $A : D(A) \subseteq E \longrightarrow 2^E$ be an m-accretive operator such that the domain of A is included in C and $B : C \longrightarrow X$ be an α-inverse strongly accretive* *operator. Let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0,1)$. Let $J^A_{r_n} = (I + r_n A)^{-1}$ be a resolvent of A for $r_n > 0$ such that $(A + B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, Let x_n be a sequence in the following process:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)J^A_{r_n}(y_n - r_n B y_n + e_n), \ \forall n \geq 0, \end{cases} \tag{3.28}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$.

Consider a mapping $S \equiv I$ and $f(x_n) \equiv u$, $\forall n \in \mathbb{N}$ in theorem 3.2, we obtain the following corollary direct.

Corollary 3.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping. Let C be a nonempty closed convex subset of E. Let $A : D(A) \subseteq E \longrightarrow 2^E$ be an m-accretive operator such that the domain of A is included in C and let $B : C \longrightarrow X$ be an α-inverse strongly accretive operator. Let $J^B_{r_n} = (I + r_n B)^{-1}$ be a resolvent of B for $r_n > 0$ such that $(A + B)^{-1}(0) \neq \emptyset$.*

For given $x_0 \in C$, Let x_n be a sequence in the following process:

$$\begin{aligned} & y_n = \alpha_n u + (1-\alpha_n)x_n, \\ & x_{n+1} = \beta_n x_n + (1-\beta_n)J^A_{r_n}(y_n - r_n B y_n + e_n), \ \forall n \geq 0, \end{aligned} \tag{3.29}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, \frac{\alpha}{K^2})$, $K > 0$ is the 2-uniformly smooth constant of E and $\{e_n\}$ is a sequence in E. Assume that the control sequence satisfy the following conditions:

(a) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(b) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(c) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in (A+B)^{-1}(0)$.

Setting $J^A_{r_n} \equiv I$, $B \equiv 0$, $f(x_n) \equiv u$, $\forall n \in \mathbb{N}$ and $e_n \equiv 0$, then we have the following corollary of the modified Mann-Halpern iteration.

Corollary 3.6. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $Fix(S) \neq \emptyset$. For given $x_0, u \in C$, Let x_n be a sequence in the following process:*

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S y_n, \ \forall n \geq 0, \end{cases} \tag{3.30}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$. Assume that the control sequence satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S)$.

4. Some applications

In this section, we give two applications of our main results in the framework of Hilbert spaces. Now, we consider theorem 3.2, in the framework of Hilbert spaces, it known that $K = \frac{\sqrt{2}}{2}$. Let H be a Hilbert space and let C be a nonempty closed convex subset of H.

Theorem 4.1. [6, Corollary 2.2] *Let $A : C \longrightarrow 2^H$ be a maximal monotone operators such that the domain of B which included in C and $B : C \longrightarrow H$ be an α-inverse strongly monotone operator. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$. Let $J^A_{r_n} = (I+r_nA)^{-1}$ be a resolvent of A for $r_n > 0$ such that $Fix(S)\cap(A+B)^{-1}(0) \neq \emptyset$.*
For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)SJ^A_{r_n}(y_n - r_nBy_n + e_n), \ \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, \frac{\alpha}{K^2})$;
(d) $\sum_{n=0}^{\infty} \| e_n \|< \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap (A + B)^{-1}(0)$. Next, we will give some related results.

4.1. **Application to projection for variational inequality.**

Let C be a nonempty, close and convex subset of a Hilbert space H. The metric projection of a point $x \in H$ onto C, denoted by $P_C(x)$, is defined as the unique solution of the problem

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C, \ \forall x \in H.$$

For each $x \in H$ and $z \in C$, the metric projection P_C is satisfied

$$z = P_C(x) \Longleftrightarrow \langle y - z, x - z\rangle \leq 0, \ \forall y \in C. \tag{4.2}$$

Note that the metric projection is nonexpansive mapping.

Let $g : H \longrightarrow (-\infty, \infty]$ is a proper convex lower semicontinuous function. Then the subdifferential ∂g of g is defined as follow:

$$\partial g(x) = \{z \in H : g(y) - g(x) \geq \langle y - x, z\rangle, \ \forall y \in H\},$$

for all $x \in H$. If $g(x) = \infty$, then $\partial g(x) \neq \emptyset$, Takahashi [16] claim that ∂g is m-accretive operator. Since we know that, an m-accretive operator is maximal monotone operators in a Hilbert space, then we claim that ∂g is maximal monotone operators. Then we define the set of minimizers of g as follow:

$$argmin_{y\in H} g(y) = \{z \in H : g(z) = min_{y\in H} g(y)\}.$$

It is easy to verify that $0 \in \partial g(x)$ if and only if $g(z) = min_{y\in H} g(y)$. Let i_C be the indicator function of C by

$$i_C(x) = \begin{cases} 0, & \forall x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then i_C is a proper lower semicontinuous convex function on H. So, we see that the subdifferential ∂i_C of i_C is maximal monotone operator; see, [16]. The resolvent J_r of ∂i_C for $r > 0$, that is $J_r x = (I + r\partial i_C)^{-1}x,\ \forall x \in H$. Next, we recall that set $N_C(u)$ is called the normal cone of C at u define by

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0,\ \forall y \in C\}.$$

Since $N_C(u) = \partial i_C(u)$. In fact, we have that for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_r x = (I + r\partial i_C)^{-1}x &\iff x \in u + r\partial i_C u \\ &\iff x \in u + rN_C(u) \\ &\iff x - u \in rN_C(u) \\ &\iff \frac{1}{r}\langle x - u, y - u\rangle \leq 0,\ \forall y \in C \\ &\iff \langle x - u, y - u\rangle \leq 0,\ \forall y \in C \\ &\iff u = P_C x. \end{aligned} \tag{4.3}$$

Then $u = (I + r\partial i_C)^{-1}x \iff u = P_C x,\ \forall x \in H,\ u \in C$.

Now, we consider the following variational inequality problem (VIP) for B is to find $x \in C$ such that

$$\langle Bx, y - x\rangle \geq 0,\ \forall y \in C. \tag{4.4}$$

The set of solutions of (4.4) is denoted by $VI(C, B)$.

$$VI(C, B) = \left\{\ x \in C : \langle Bx, y - x\rangle \geq 0,\ \forall y \in C\ \right\}. \tag{4.5}$$

Theorem 4.2. *Let $B : C \longrightarrow H$ be an α-inverse strongly monotone mapping. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0, 1)$. Assume that $Fix(S) \cap VI(C, B) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C(y_n - r_n By_n + e_n),\ \forall n \geq 0, \end{cases} \tag{4.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H. Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$;
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, 2\alpha)$;
(d) $\sum_{n=0}^{\infty} \parallel e_n \parallel < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap VI(C, A)$, where $\bar{x} = P_{Fix(S)\cap VI(C,B)} f(\bar{x})$.

Proof. By lemma 2.4 we know that $Fix(J_r^A(I - rB)) = (A + B)^{-1}(0)$. Put $A = \partial i_C$, and we to show that $VI(C, B) = (\partial i_C + B)^{-1}(0)$. Note that

$$\begin{aligned} x \in (\partial i_C + B)^{-1}(0) &\iff 0 \in \partial i_C x + Bx \\ &\iff 0 \in N_C x + Bx \\ &\iff -Bx \in N_C x \\ &\iff \langle -Bx, y - x\rangle \leq 0 \\ &\iff \langle Bx, y - x\rangle \geq 0 \\ &\iff x \in VI(C, B). \end{aligned} \tag{4.7}$$

From (4.3), therefore, we can conclude the desired conclusion immediately. □

4.2. **Application for equilibrium problems.** Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for finding $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C. \tag{4.8}$$

The set of solutions of (4.8) is denoted by $EP(F)$.

For solving the equilibrium problem, we assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\limsup_{t \longrightarrow 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;
(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 4.3. [17] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $z \in H$. Then, there exists $x \in C$ such that*

$$F(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C. \tag{4.9}$$

Lemma 4.4. [18] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $z \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(z) = \{x \in C : F(x, y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C\}, \forall z \in H. \tag{4.10}$$

Then, the following hold:
(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) $Fix(T_r) = EP(F)$;
(4) $EP(F)$ is closed and convex.

Lemma 4.5. [19] *Let C be a nonempty closed and convex subset of a real Hilbert space H and let $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4) and A_F be a multi-valued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then $EP(F) = A_F^{-1}(0)$ and $A_F x$ is a maximal monotone operator with the domain $D(A_F) \subset C$. Furthermore, the resolvent T_r of F coincides with the resolvent of A_F, i.e.,

$$T_r x = (I + rA_F)^{-1}(x), \ \forall x \in H, \ r > 0, \tag{4.11}$$

where T_r is defined as in (4.10)

We recalled that T_r is the resolvent of A_F for $r > 0$. Since $A = A_F$, we will show that $J_r x = T_r x$. Indeed, for $x \in H$, we have

$$\begin{aligned} z \in J_r x = (I + rA_F)^{-1}(x) &\iff x \in (I + rA_F)z \\ &\iff x \in z + rA_F z \\ &\iff x - z \in rA_F z \\ &\iff \frac{1}{r}(x - z) \in A_F z \end{aligned}$$

$$
\begin{aligned}
&\Longleftrightarrow \quad F(z,y) \geq \langle y-z, \frac{1}{r}(x-z)\rangle \\
&\Longleftrightarrow \quad F(z,y) \geq \langle y-z, \frac{-1}{r}(z-x)\rangle \\
&\Longleftrightarrow \quad F(z,y) \geq \frac{-1}{r}\langle y-z, z-x\rangle \\
&\Longleftrightarrow \quad F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \ \forall y \in C \\
&\Longleftrightarrow \quad z \in T_r x. \qquad (4.12)
\end{aligned}
$$

Using lemmas 4.3, 4.4, 4.5 and theorem 4.1, we also obtain the following result.

Theorem 4.6. *Let $F : C \times C \longrightarrow \mathbb{R}$ which satisfies $(A1)-(A4)$. Let $S : C \longrightarrow C$ be a nonexpansive mapping and let $f : C \longrightarrow C$ be a contraction mapping with the constant $k \in (0,1)$. Assume that $Fix(S) \cap EP(F) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be a sequence defined by following:*

$$
\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)ST_{r_n}(y_n + e_n), \ \forall n \geq 0, \end{cases} \qquad (4.13)
$$

where $\{\alpha_n\}, \{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequences in $(0, 2\alpha)$ and $\{e_n\}$ is a sequence in H.

Assume that the control sequences satisfy the following conditions:

(a) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(b) $0 < \liminf_{n\longrightarrow\infty} \beta_n \leq \limsup_{n\longrightarrow\infty} \beta_n < 1$*;*
(c) $\lim_{n\longrightarrow\infty} r_n = r$, *and* $r \in (0, 2\alpha)$*;*
(d) $\sum_{n=0}^{\infty} \| e_n \| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(S) \cap EP(F)$, where $\bar{x} = P_{Fix(S)\cap EP(F)} f(\bar{x})$.

Proof. Put $A \equiv A_F$ and $B \equiv 0$ in $(A+B)^{-1}(0)$ from theorem 4.1. Furthermore, for bifunction $F : C \times C \longrightarrow \mathbb{R}$, we define $A_F x$ as in lemma 4.5, we have $EP(F) = A_F^{-1}(0)$ and let T_{r_n} be the resolvent of A_F for $r_n > 0$. Therefore, we can conclude the desired conclusion immediately. □

5. Conclusion and remarks

Our main results extends and improves in the following:

(i) Theorem 3.2 extends and improves Theorem 3.1 of Manaka and Takahashi [4, Theorem 3.1] from a Hilbert space to a Banach space and from weak convergence to strong convergence.

(ii) Theorem 3.2 partially extends and improves Theorem 2.1 of Cho et al. [6, Theorem 2.1] from a Hilbert space to a Banach space with uniformly convex and 2-uniformly smooth.

(iii) Theorem 3.2 extends and improves Theorem 3.1 of Qing and Cho [20, Theorem 3.1] from the problems of finding an element of $A^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

(iv) Theorem 3.2 extends and improves Theorem 3.7 of Sahu and Yao [3, Theorem 3.7] from the problems of finding an element of $A^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

(v) Theorem 3.2 extends and improves Theorem 3.7 of López et al. [5, Theorem 3.7] from the problems of finding an element of $(A+B)^{-1}(0)$ to the problem of finding an element of $Fix(S) \cap (A+B)^{-1}(0)$.

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