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SOLVE ONE-DIMENSIONAL OPTIMIZATION PROBLEMS USING NEWTON-RAPHSON METHOD

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ABSTRACT. This research studied solve one-dimensional optimization problems using Newton – Raphson Method. This method uses first derivatives to solve solutions. It was tested with 12 function tests and tested the efficiency of the solution convergence rate Newton – Raphson method and the random search method. It was found that the Newton-Raphson method was more effective in converging rate to better answers than random search methods. The error value and the iteration are indicative.

KEYWORDS: Optimization, Newton – Raphson method, Random Search Method.
AMS Subject Classification: 47H09, 49M05.

1. Introduction

Optimization is the act of achieving the best possible result under given circumstances. In design, construction, maintenance, and engineers, etc. Have to make decisions. The goal of all such decisions is either to minimize effort or to maximize benefit. The effort or benefit can be usually expressed as a function of certain design variables. Hence, optimization is the process of finding the conditions that give the maximum or the minimum value of a function. It is obvious that if a point x^* corresponds to the minimum value of a function $f(x)$, the same point corresponds to the maximum value of the function. Thus, optimization can be taken to be minimization. There is no single method available for solving all optimization problems efficiently. Hence, a number of methods have been developed for solving different types of problems. Optimum seeking methods are also known as mathematical programming techniques, which are a branch of operations search. The Newton – Raphson (NR) method is an iterative scheme used to solve non-linear simultaneous equations. France (1991) described the application of NR method to solve various

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hydraulic problems. Martin and Peters (1963) were the first to propose the application of NR method for analysis of water distribution network (WDN) having pipes and reservoirs only. McCormick and Bellamy (1968) and Zarghamee (1971) extended its use to include other network elements such as pumps and valves. Shamir and Howard (1968) applied this method to solve for all types of unknowns in a network including pipe resistances using head equations. Epp and Fowler (1970) and Gofman and Rodeh (1981) used this method for a solution of loop equations. Some investigators proposed modifications to the classical NR method. Lam and Wolla (1972 a, b) modified the algorithm so that it does not require the Jacobian matrix or its inverse in the iterative process and also suggested a change in step size to minimize the error. The modification presented by Lemieux (1972) ensures the convergence of the algorithm irrespective of the starting assumption. Donachie (1974) suggested halving the step size at any node when oscillation occurs. Neilson (1989) compared the NR method with linear theory method and suggested starting of the NR method with a single iteration by the linear theory method followed by NR iterations. Andersen and Powell (1999) used a linear headloss formula in the NR method for the first iteration. Several computer programs are also available for analysis of WDNs using Newton-Raphson method. The application of NR method in optimal design of WDNs is however very limited. Young (1994) used NR method to solve nonlinear simultaneous equations which were generated through Lagrangian multiplier method for the optimal design of branched 3 WDNs. Johnson et al. (1995) discussed the limitations of the Lagrangian multiplier method proposed by Young (1994). Bhave (1978, 1985) developed the cost head loss ratio, criterion method for the optimal design of WDNs. This method can be used for optimal design and expansion of single as well as multi-source branched or looped networks including pumped source nodes. Herein, the cost head loss ratio criterion method is modified for faster convergence using the NR method.
The NR method is used to solve simultaneous non-linear equations iteratively. It expands the non-linear terms in Taylor's series, neglects the residues after two terms and thereby considers only the linear terms (Bhave 1991). Thus, the NR method linearizes the non-linear equations through partial differentiation and solves. Naturally, the solution is approximate and therefore is successively corrected. The iterative procedure is continued until satisfactory accuracy is reached. Thus, while applying NR method for obtaining correction in the cost head loss ratio criterion method, all correction equations would be considered simultaneously and solved at a time.

2. Preliminaries

2.1. **Optimal Conditions.**

We begin with a formal statement of the conditions which held at a minimum of a one-variable differentiable function. We have already made use of these conditions.

Definition 1. Suppose that $f(x)$ is a continuously differentiable function of the scalar variable x, and that, when $x = x^*$,

$$\frac{df}{dx} = 0 \text{ and } \frac{d^2 f}{dx^2} > 0. \tag{2.1}$$

The function $f(x)$ is then said to have a *local minimum* at x^*. Conditions (1) imply that $f(x^*)$ is the smallest value of f in some region near x^*. It may also be true that $f(x^*) \leq f(x)$ for all x but condition (1) does not guarantee this.

Definition 2. If conditions (1) hold at $x = x^*$ and if $f(x^*) \leq f(x)$ for all x then x^* is said to be the *global minimum*. In practice, it is usually hard to establish that x^* is a global minimum and so we shall chiefly be concerned with methods of finding local minima. Conditions (1) are called optimality conditions.

2.2. **Optimal Solution.**

A globally optimal solution for an optimization problem is defined as the solution $x^* \in X$, where $f(x^*) \leq f(x)$ for all $x \in X$ (minimization problem). For the definition of a globally optimal solution, it is not necessary to define the structure of the search space, a metric, or a neighborhood.
Given a problem instance (X, f) and a neighborhood function N^*, a feasible solution $x^\circ \in X$ is called locally optimal (minimization problem) with respect to N^* if

$$f(x^\circ) \leq f(x) \text{ for all } x \in N^*(x^\circ)$$

Therefore, locally optimal solutions do not exist if no neighborhood is defined. Furthermore, the existence of local optima is determined by the neighborhood the definition used as different neighborhoods can result in different locally optimal solutions.

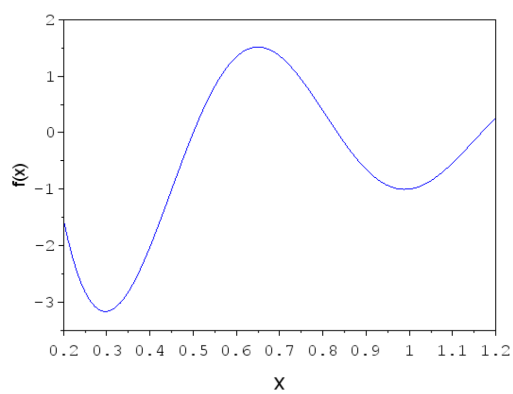

Fig.1. Locally and globally optimal solutions

Figure 1. illustrates the differences between locally and globally optimal solutions and shows how local optimal depend on the definition of N^*. We have a one-dimensional minimization problem with $x \in [a, b] \in R$. We assume an objective function f that assigns objective values to all $x \in X$. The modality of a problem describes the number of local optima in the problem. Unimodal problems have only one local optimum(which is also the global optimum), whereas muti-modal problems have multiple local optima. In general, multi-modal problems are more difficult for guided search methods to solve than unimodal problems.

2.3. **Objective function.**

The classical design procedure aims at finding an acceptable design, *i.e.* a design which satisfies the constraints. In general, there are several acceptable designs, and purpose

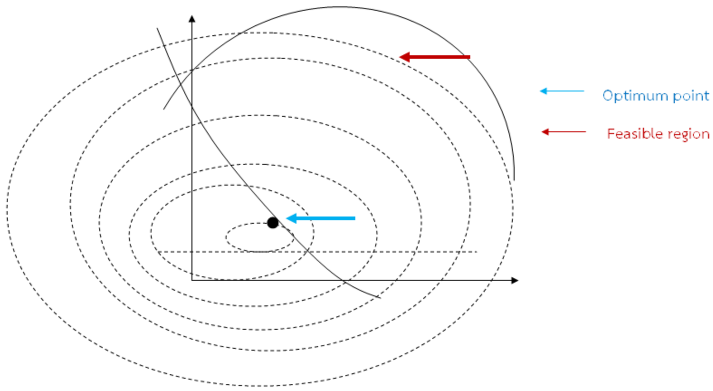

Fig.2. Design space, and optimum point

of the optimization is to single out the best possible design. Thus, a criterion has to be selected for comparing different designs. This criterion, when expressed as a function of the design variables are known as an objective function. The objective function is in general specified by physical or economic considerations. However, the selection of an objective function is not trivial, because what is the optimal design with respect to a certain criterion may be unacceptable with respect to another criterion. Typically, there is a trade-off performance-cost, performance reliability, hence the selection of the objective function is one of the most important decisions in the whole design process. If more than one criterion has to be satisfied, we have multiobjective optimization problem, that may be approximately solved considering a cost function which is a weighted sum of several objective functions.

Let $f : D \subseteq R \longrightarrow R$ is the objective function by a vector $x = (x_1, x_2, x_3, , ... , x_n)$ such as Styblinski-Tang Function $f(x) = 0.5(x^4 - 16x^2 + 5x)$ for $n = 1$. The function is usually evaluated on the hypercube $x \in [-5, 5]$ and global optimal is $f(x) = -39.16599n$

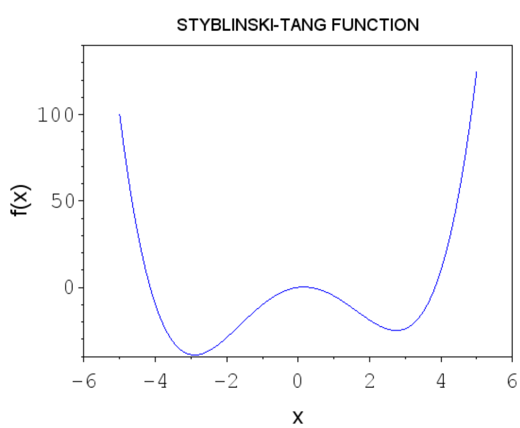

Fig.3. Styblinski-Tang Function

3. Random Search Method

This method generates trial solutions for the optimization model using random number generators for decision variables. Random search method includes random jump method and the random walk method with direction exploitation. Random jump method generates a huge number of data points for the decision variable assuming a uniform distribution for them and finds out the best solution by comparing the corresponding objective function values. Random walk method generates a trial solution with sequential improvements which is governed by a scalar step length and a unit random vector. The random walk method with direct exploitation is an improved version of a random walk method, in which, first the successful direction of generating trial solutions is found out and the maximum possible steps are taken along this successful direction. A generalized flowchart of the search algorithm in solving a nonlinear optimization with decision variable x_i, is presented in Fig.4.

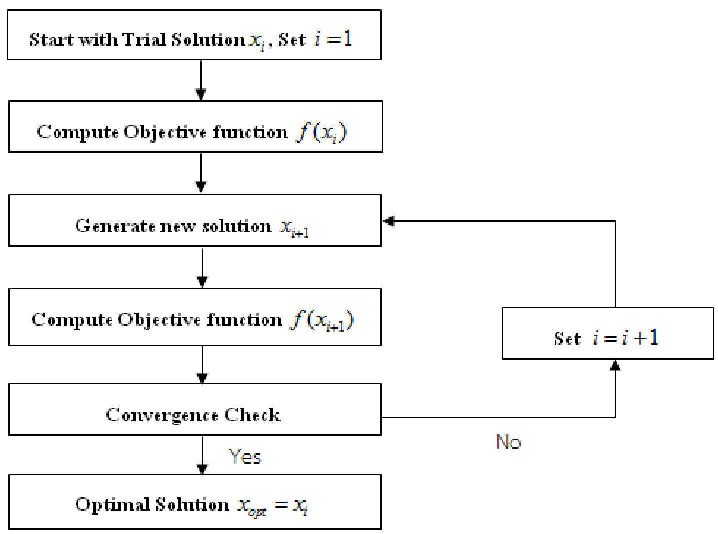

Fig.4. Flowchart Random Search Method

4. Main Results

4.1. **The Newton – Raphson Method.**

Newton – Raphson method is an open approach to find the minimum of function $f(x)$. Derivative using Taylor series recall Taylor series expansion as follows.

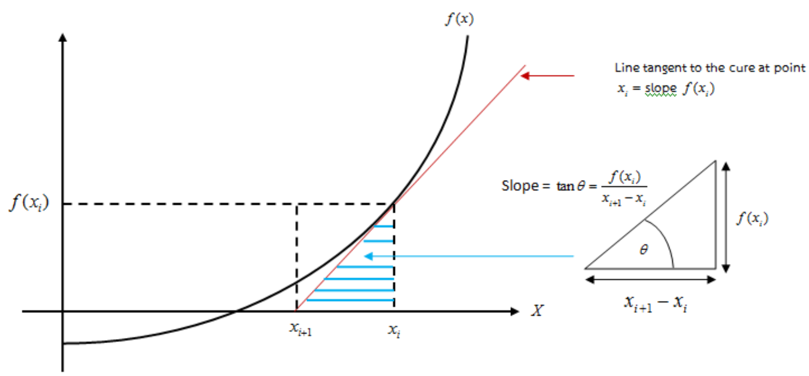

Fig.5. Newton - Raphson Method

from the previous figure 5.

$$\text{slope} = f^{'}(x_i) = \frac{df(x)}{dx}|_{x=x_i} = \frac{f(x_i)-0}{x_i - x_{i+1}}$$

or

$$f^{'}(x_i) = \frac{f(x_i)}{x_i - x_{i+1}} \qquad \text{for } i = 0, 1, 2, 3, ...$$

Thus

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

This algorithm is derived by expanding $f(x)$ as a Taylor series

$$f(x_0 + h) = f(x_0) + hf^{(1)}(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + ... + \frac{h^n}{n!}f^{(n)}(x_0) + R_{n+1}$$

If we let $x_0 + h = x_i + h = x_{i+1}$ and terminate the series at its linear term, then

$$f(x_i + h) = f(x_i) + (x_{i+1} - x_i)f^{(1)}(x_i)$$

or

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i)f^{'}(x_i)$$

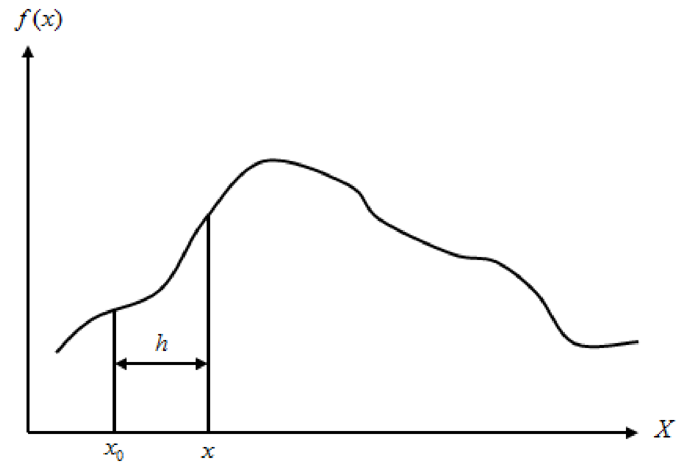

Fig.6. Interval point x and x_0 design

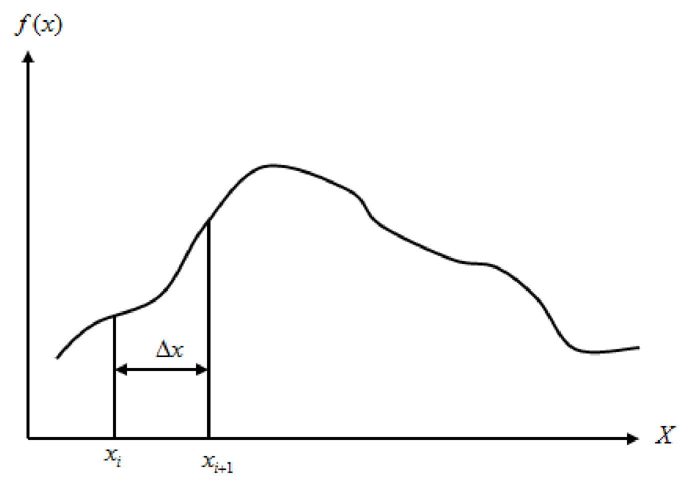

Fig.7. Interval point x_i and x_{i+1} design

Note that since the root of the function relating $f(x)$ and x is the value of x when $f(x_{i+1}) = 0$ at the intersection, hence,

$$\begin{aligned} f(x_{i+1}) &= 0 \\ f(x_i) + (x_{i+1} - x_i)f^{'}(x_i) &= 0 \\ (x_{i+1} - x_i)f^{'}(x_i) &= -f(x_i) \\ x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)} \end{aligned}$$

Newton – Raphson Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad \text{for} \quad \text{i=0,1,2,3,...} \tag{4.1}$$

where
x_i = value of the root at iteration i
x_{i+1} = a revised value of the root at iteration $i+1$
$f(x_i)$ = value of the function at iteration i

$f^{'}(x_i)$ = derivative of $f(x)$ evaluated at iteration i

4.2. **Iterations.**

The Newton – Raphson method uses the slope(tangent) of the function $f(x)$ of the current iterative solution (x_i) to find the solution (x_{i+1}) in the next iteration. The slope at $(x_i.f(x_i))$ is given by

$$f^{'}(x_i) = \frac{f(x_i)-0}{x_i-x_{i+1}}$$

Then x_{i+1} can be solved as

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

Which is known as the Newton – Raphson formula.
Relative error: $|E_{rr}| = |\frac{x_{i+1}-x_i}{x_{i+1}}|$ and Iterations stop if $E_{rr} \leq E$ is presented in Fig.8.

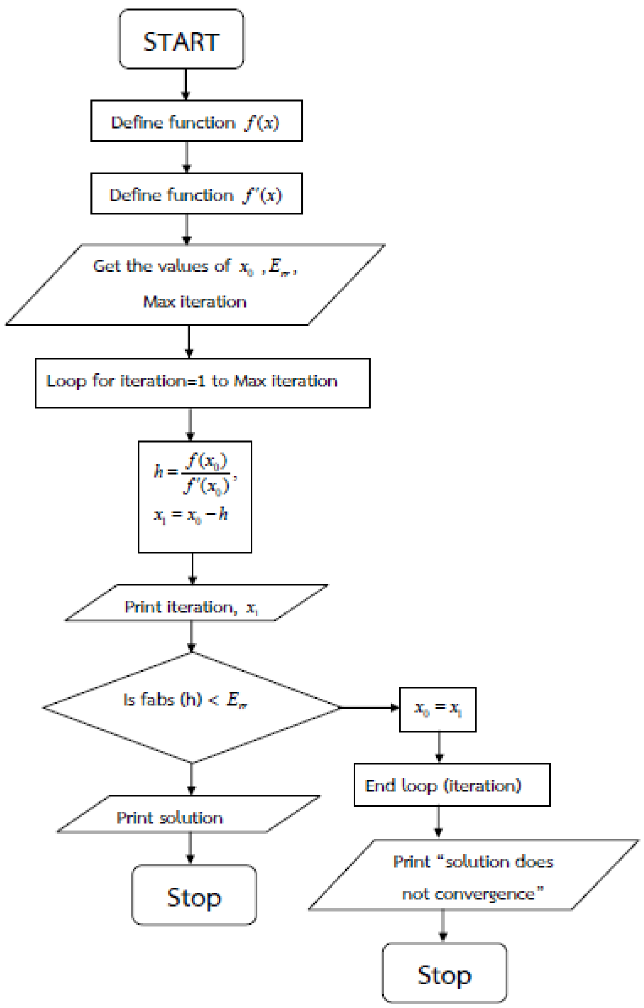

Fig.8. Flowchart of Newton – Raphon Method

Example 4.1. Use the Newton – Raphson iteration method to estimate the root of the following function employing an initial guess of $x_0 = 0 : f(x) = e^x - 5x$
Solution: Let's find the derivative of the function first,

$$f^{'}(x) = \frac{df(x)}{dx} = e^x - 5$$

The initial guess is $x_0 = 0$, hence,
$i = 0$

$$\begin{array}{rcl} f(x) = e^x - 5x & , & f(0) = e^0 - 5(0) = 1 \\ f^{'}(x) = e^x - 5 & , & f(0) = e^0 - 5 = 1 \\ x_{i+1} & = & x_i - \frac{f(x_i)}{f^{'}(x_i)} \\ x_1 & = & x_0 - \frac{f(x_0)}{f^{'}(0)} \\ x_1 & = & 0 - \frac{1}{1} \\ x_1 & = & -1 \end{array}$$

$i = 1$ Now $x_1 = -1$, hence,

$$\begin{array}{rcl} f(x) = e^x - 5x & , & f(-1) = e^{-1} - 5(-1) = 5.3678794412 \\ f^{'}(x) = e^x - 5 & , & f(-1) = e^{-1} - 5 = -4.632120559 \\ x_2 & = & x_1 - \frac{f(x_1)}{f^{'}(1)} \\ x_2 & = & -1 - \frac{5.3678794412}{-4.632120559} \\ x_2 & = & -1 + 1.158838457 \\ x_2 & = & 0.158838457 \end{array}$$

$i = 2$ Now $x_2 = 0.158838457$, hence,

$$\begin{array}{rcl} f(x) = e^x - 5x & , & f(0.158838457) = e^{0.158838457} - 5(0.158838457) \\ & = & 0.3787956294 \\ f^{'}(x) = e^x - 5 & , & f(0.158838457) = e^{0.158838457} - 5 = -3.827851421 \\ x_3 & = & x_2 - \frac{f(x_2)}{f^{'}(2)} \\ x_3 & = & 0.158838457 - \frac{0.3787956294}{-3.827851421} \\ x_3 & = & 0.158838457 + 0.0989577671 \\ x_3 & = & 0.2577962241 \end{array}$$

$i = 3$ Now $x_3 = 0.2577962241$, hence,

$$\begin{array}{rcl} f(x) = e^x - 5x & , & f(0.2577962241) = e^{0.2577962241} - 5(0.2577962241) \\ & = & 0.0050939696 \\ f^{'}(x) = e^x - 5 & , & f(0.2577962241) = e^{0.2577962241} - 5 = -3.70592491 \\ x_4 & = & x_3 - \frac{f(x_3)}{f^{'}(3)} \\ x_4 & = & 0.2577962241 - \frac{0.0050939696}{-3.70592491} \\ x_4 & = & 0.2577962241 + 0.00137454744 \\ x_4 & = & 0.25917077154 \end{array}$$

$i = 4$ Now $x_4 = 0.25917077154$, hence,

$$
\begin{aligned}
f(x) = e^x - 5x \quad &, \quad f(0.25917077154) = e^{0.25917077154} - 5(0.25917077154) \\
&= 1.2234E-6 \\
f^{'}(x) = e^x - 5 \quad &, \quad f(0.25917077154) = e^{0.25917077154} - 5 = -3.704144919 \\
x_5 &= x_4 - \frac{f(x_4)}{f^{'}(4)} \\
x_5 &= 0.25917077154 - \frac{1.2234E-6}{-3.704144919} \\
x_5 &= 0.25917077154 + 3.3027865e-7 \\
x_5 &= 0.2591711018
\end{aligned}
$$

Thus, the approach rapidly convergences on the true root of 0.2591 to four significant digits in table 1.

TABLE 1. Newton - Raphson iteration method $f(x) = e^x - 5x$

i	x_i	x_{i+1}	$f(x_i)$	$\lvert E_{rr}\rvert = \lvert\frac{x_{i+1}-x_i}{x_{i+1}}\rvert$
0	0	-1	1	-
1	-1	0.158838457	5.3678794412	7.29570457235
2	0.158838457	0.2577962241	0.3787956294	0.38386042097
3	0.2577962241	0.25917077154	0.0050939696	0.00530363602
4	0.25917077154	0.2591711018	1.2234E-6	0.00000127429
	0.2591696			

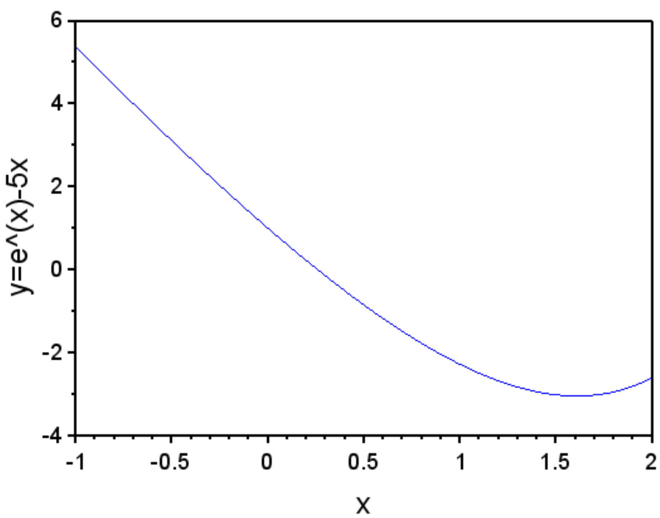

Fig.9. Results of $f(x) = e^x - 5x$

TABLE 2. The $f_1(x) - f_{12}(x)$ function is shown the convergence solution value and an error value of the function.

function	Methods	Solution	Iteration	$\lvert E_{rr} \rvert$
$f_1(x) : x^3 - 3x^2$	Newton-Raphson Method	0.0012125	8	0.995977
	Random Search method	0.0278956	9	0.924593
$f_2(x) : e^{2x} - x - 6$	Newton-Raphson Method	0.9042487	9	0.001386
	Random Search method	0.9996868	9	0.001543
$f_3(x) : y = x + tan^{-1}(x) - 1$	Newton-Raphson Method	0.5202264	4	0.000157
	Random Search method	0.8722966	5	0.003041
$f_4(x) : y = log(x)$	Newton-Raphson Method	0.999991	3	0.604936
	Random Search method	0.997933	4	0.516728
$f_5(x) : y = 4cos(x^2) - sin(x^2) - 3$	Newton-Raphson Method	4.912412	3	0.000036
	Random Search method	2.298318	6	0.168945
$f_6(x) : y = sin(x) - \frac{1}{2}$	Newton-Raphson Method	0.5235988	2	0.000243
	Random Search method	0.9999733	4	0.287146
$f_7(x) : y = cos(x) - x^3$	Newton-Raphson Method	0.8664923	7	0.037025
	Random Search method	0.7596625	7	0.056987
$f_8(x) : y = x^6 - x - 1$	Newton-Raphson Method	-0.778089	4	0.000365
	Random Search method	-0.159662	7	0.046713
$f_9(x) : y = x^3 + x - 4$	Newton-Raphson Method	1.3788174	5	0.004193
	Random Search method	1.9658429	8	0.719548
$f_{10}(x) : y = 2x^2 - 6$	Newton-Raphson Method	1.7320511	10	0.000532
	Random Search method	0.4265912	15	0.195623
$f_{11}(x) : y = x^2 - 3$	Newton-Raphson Method	1.7320521	6	0.001224
	Random Search method	0.9521834	12	0.985624
$f_{12}(x) : y = cos(x) - x$	Newton-Raphson Method	0.7390851	4	0.000009
	Random Search method	0.5831952	8	0.056731

5. CONCLUSION

A Newton – Raphson method is a basic tool in numerical analysis and numerous applications, including operations research. We survey the optimization problems, Random Search and Newton – Raphson method its main ideas convergence results in its global behavior. This research studied solve one-dimensional optimization problems using Newton – RaphsonMethod. This method uses first derivatives to solve solutions. It was tested with 12 function test and tested the converging rate of the solution convergence rate The Newton – Raphson method and random search method. It was found that Newton – Raphson a method was more effective in converging rate to better answers than random search methods. The error value and the iteration are indicative.

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