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HARDY-ROGERS TYPE MAPPINGS ON DISLOCATED QUASI METRIC SPACES

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ABSTRACT. In this paper, We prove some common fixed point results for two α-dominated mappings satisfying Hardy-Rogers Type on a closed ball of left (right) K-sequentially complete dislocated quasi-metric space and give some example for support our result.

KEYWORDS: Hardy-Rogers Type, Dislocated Quasi Metric Spaces, Quasi Metric Spaces.
AMS Subject Classification: 47H09, 47H10, 49M05

1. Introduction

The partial metric spaces have applications in theoretical computer science (see [14]). The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [15]). Dislocated metric (metric-like) spaces (see [4, 16, 17, 18]) are generalizations of partial metric spaces. Furthermore, dislocated quasi metric spaces (quasi-metric-like spaces) generalize the idea of dislocated metric spaces and quasi-partial metric spaces.

Samet et al. [19] introduced the notion of α-admissible mappings. They weakened and generalized the contractive condition and several other known results.

In this paper, we proof common fixed point results for two α-dominated mappings in a closed ball in complete dislocated quasi metric space, under Hardy-Rogers Type.

2. Preliminaries

Definition 2.1. [10] Let X be a nonempty set. A quasi-partial metric on X is a function $q : X \times X \to \mathbb{R}^+$ satisfying, for all $x, y, z \in X$,

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(a) $0 \leq q(x,x) = q(x,y) = q(y,y)$ implies $x = y$ (equality),
(b) $q(x,x) \leq q(y,x)$ (small self-distances),
(c) $q(x,x) \leq q(x,y)$ (small self-distances),
(d) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$ (triangleinequality).

The pair (X, q) is called a quasi-partial metric space.

Definition 2.2. [13] Let X be a nonempty set. A function $d_q : X \times X \to [0,\infty)$ is called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x, y, z \in X$:

(a) If $d_q(x,y) = d_q(y,x) = 0$, then $x = y$,
(b) $d_q(x,y) \leq d_q(x,z) + d_q(z,y)$.

In this case, the pair (X, d_q) is called a dislocated quasi metric space.

It is clear that, if $d_q(x,y) = d_q(y,x) = 0$, then from (a) we have $x = y$. But, if $x = y$, then $d_q(x,y)$ may not be 0. It can be observed that, if $d_q(x,y) = d_q(y,x)$ for all $x, y \in X$, then (X, d_q) becomes a dislocated metric space (metric-like space)[1, 4, 5, 6, 9]. We will denote by (X, d_l) a dislocated metric space. For $x \in X$ and $\epsilon > 0, \overline{B_{d_q}(x,\epsilon)} = \{y \in X : d_q(x,y) \leq \epsilon\}$ is a closed ball in (X, d_q). Every quasi-partial metric space is a dislocated quasi metric space, but the converse is not true in general.

Example 2.3. If $X = \mathbb{R}^+ \cup \{0\}$, then $d_q(x,y) = x + \max\{x,y\}$ defines a dislocated quasi metric d_q on X. But, it is not a quasi-partial metric space. Indeed,

$$d_q(3,3) = 6 > d_q(2,3) = 5.$$

Reilly et al. [11] introduced the notion of left (right) K-Cauchy sequence and left (right) K-sequentially complete spaces.

Definition 2.4. Let (X, d_q) be a dislocated quasi metric space.

(a) A sequence $\{x_n\}$ in (X, d_q) is called left(right) K-Cauchy if $\forall\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall n > m \leq n_0, d_q(x_m, x_n) < \epsilon$ (respectively $d_q(x_n, x_m) < \epsilon$).
(b) A sequence $\{x_n\}$ in (X, d_q) dislocated quasi-converges (for short d_q-converges) to x
if $\lim_{n\to\infty} d_q(x_n, x) = \lim_{n\to\infty} d_q(x, x_n) = 0$. In this case, the point x is called a d_q-limit of $\{x_n\}$.
(c) (X, d_q) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in $(X, d_q), d_q$ -converges to a point $x \in X$ such that $d_q(x,x) = 0$.

One can easily observe that every complete dislocated quasi metric space is also left K-sequentially complete dislocated quasi metric space, but the converse is not true in general.

Remark 2.5. [3] It is easy to see that, if $x_n \in B_{d_q}(x_0, r)$ for all $n \in \mathbb{N}$ and for some $x_0 \in X$, $r > 0$, and the sequence $\{x_n\}$, d_q-converges to a point $z \in X$, then $z \in B_{d_q}(x_0, r)$.

Definition 2.6. [12] Let (X, q) be a quasi-partial metric space.

(a) A sequence $\{x_n\}$ in (X, q) is called 0-Cauchy if $\lim_{n,m\to\infty} q(x_n, x_m) = 0$ or $\lim \lim_{n,m\to\infty} q(x_m, x_n) = 0$.
(b) The space (X, q) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $q(x,x) = 0$.

Remark 2.7. [3] By definitions, one can easily observe that if X is a 0-complete quasi-partial metric space then it is also a K-sequentially complete dislocated quasi metric space. But a K-sequentially complete dislocated quasi metric space may not be a 0-complete quasi-partial metric space. Therefore, the results in a K-sequentially complete dislocated quasi metric space are more general than those in a 0-complete quasi-partial metric space.

Let X be a non-empty set and $T, f : X \to X$ be two mappings. A point $y \in X$ is called a point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$, here x is called a coincidence point of T and f. The mappings T, f are said to be weakly compatible if they commute at their coincidence points i.e., $Tfx = fTx$ whenever $Tx = fx$.

Let Ψ denote the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t \geq 0$, where ψ^n is the n^{th} iterate of ψ. The following lemma is a consequence of definition of Ψ.

Lemma 2.8. *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

Definition 2.9. [3] Let (X, d_q) be a dislocated quasi metric space, $A \subseteq X$, $T : X \to X$ be a selfmapping and $\alpha : X \times X \to [0, +\infty)$. Then:

(a) The mapping T is said to be α-dominated on A, if $\alpha(x, Tx) \geq 1$ for all $x \in A$.

(b) The function α is said to be a triangular function on A, if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies that $\alpha(x, z) \geq 1$ for all $x, y, z \in A$.

(b) (X, d_q) $is\alpha$-regularon A if for any sequence $\{x_n\}$ in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x \in A$ as $n \to \infty$ we have $\alpha(x_n, x) \geq 1$ for all $n \geq 0$.

It is clear that if T is an α-dominated mapping on X then T is α-dominated on each subset of X, but T can be α-dominated on some $A \subseteq X$, without being α-dominated mapping on X.

3. Main Results

Theorem 3.1. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S : X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that $x_0 \in B_{d_q}(x_0, r)$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in \overline{B_{d_q}(x_0, r)}$, then*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)] \quad (3.1)$$

$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)] \quad (3.2)$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r, \quad (3.3)$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d_q) is α-regular on $\overline{B_{d_q}(x_0, r)}$. Then there exists a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ of S and T. Moreover, $d_q(z, z) = 0$.

Proof. Let $x_0 \in X$, define $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence $\{x_n\}$ of points in X, such that

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1}, \ \forall k = 0, 1, 2, \ldots.$$

By mathematical induction, we can show that

$$\begin{cases} x_{n+1} \in \overline{B_{d_q}(x_0, r)},\ \alpha(x_n, x_{n+1}) \geq 1, \\ d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \quad \forall n \in \mathbb{N}. \end{cases} \tag{P_n}$$

By using (3.3) and $0 < \lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} < 1$, we obtain

$$d_q(x_0, x_1) = d_q(x_0, Sx_0) \leq (1-\lambda)r \leq r.$$

Hence, $x_1 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_0, x_1) \geq 1$. Therefore, using (3.1), we get that

$$\begin{aligned} d_q(x_1, x_2) &= d_q(Sx_0, Tx_1) \\ &\leq \beta d_q(x_0, x_1) + \gamma[d_q(x_0, Sx_0) + d_q(x_1, Tx_1)] \\ &\quad + \delta[d_q(x_1, Sx_0) + d_q(x_0, Tx_1)] \\ &= \beta d_q(x_0, x_1) + \gamma[d_q(x_0, x_1) + d_q(x_1, x_2)] \\ &\quad + \delta[d_q(x_1, x_1) + d_q(x_0, x_2)] \\ &\leq \beta d_q(x_0, x_1) + \gamma[d_q(x_0, x_1) + d_q(x_1, x_2)] \\ &\quad + \delta[d_q(x_0, x_1) + d_q(x_1, x_2)] \end{aligned}$$

Thus,

$$d_q(x_1, x_2) \leq \lambda d_q(x_0, x_1). \tag{3.4}$$

By using (3.4), we get that

$$\begin{aligned} d_q(x_0, x_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) \leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) \\ &= (1+\lambda) d_q(x_0, x_1) \leq (1+\lambda)(1-\lambda)r = (1-\lambda^2)r \leq r. \end{aligned}$$

Hence, $x_2 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_1, Tx_1) \geq 1$. Therefore, from (P_1) holds and using (3.2), we get that

$$\begin{aligned} d_q(x_2, x_3) &= d_q(Tx_1, Sx_2) \\ &\leq \beta d_q(x_1, x_2) + \gamma[d_q(x_1, Tx_1) + d_q(x_2, Sx_2)] \\ &\quad + \delta[d_q(x_2, Tx_1) + d_q(x_1, Sx_2)] \\ &= \beta d_q(x_1, x_2) + \gamma[d_q(x_1, x_2) + d_q(x_2, x_3)] \\ &\quad + \delta[d_q(x_2, x_2) + d_q(x_1, x_3)] \\ &\leq \beta d_q(x_1, x_2) + \gamma[d_q(x_1, x_2) + d_q(x_2, x_3)] \\ &\quad + \delta[d_q(x_1, x_2) + d_q(x_2, x_3)] \end{aligned}$$

By using (3.4), we get that

$$d_q(x_2, x_3) \leq \lambda d_q(x_1, x_2) \leq \lambda^2 d_q(x_0, x_1). \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} d_q(x_0, x_3) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, x_3) \\ &\leq= d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 d_q(x_0, x_1) \\ &= (1+\lambda+\lambda^2) d_q(x_0, x_1) = \frac{1-\lambda^3}{1-\lambda} d_q(x_0, x_1) \\ &\leq \frac{1-\lambda^3}{1-\lambda}(1-\lambda)r = (1-\lambda^3)r \leq r. \end{aligned}$$

Hence, $x_3 \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(x_1, Tx_1) \geq 1$. Therefore, from (P_2) holds. Suppose, $(P_1), (P_2), \ldots, (P_i)$

be the inductive hypothesis. We shall show that (P_{i+1}) holds. For this, we consider two possible cases. First, suppose that i is even. Then, since $\alpha(x_i, x_{i+1}) \geq 1$ and using (3.1), we get that

$$\begin{aligned} d_q(x_{i+1}, x_{i+2}) &= d_q(Sx_i, Tx_{i+1}) \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, Sx_i) + d_q(x_{i+1}, Tx_{i+1})] \\ &\quad + \delta[d_q(x_{i+1}, Sx_i) + d_q(x_i, Tx_{i+1})] \\ &= \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, x_{i+1}) + d_q(x_i, x_{i+2})] \\ &\quad + \delta[d_q(x_{i+1}, x_{i+1}) + d_q(x_i, x_{i+2})] \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma[d_q(x_i, x_{i+1}) + d_q(x_{i+1}, x_{i+2})] \\ &\quad + \delta[d_q(x_i, x_{i+1}) + d_q(x_{i+1}, x_{i+2})] \end{aligned}$$

Since (P_i) holds, we get that

$$d_q(x_{i+1}, x_{i+2}) \leq \lambda d_q(x_i, x_{i+1}) \leq \lambda^{i+1} d_q(x_0, x_1).$$

Thus,

$$\begin{aligned} d_q(x_0, x_{i+2}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_{i+1}, x_{i+2}) \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^{i+2}) d_q(x_0, x_1) \\ &\leq \frac{1 - \lambda^{i+2}}{1 - \lambda}(1 - \lambda)r \leq (1 - \lambda^{i+2})r \leq r. \end{aligned}$$

Hence, $x_{i+2} \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$., we have $\alpha(x_{i+1}, Sx_{i+1}) = \alpha(x_{i+1}, x_{i+2}) \geq 1$. Therefore, (P_{i+1}) holds. Similarly, one can see that if i is odd, then (P_{i+1}) holds, which completes the inductive proof. Thus, we can write

$$d_q(x_n, x_{n+1}) \leq \lambda^n d_q(x_0, x_1), \ \ \forall n \in \mathbb{N}. \tag{3.6}$$

Next, we will show that the sequence $\{x_n\}$ is a left K-Cauchy sequence. Indeed, for $n, m \in \mathbb{N}$ with $m > n$ using (3.7) we have

$$\begin{aligned} d_q(x_n, x_m) &\leq d_q(x_n, x_{n+1}) + d_q(x_{n+1}, x_{n+2}) + \cdots + d_q(x_{m-1}, x_m) \\ &\leq \lambda^n d_q(x_0, x_1) + \lambda^{n+1} d_q(x_0, x_1) + \cdots + \lambda^{m-1} d_q(x_0, x_1). \end{aligned}$$

Thus,

$$d_q(x_n, x_m) \leq \frac{\lambda^n}{1 - \lambda} d_q(x_0, x_1), \ \forall n, m \in \mathbb{N}, \ m > n. \tag{3.7}$$

Since $0 < \lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} < 1$, for every $\epsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such tha $\lambda^n < \frac{1-\lambda}{d_q(x_0,x_1)}\epsilon$ for all $n > n_0$. Therefore, it follows from (3.7) that

$$d_q(x_n, x_m) < \epsilon, \ \ \forall m > n > n_0.$$

Therefore, the sequence $\{x_n\}$ is a left K-Cauchy sequence in X. By left K-sequential completeness of X, there exists $x \in X$ such that

$$\lim_{n\to\infty} d_q(x_n, z) = \lim_{n\to\infty} d_q(z, x_n) = 0. \tag{3.8}$$

We will show that z is a common fixed point of the mappings S and T. By Remark 2.5, we have $z \in B_{d_q}(x_0, r)$. Now, by the assumption we have for all $n \in \mathbb{N}$, therefore for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_q(z, Sz) &\leq d_q(z, x_{2n+2}) + d_q(x_{2n+2}, Sz) \\ &\leq d_q(z, x_{2n+2}) + d_q(Tx_{2n+1}, Sz) \\ &\leq d_q(z, x_{2n+2}) + \beta d_q(x_{2n+1}, z) \\ &\quad + \gamma[d_q(x_{2n+1}, Sx_{n+1}) + d_q(z, Sz)] \\ &\quad + \delta[d_q(z, Tx_{2n+1}) + d_q(x_{2n+1}, Sz)] \\ &\leq d_q(z, x_{2n+2}) + \beta d_q(x_{2n+1}, z) \\ &\quad + \gamma[d_q(x_{2n+1}, Sx_{n+1}) + d_q(z, Sz)] \\ &\quad + \delta[d_q(z, x_{2n+2}) + d_q(x_{2n+1}, z) + d_q(z, Sz)]. \end{aligned}$$

By using (3.7) and (3.8), we obtain

$$(1 - \gamma + \delta) d_q(z, Sz) \leq 0 \tag{3.9}$$

which implies that $d_q(z, Sz) = 0$. Similarly, one can show that $d_q(Sz, z) = 0$. Thus, $d_q(z, Sz) = d_q(Sz, z) = 0$, i.e., $z = Sz$ Similarly, one can show that $z = Tz$.

Hence, S and T have a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$. As is an dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(z, Sz) = \alpha(z, z) \geq 1$. Therefore,

$$\begin{aligned} d_q(z, z) &\leq d_q(Sz, Tz) \\ &\leq \beta d_q(z, z) + \gamma[d_q(z, Sz) + d_q(z, Tz)] \\ &\quad + \delta[d_q(z, Sz) + d_q(z, Tz)] \\ &\leq (\beta + 2\gamma + 2\delta) d_q(z, z), \end{aligned}$$

and this implies that

$$d_q(z, z) = 0.$$

□

Example 3.2. Let $X = \mathbb{Q}^+ \cup \{0\}$ and let $d_q : X^2 \times X^2 \to X$ be defined by $d_q((x_1, y_1), (x_2, y_2)) = x_1 + 4y_1 + \frac{x_2}{4} + y_2$. Then it is easy to show that (X^2, d_q) is a left K-sequentially complete dislocated quasi metric space. If $(x_0, y_0) = (4, 1), r = 28$, then

$$\overline{B_{d_q}((4,1), 28)} = \{(x, y) \in X : x + 4y \leq 42\}.$$

In particular, $(4, 1) \in \overline{B_{d_q}((4,1), 28)}$.

Let $S, T : X^2 \to X^2$ be defined by

$$S(x, y) = \begin{cases} \left(\dfrac{x}{7}, \dfrac{y}{7}\right), & \text{if } x + 4y \leq 42 \\ \\ (2x^2 - 2, 4x + 5), & \text{if } x + 4y > 42 \end{cases}$$

and

$$T(x, y) = \begin{cases} \left(\dfrac{x}{6}, \dfrac{y}{9}\right), & \text{if } x + 4y \leq 42 \\ \\ (3x^2 - 3, y), & \text{if } x + 4y > 42. \end{cases}$$

Also, define $\alpha : X^2 \times X^2 \to [0, \infty)$ by

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } \frac{x_1}{4} + y_1 + x_2 + y_2 \le 42 \\ \\ 0, & \text{if } \frac{x_1}{4} + y_1 + x_2 + y_2 > 42. \end{cases}$$

Clearly, S and T are α-dominated mappings on $\overline{B_{d_q}((4,1),28)}$. Let $\beta = \frac{1}{7}$, $\gamma = \delta = \frac{1}{10}$, then $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta} = \frac{3}{10} \in (0,1)$ and $(1-\lambda)r = 16$, $d_q((x_0, y_0), S(x_0, y_0) = d_q((4,1), S(4,1)) = \frac{104}{7} < 16 = (1-\lambda)r$. Observe that, for $(43, 0) \notin \overline{B_{d_q}((4,1),28)}$, we have $d_q(S(43,0), T(43,0)) = d_q((3696,5),(5544,0)) = 5104$, $d_q((43,0),T(43,0)) + d_q((43,0),S(43,0)) = 2401$, and $d_q((43,0),(43,0)) = \frac{215}{4}$. Hence, there are no β, γ, δ such that $\beta + 2\gamma + \delta \in (0,1)$ and (3.1) is satisfied. So the contractive condition does not hold on X^2. On the other hand, if $(x_1, y_1), (x_2, y_2) \in \overline{B_{d_q}((4,1),28)}$, then

$$\begin{aligned} d_q(S(x_1,y_1), T(x_2,y_2)) &= d_q\Big(\Big(\frac{x_1}{7}, \frac{y_1}{7}\Big), \Big(\frac{x_2}{6}, \frac{y_2}{9}\Big)\Big) \\ &= \frac{x_1}{7} + \frac{4y_1}{7} + \frac{x_2}{24} + \frac{y_2}{9} \\ &\le \frac{1}{7} d_q((x_1,y_1),(x_2,y_2)) \\ &\quad + \frac{1}{10}[d_q((x_1,y_1), S(x_1,y_1)) + d_q((x_2,y_2), T(x_2,y_2))] \\ &\quad + \frac{1}{10}[d_q((x_2,y_2), S(x_1,y_1)) + d_q((x_1,y_1), T(x_2,y_2))]. \end{aligned}$$

Also,

$$\begin{aligned} d_q(T(x_1,y_1), S(x_2,y_2)) &= d_q\Big(\Big(\frac{x_1}{6}, \frac{y_1}{9}\Big), \Big(\frac{x_2}{7}, \frac{y_2}{7}\Big)\Big) \\ &= \frac{x_1}{6} + \frac{4y_1}{9} + \frac{x_2}{28} + \frac{y_2}{7} \\ &\le \frac{1}{7} d_q((x_1,y_1),(x_2,y_2)) \\ &\quad + \frac{1}{10}[d_q((x_1,y_1), S(x_1,y_1)) + d_q((x_2,y_2), T(x_2,y_2))] \\ &\quad + \frac{1}{10}[d_q((x_2,y_2), S(x_1,y_1)) + d_q((x_1,y_1), T(x_2,y_2))]. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied. Moreover, $(0,0)$ is the common fixed point of S and T.

Corollary 3.3. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $S : X \to X$ be a mapping. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S be an α-dominated mappings on $B_{d_q}(x_0, r)$. Suppose that $x_0 \in B_{d_q}(x_0, r)$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + \delta \in (0,1)$ and the following condition holds: if $\alpha(x,y) \ge 1$ or $\alpha(y,x) \ge 1$ and $x, y \in B_{d_q}(x_0, r)$, then*

$$d_q(Sx, Sy) \le \beta d_q(x,y) + \gamma[d_q(x, Sx) + d_q(y, Sy)] + \delta[d_q(y, Sx) + d_q(x, Sy)]$$

and

$$d_q(x_0, Sx_0) \le (1-\lambda) r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d_q) is α-regular on $\overline{B_{d_q}(x_0, r)}$. Then there exists a point $z \in \overline{B_{d_q}(x_0, r)}$ such that $z = Sz$ and $d_q(z, z) = 0$.

Proof. Letting $T = S$ in Theorem 3.1, we obtain the following result. □

Corollary 3.4. *Let (X, d) be a complete dislocated metric space and $S, T : X \to X$ be two mappings. Let $x_0 \in X$, $r > 0$ and there exists a function $\alpha : X \times X \to [0, +\infty)$ such that S and T are α-dominated mappings on $\overline{B_d(x_0, r)}$. Suppose that $x_0 \in \overline{B_d(x_0, r)}$ and there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + \delta \in (0, 1)$ and the following condition holds: if $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$ and $x, y \in B_d(x_0, r)$, then*

$$d(Sx, Ty) \leq \beta d(x, y) + \gamma[d(x, Sx) + d(y, Ty)] + \delta[d(y, Sx) + d(x, Ty)]$$

and

$$d(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Suppose that (X, d) is α-regular on $\overline{B_d(x_0, r)}$. Then there exists a point $z \in \overline{B_d(x_0, r)}$ such that $z = Sz$ and $d(z, z) = 0$.

Proof. By Theorem 3.1, we obtain the following result. □

Theorem 3.5. *Suppose that all the conditions of Theorem 3.1 are satisfied. In addition suppose that:*

(a) *The function α is a triangular function on $\overline{B_{d_q}(x_0, r)}$.*
(b) *For $x, y \in \overline{B_{d_q}(x_0, r)}$ there exists $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(x, u_0) \geq 1, \alpha(y, u_0) \geq 1$.*
(c) *For all $u \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(Sx_0, u) \geq 1$ the following condition holds*

$$d_q(x_0, Sx_0) + d_q(u, Tu) + d_q(u, Sx_0) + d_q(x_0, Tu) \leq d_q(x_0, u) + d_q(Sx_0, Tu).$$

Then S and T have a unique common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ and $d_q(z, z) = 0$.

Proof. Define the sequence fxng as in the proof Theorem 3.1. Then, $\{x_n\}, d_q$-converges to a common fixed point $z \in \overline{B_{d_q}(x_0, r)}$ of the mappings S and T such that $\alpha(x_n, z) \geq 1$ for all $n \geq 0, (P_n)$ holds and $d_q(z, z) = 0$. In order to prove uniqueness of z, suppose that z^* is another point in $\overline{B_{d_q}(x_0, r)}$ such that $z^* = Sz^* = Tz^*$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(z^*, Sz^*) = \alpha(z^*, z^*) \geq 1$. Therefore,

$$\begin{aligned} d_q(z^*, z^*) &\leq d_q(Sz^*, Tz^*) \\ &\leq \beta d_q(z^*, z^*) + \gamma[d_q(z^*, Sz^*) + d_q(z^*, Tz^*)] \\ &\quad + \delta[d_q(z^*, Sz^*) + d_q(z^*, Tz^*)] \\ &\leq (\beta + 2\gamma + 2\delta) d_q(z^*, z^*), \end{aligned}$$

and this implies that

$$d_q(z^*, z^*) = 0.$$

By assumption, there exists a point $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that $\alpha(z, u0) \geq 1$ and $\alpha(z^*, u_0) \geq 1$. Define a sequence $\{u_n\}$ in X such that,

$$u_{2k+1} = Su_{2k} \text{ and } u_{2k+2} = Tu_{2k+1}, \ \forall k = 0, 1, 2, \ldots.$$

By mathematical induction, we can show that

$$\begin{cases} \alpha(u_n, u_{n+1}) \geq 1, \alpha(x_n, u_n) \geq 1, \ \forall n \in \mathbb{N}; \\ d_q(u_n, u_{n+1}) \leq \lambda^n d_q(u_0, u_1), \ \forall n \in \mathbb{N}; \\ d_q(x_n, z_n) \leq \lambda^n r, \ u_n \in \overline{B_{d_q}(x_0, r)}, \ \forall n \in \mathbb{N}. \end{cases} \qquad (P'_n)$$

Since T is α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$, we have $\alpha(u_0,Tu_0)=\alpha(u_0,u_1)\geq 1$. Since α is triangular function on $\overline{B_{d_q}(x_0,r)}$, and $\alpha(x_n,z)\geq 1, \alpha(z,u_0)\geq 1$, we have $\alpha(x_n,u_0)\geq 1$ for all $n\geq 0$. Therefore, using (c), we get that

$$\begin{aligned} d_q(x_1,u_1) &= d_q(Sx_0,Tu_0) \\ &\leq \beta d_q(x_0,u_0)+\gamma[d_q(x_0,Sx_0)+d_q(u_0,Tu_0)] \\ &\quad +\delta[d_q(u_0,Sx_0)+d_q(x_0,Tu_0)] \\ &\leq \beta d_q(x_0,u_0)+\gamma[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)] \\ &\quad +\delta[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)] \\ &\leq \beta d_q(x_0,u_0)+\gamma[d_q(x_0,u_0)+d_q(x_1,u_1)] \\ &\quad +\delta[d_q(u_0,x_0)+d_q(x_1,u_1)]. \end{aligned}$$

Thus,

$$d_q(x_1,u_1)\leq \frac{\beta+\gamma+\delta}{1-\gamma-\delta}d_q(x_0,u_0)=\lambda d_q(x_0,u_0)\leq \lambda r. \tag{3.10}$$

Since $u_0\in\overline{B_{d_q}(x_0,r)}$, using (3.10), we get

$$\begin{aligned} d_q(x_0,u_1) &\leq d_q(x_0,x_1)+d_q(x_1,u_1) \\ &\leq (1-\lambda)r+\lambda d_q(x_0,u_0) \\ &\leq (1-\lambda)r+\lambda r\leq r \end{aligned}$$

Hence, $u_1\in\overline{B_{d_q}(x_0,r)}$. Since $\alpha(u_0,u_1)\geq 1$, by using (3.2), we get that

$$d_q(u_1,u_2)\leq \frac{\beta+\gamma+\delta}{1-\gamma-\delta}d_q(u_0,u_1)=\lambda d_q(u_0,u_1).$$

Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$, we have $\alpha(u_1,Su_1)=\alpha(u_1,u_1)\geq 1$. As, α is a triangular function on $\overline{B_{d_q}(x_0,r)}$, and $\alpha(x_1,u_0)\geq 1$, $\alpha(u_0,u_1)\geq 1$, we have $\alpha(x_1,u_1)\geq 1$. Therefore, from (P_1') holds. Since $\alpha(u_1,u_2)\geq 1$ and using (3.1), we get that

$$d_q(u_2,u_3)\leq \lambda d_q(u_1,u_2)\leq \lambda^2 d_q(u_0,u_1).$$

Since $\alpha(x_1,u_1)\geq 1$, using (3.2) that

$$\begin{aligned} d_q(x_2,u_2) &= d_q(Tx_1,Su_1) \\ &\leq \beta d_q(x_1,u_1)+\gamma[d_q(x_1,Tx_1)+d_q(u_1,Su_1)] \\ &\quad +\delta[d_q(u_1,Tx_1)+d_q(x_1,Su_1)] \\ &\leq \beta d_q(x_1,x_2)+\gamma\lambda[d_q(x_0,Sx_0)+d_q(u_0,Tu_0)] \\ &\quad +\delta\lambda[d_q(u_0,Sx_0)+d_q(x_0,Tu_0)] \end{aligned}$$

which gives wiyh (c)

$$\begin{aligned} d_q(x_2,u_2) &\leq \beta d_q(x_1,x_2)+\gamma\lambda[d_q(x_0,u_0)+d_q(Sx_0,Tu_0)] \\ &\quad +\delta\lambda[d_q(u_0,x_0)+d_q(Sx_0,Tu_0)] \\ &\leq (\beta+\lambda\gamma+\lambda\delta)d_q(x_1,u_1)+(\gamma\lambda+\delta\lambda)r. \end{aligned}$$

By using (3.10) and fact that $u_0\in\overline{B_{d_q}(x_0,r)}$, in above inequality we obtain

$$\begin{aligned} d_q(x_2,u_2) &\leq (\beta+\lambda\gamma+\lambda\delta)\lambda r+(\gamma\lambda+\delta\lambda)r \\ &= (\beta+\lambda\gamma+\lambda\delta+\gamma+\delta)\lambda r=\lambda^2 r. \end{aligned}$$

Thus,

$$\begin{aligned} d_q(x_0, u_2) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + d_q(x_2, u_2) \\ &\leq d_q(x_0, x_1) + \lambda d_q(x_0, x_1) + \lambda^2 \leq r. \end{aligned}$$

Hence, $u_2 \in \overline{B_{d_q}(x_0, r)}$. Since T is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$, we have $\alpha(x_0, Sx_0) = \alpha(u_2, u_3) \geq 1$. Therefore, from (P_2') holds. Suppose, $(P_1'), (P_2'), \ldots, (P_i')$ be the inductive hypothesis. We shall show that (P_{i+1}') holds. For this, we consider two possible cases. First, suppose that i is even. Then, since $\alpha(u_i, u_{i+1}) \geq 1$ and using (3.2), we get that

$$d_q(u_{i+1}, u_{i+2}) \leq \frac{\beta + \gamma + \delta}{1 - \gamma - \delta} d_q(u_i, u_{i+1}) = \lambda^{i+1} d_q(u_0, u_1).$$

Since $\alpha(x_i, u_i) \geq 1$, using (3.1) that

$$\begin{aligned} d_q(x_{i+1}, u_{i+1}) &= d_q(Sx_i, Tu_i) \\ &\leq \beta d_q(x_i, u_i) + \gamma[d_q(x_i, Sx_i) + d_q(u_i, Tu_i)] \\ &\quad + \delta[d_q(u_i, Sx_i) + d_q(x_i, Tu_i)] \\ &\leq \beta d_q(x_i, x_{i+1}) + \gamma\lambda[d_q(x_0, Sx_0) + d_q(u_0, Tu_0)] \\ &\quad + \delta\lambda[d_q(u_0, Sx_0) + d_q(x_0, Tu_0)] \end{aligned}$$

which gives wiyh (c) and P_i'

$$\begin{aligned} d_q(x_{i+1}, u_{i+1}) &\leq \beta d_q(x_i, u_i) + \gamma\lambda^i[d_q(x_0, u_0) + d_q(Sx_0, Tu_0)] \\ &\quad + \delta\lambda^i[d_q(u_0, x_0) + d_q(Sx_0, Tu_0)] \\ &\leq \beta\lambda^i r + \gamma\lambda^i[r + \lambda r] + \delta\lambda^i[r + \lambda r] \\ &= (\beta + \lambda\gamma + \lambda\delta + \gamma + \delta)\lambda^i r = \lambda^{i+1} r. \end{aligned}$$

Thus,

$$\begin{aligned} d_q(x_0, u_{i+1}) &\leq d_q(x_0, x_1) + d_q(x_1, x_2) + \cdots + d_q(x_i, x_{i+1}) + d_q(x_{i+1}, u_{i+1}) \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^i) d_q(x_0, x_1) + \lambda^{i+1} r \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^i)(1 - \lambda) r + \lambda^{i+1} r = r. \end{aligned}$$

Hence, $u_{i+1} \in \overline{B_{d_q}(x_0, r)}$. Since S is an α-dominated mapping on $\overline{B_{d_q}(x_0, r)}$., we have $\alpha(u_{i+1}, Su_{i+1}) = \alpha(u_{i+1}, u_{i+2}) \geq 1$. Also, since $\alpha(x_{i+1}, u_0) \geq 1$, $\alpha(u_n, u_{n+1}) \geq 1$, $n = 0, 1, 2, \ldots, i+1$, by triangular nature of α, we have $\alpha(x_{i+1}, u_{i+1}) \geq 1$. Therefore, (P_{i+1}') holds. Similarly, one can see that if i is odd, then (P_{i+1}') holds, which completes the inductive proof. Thus, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d_q(z, u_{2n}) &= d_q(Tz, Su_{2n-1}) \\
&\leq \beta d_q(z, u_{2n-1}) + \gamma[d_q(z, Tz) + d_q(u_{2n-1}, Su_{2n-1})] \\
&\quad + \delta[d_q(u_{2n-1}, Tz) + d_q(z, Su_{2n-1})] \\
&\leq \beta d_q(z, u_{2n-1}) + \gamma d_q(u_{2n-1}, u_{2n}) \\
&\quad + \delta[d_q(u_{2n-1}, z) + d_q(z, u_{2n})] \\
&= (\beta + \delta) d_q(z, u_{2n-1}) + \gamma d_q(u_{2n-1}, u_{2n}) + \delta d_q(z, u_{2n}) \\
&\leq (\beta + 2\delta) d_q(z, u_{2n-1}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta) d_q(Tz, Su_{2n-2}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta)^2 d_q(z, u_{2n-2}) + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1})] \\
&\quad + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta)^2 d_q(Tz, Su_{2n-3}) + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1}) \\
&\quad + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\leq (\beta + 2\delta)^3 d_q(z, u_{2n-3}) + (\beta + 2\delta)^2 (\gamma + \delta) d_q(u_{2n-3}, u_{2n-2}) \\
&\quad + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n}) \\
&\quad \vdots \\
&\leq (\beta + 2\delta)^{2n} d_q(z, u_0) + (\beta + 2\delta)^{2n-1} (\gamma + \delta) d_q(u_0, u_1) + \dots \\
&\quad + (\beta + 2\delta)(\gamma + \delta) d_q(u_{2n-2}, u_{2n-1}) + (\gamma + \delta) d_q(u_{2n-1}, u_{2n})
\end{aligned}$$

Since $\frac{\beta+2\delta}{\lambda} = \frac{(\beta+2\delta)(1-\gamma-\delta)}{\beta+\gamma+\delta} < 1$, using (P_n') in the above inequality we obtain

$$\begin{aligned}
d_q(z, u_{2n}) &\leq (\beta + 2\delta)^{2n} d_q(z, u_0) + (\beta + 2\delta)^{2n-1} (\gamma + \delta) d_q(u_0, u_1) + \dots \\
&\quad + (\beta + 2\delta)(\gamma + \delta) \lambda^{2n-2} d_q(u_0, u_1) + (\gamma + \delta) \lambda^{2n-1} d_q(u_0, u_1) \\
&= (\beta + 2\delta)^{2n} d_q(z, u_0) \\
&\quad + (\gamma + \delta) \lambda^{2n-1} d_q(u_0, u_1) \Big[1 + \frac{\beta + 2\delta}{\lambda} + \dots + \Big(\frac{\beta + 2\delta}{\lambda} \Big)^{2n-1} \Big] \\
&\leq (\beta + 2\delta)^{2n} d_q(z, u_0) + \frac{(\gamma + \delta) \lambda^{2n-1} d_q(u_0, u_1)}{1 - \frac{\beta+2\delta}{\lambda}}
\end{aligned}$$

Since $\beta + 2\delta, \lambda \in [0, 1)$, it follows from the above inequality that

$$\lim_{n \to \infty} d_q(z, u_{2n}) = 0. \tag{3.11}$$

Similarly, we can show that

$$\lim_{n \to \infty} d_q(u_{2n}, z) = \lim_{n \to \infty} d_q(u_{2n}, z^*) = \lim_{n \to \infty} d_q(z^*, u_{2n}) = 0. \tag{3.12}$$

By using (3.11) and (3.12), we obtain

$$d_q(z, z^*) \leq d_q(z, u_{2n}) + d_q(u_{2n}, z^*) \to 0, \text{ as } n \to \infty.$$

$$d_q(z^*, z) \leq d_q(z^*, u_{2n}) + d_q(u_{2n}, z) \to 0, \text{ as } n \to \infty.$$

Hence, $d_q(z, z^*) = d_q(z^*, z) = 0$, i.e., $z = z^*$ □

Corollary 3.6. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space and $T, S : X \to X$ be two mappings. Let $x_0 \in X, r > 0, x_0 \in \overline{B_{d_q}(x_0, r)}$ and* *there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$. Then there exists a unique point $z \in \overline{B_{d_q}(x_0, r)}$ such that $z = Sz = Tz$ and $d_q(z, z) = 0$. Moreover, S and T have no fixed point in $\overline{B_{d_q}(x_0, r)}$ other than z.

Proof. The proof follows by the previous results, taking $\alpha : X \times X \to [0, \infty)$ with $\alpha(x, y) = 1$ for all $x, y \in X$. □

Theorem 3.7. *Let (X, d_q) be a left K-sequentially complete dislocated quasi metric space. Suppose, there exist a function $\alpha : X \times X \to [0, +\infty)$ and nonnegative constants β, γ, δ such that $\beta + 2\gamma + 2\delta \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

for all $x, y \in X$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$. If (X, d_q) is α-regular,then there exists a point z in X such that $z = Sz = Tz$ and $d_q(z, z) = 0$.

Proof. By Theorem 3.1, the condition (3.3) is imposed in order to restrict the contractive conditions (3.1) and (3.2) to $\overline{B_{d_q}(x_0, r)}$. However, the condition (3.3) can be relaxed by imposing the conditions (3.1) and (3.2) to all elements $x, y \in X$ such that $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$, we obtain the following result. □

Recall that, if $(X, \preceq)$ is a pre-ordered set and $T : X \to X$ is such that $Tx = x$ for all $x \in A \subseteq X$, then the mapping T is said to be dominated on A. Define the set ∇ by

$$\nabla = \{(x, y) \in X \times X : x \preceq y \,\text{or}\, y \preceq x\}.$$

Theorem 3.8. *Let $(X, \preceq, d_q)$ be a pre-ordered left K-sequentially complete dislocated quasi metric space, $x_0 \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings on $\overline{B_{d_q}(x_0, r)}$. Suppose that there exist nonnegative real numbers β, γ, δ such that $\beta + 2\gamma + \delta \in (0, 1)$ and the following conditions hold:*

$$d_q(Sx, Ty) \leq \beta d_q(x, y) + \gamma[d_q(x, Sx) + d_q(y, Ty)] + \delta[d_q(y, Sx) + d_q(x, Ty)],$$
$$d_q(Tx, Sy) \leq \beta d_q(x, y) + \gamma[d_q(x, Tx) + d_q(y, Sy)] + \delta[d_q(y, Tx) + d_q(x, Sy)],$$

for all $(x, y) \in \overline{B_{d_q}(x_0, r)} \times \overline{B_{d_q}(x_0, r)} \cap \nabla$ and

$$d_q(x_0, Sx_0) \leq (1 - \lambda)r,$$

where $\lambda = \frac{\beta+\gamma+\delta}{1-\gamma-\delta}$ If for any sequence $\{x_n\} \in \overline{B_{d_q}(x_0, r)}$ such that $(x_n, x_{n+1}) \in \nabla$, $x_n \to w$ as $n \to \infty$ implies that $(w, x_n) \in \nabla$ for all $n \geq 0$, then there exists a point $z \in \overline{B_{d_q}(x_0, r)}$ such that $z = Sz = Tz$ and $d_q(z, z) = 0$. In addition, suppose that:

(a) *$(x, y), (y, z) \in \nabla$ implies $(x, z) \in \nabla$.*
(b) *For x, $y \in \overline{B_{d_q}(x_0, r)}$ there exists $u_0 \in \overline{B_{d_q}(x_0, r)}$ such that (x, u_0), $(y, u_0) \in \nabla$.*
(c) *For all $u \in \overline{B_{d_q}(x_0, r)}$ such that $(u, Sx_0) \in \nabla$ the following condition holds*

$$d_q(x_0, Sx_0) + d_q(u, Tu) + d_q(u, Sx_0) + d_q(x_0, Tu) \leq d_q(x_0, u) + d_q(Sx_0, Tu).$$

Then, z is the unique common fixed point of S and T in $\overline{B_{d_q}(x_0, r)}$.

Proof. This follows from Theorem 3.6 taking $\alpha : X \times X \to [0, +\infty)$ defined as

$$\alpha(x, y) = \begin{cases} 1, & \text{If } (x, y) \in \nabla, \\ 0, & \text{otherwise.} \end{cases}$$

□

4. Conclusions

We prov some common fixed point theorems for mappings under Hardy Rogers contractive conditions on a left K-sequentially complete dislocated quasi metric space.

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