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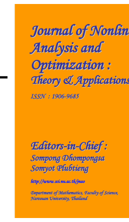

AN HYBRID EXTRAGRADIENT ALGORITHM FOR VARIATIONAL INEQUALITIES WITH PSEUDOMONOTONE EQUILIBRIUM CONSTRAINTS

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ABSTRACT. In this paper, we propose a new hybrid extragradient algorithm for solving a variational inequality problem over the solution set of an equilibrium problem in Euclidean space. By using fixed point and hybrid plane cutting techniques, we show that this problem can be solved by an explicit extragradient method. Under certain conditions on parameters, the convergence of the iteration sequences generated by the algorithm are obtained.

KEYWORDS: Variational inequalities; equilibrium problems; KyFan inequality; auxiliary subproblem principle; projection method; Armijo linesearch; pseudomonotone.
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1. Introduction and Motivation

Let $\mathbb{R}^n$ be a n-dimentional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. Let C be a nonempty closed convex subset in $\mathbb{R}^n$ and $G : C \longrightarrow \mathbb{R}^n$ be an operator, and $f : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$ for every $x \in C$. We consider the following variational inequality problem over the solution set of the equilibrium problem (shortly VIEP(C, f, G)):

$$\text{Find } x^* \in S_f \text{ such that } \langle G(x^*), y - x^* \rangle \geq 0 \ \forall y \in S_f, \tag{1.1}$$

where $S_f = \{u \in C : f(u, y) \geq 0, \ \forall y \in C\}$, i.e., S_f is the solution set of the following equilibrium problems (EP(C, f) for short):

$$\text{Find } u \in C \text{ such that } f(u, y) \geq 0 \ \forall y \in C. \tag{1.2}$$

As usual, we call problem (1.1) the upper problem and (1.2) the lower one. Problem (1.1) can be consider as a special case of mathematical programs with equilibrium constraints. Sources for such problems can be found in [11, 15, 17]. Bilevel variational inequalities were considered in [1], Moudafi in [16] and Yao et al

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in [22] suggested the use of the proximal point method for monotone bilevel equilibrium problems, which contain monotone variational inequalites as a special case. Recently, Ding in [6] used the auxiliary problem principle to monotone bilevel equilibrium problems. In those papers, the lower problem is required to be monotone. In this case the subproblems to be solved are monotone.

It should be noticed that the solution set S_f of the lower problem (1.2) is convex whenever f is pseudomonotone on C. However, the main difficulty is that, even the constrained set S_f is convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the available methods of convex optimization and variational inequality cannot be applied directly to problem (1.1).

In our recent paper [4] we proposed penalty and gap function methods for solving bilevel equilibrium problems which contains (1.1) as a special case. Under a certain strictly ∇-pseudomonotonicity, it has been proved that any stationary point of the gap function over C is a solution of the penalized problem. This assumption is satisfied for strict monotonicity case, but it may fail to hold for problem (1.1) when the lower equilibrium problem is pseudomonotone. The reason is that the sum of a strongly monotone and a pseudomonotone bifunction, in general, is not pseudomonotone, even not strongly monotone.

In this paper, we continue our work in [4] by further extend the hybrid extragradient-viscosity methods introduced by Maingé in [13] for solving bilevel problem (1.1) when the lower problem is pseudomonotone with respect to its solution set equilibrium problems rather than monotone variational inequalities as in [13], the later pseudomonotonicity is somewhat general than pseudomonotone. We show that the sequence of iterates generated by the proposed algorithm converges to the unique solution of the bilevel problem (1.1).

The paper is organized as follows. The next section contains some preliminaries on the Euclidean projection and equilibrium problems. The third section is devoted to presentation of the algorithm and its convergence. In the last section, we describe a special case of minimizing the Euclidean norm over the solution set of an equilibrium problem, where the bifunction is pseudomonotone with respect to its solution set. The latter problem arises from the Tikhonov regularization method for pseudomonotone equilibrium problems [8].

2. Preliminaries

Throughout the paper, by P_C we denote the projection operator on C with the norm $\|.\|$, that is

$$P_C(x) \in C: \ \|x - P_C(x)\| \le \|y - x\| \ \forall y \in C.$$

The following well known results on the projection operator onto a closed convex set will be used in the sequel.

Lemma 2.1. *Suppose that C is a nonempty closed convex set in $\mathbb{R}^n$. Then*
(i) $P_C(x)$ is singleton and well defined for every x;
(ii) $\pi = P_C(x)$ if and only if $\langle x - \pi, y - \pi\rangle \le 0, \forall y \in C$;
(iii) $\|P_C(x) - P_C(y)\|^2 \le \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \ \forall x, y \in C$.

We recall some well known definitions on monotonicity (see e.g., [2, 7, 9, 17, 21])

Definition 2.2. A bifunction $\varphi : C \times C \longrightarrow \mathbb{R}$ is said to be
(a) strongly monotone on C with modulus $\beta > 0$, if

$$\varphi(x, y) + \varphi(y, x) \le -\beta\|x - y\|^2 \ \forall x, y \in C;$$

(b) monotone on C if

$$\varphi(x,y) + \varphi(y,x) \leq 0 \ \forall x,y \in C;$$

(c) pseudomonotone on C if

$$\varphi(x,y) \geq 0 \Longrightarrow \varphi(y,x) \leq 0 \ \forall x,y \in C;$$

(d) pseudomonotone on C with respect to x^* if

$$\varphi(x^*,y) \geq 0 \Longrightarrow \varphi(y,x^*) \leq 0 \ \forall y \in C.$$

We say that φ is pseudomonotone on C with respect to a set S if it is pseudomonotone on C with respect to every point $x^* \in S$.

From the definitions it follows that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \ \forall x^* \in C$.

When $\varphi(x,y) = \langle \phi(x), y-x \rangle$, where $\phi : C \longrightarrow \mathbb{R}^n$ is an operator then the definition (a) becomes:

$$\langle \phi(x) - \phi(y), x-y \rangle \geq \beta \|x-y\|^2 \ \forall x,y \in C$$

i.e., ϕ is β-strongly monotone on C. Similarly, if φ satisfies (b) ($(c), (d)$ resp) on C then ϕ becomes monotone, (pseudomonotone, pseudomonotone with respect to x^* resp) on C.

In the sequel, we need the following blanket assumptions

$(A1)$ $f(.,y)$ is continuous on Ω for every $y \in C$;
$(A2)$ $f(x,.)$ is convex on Ω for every $x \in C$;
$(A3)$ f is pseudomonotone on C with respect to the solution set S_f of EP(C,f);
$(A4)$ G is L-Lipschitz and β-strongly monotone on C;
$(B1)$ $h(.)$ is δ-strongly convex, continuously differentiable on Ω;
$(B2)$ $\{\lambda_k\}$ is a positive sequence such that $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\sum_{k=0}^{\infty} \lambda_k^2 < \infty$.

Lemma 2.3. *Suppose Problem EP(C,f) has a solution. Then under Assumptions $(A1)$, $(A2)$ and $(A3)$ the solution set S_f is closed, convex and*

$$f(x^*,y) \ \geq 0 \ \forall y \in C \textit{ if and only if } f(y,x^*) \leq 0 \ \forall y \in C.$$

The proof of this lemma when f is pseudomonotone on C can be found, for instance, in [9, 17]. When f is pseudomonotone with respect to the solution set of EP(C,f), it can be done by the same way. So we omit it.

The following lemmas are well-known from the auxiliary problem principle for equilibrium problems.

Lemma 2.4. *([14]) Suppose that h is a continuously differentiable and strongly convex function on C with modulus $\delta > 0$. Then under Assumptions $(A1)$ and $(A2)$, a point $x^* \in C$ is a solution of EP(C,f) if and only if it is a solution to the equilibrium problem:*

$$\textit{Find } x^* \in C : f(x^*,y) + h(y) - h(x^*) - \langle \nabla h(x^*), y-x^* \rangle \geq 0 \ \forall y \in C. \qquad (AEP)$$

The function

$$D(x,y) := h(y) - h(x) - \langle \nabla h(x), y-x \rangle$$

is called Bregman function. Such a function was used to define a generalized projection, called D-projection, which was used to develop algorithms for particular problems, see e.g., [3]. An important case is $h(x) := \frac{1}{2}\|x\|^2$. In this case D-projection becomes the Euclidean one.

Lemma 2.5. *([14]) Under Assumptions $(A1)$, $(A2)$, a point $x^* \in C$ is a solution of Problem (AEP) if and only if*

$$x^* = \text{argmin}\{f(x^*, y) + h(y) - h(x^*) - \langle \nabla h(x^*), y - x^* \rangle : \; y \in C\}. \qquad (CP)$$

Note that, since $f(x,.)$ is convex and h is strongly convex, Problem (CP) is a strongly convex program.

For each $z \in C$, by $\partial_2 f(z,z)$ we denote the subgradient of the convex function $f(z,.)$ at z, i.e.,

$$\partial_2 f(z,z) := \{w \in \mathbb{R}^n : f(z,y) \geq f(z,z) + \langle w, y - z \rangle, \; \forall y \in C\}$$

$$= \{w \in \mathbb{R}^n : f(z,y) \geq \langle w, y - z \rangle, \; \forall y \in C\},$$

and we define the halfspace H_z as

$$H_z := \{x \in \mathbb{R}^n : \; \langle w, x - z \rangle \leq 0\} \qquad (2.1)$$

where $w \in \partial_2 f(z,z)$. Note that when $f(x,y) = \langle F(x), y - x \rangle$, this halfspace becomes the one introduced in [21]. The following lemma says that the hyperplane does not cut off any solution of problem EP(C, f).

Lemma 2.6. *([5]) Under Assumptions (A2) and (A3), one has $S_f \subseteq H_z$ for every $z \in C$.*

Lemma 2.7. *([5]) Under Assumptions (A1) and (A2), if $\{z^k\} \subset C$ is a sequence such that $\{z^k\}$ converges to $\bar{z}$ and the sequence $\{w^k\}$ with $w^k \in \partial_2 f(z^k, z^k)$ converges to $\bar{w}$, then $\bar{w} \in \partial_2 f(\bar{z}, \bar{z})$.*

The following lemma is in [21] (see also [5]).

Lemma 2.8. *([21], [5]) Suppose that $x \in C$ and $u = P_{C \cap H_z}(x)$. Then*

$$u = P_{C \cap H_z}(\bar{x}), \text{ where } \bar{x} = P_{H_z}(x).$$

Lemma 2.9. *(Lemma 3.1 [12]) Let $\{a_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ such that*

$$a_{k_j} < a_{k_j+1} \text{ for all } j \geq 0$$

Also consider the sequence of integers $\{\sigma(k)\}_{k \geq k_0}$ defined by

$$\sigma(k) = \max\{j \leq k \mid a_j < a_{j+1}\}.$$

Then $\{\sigma(k)\}_{k \geq k_0}$ is a nondecreasing sequence verifying

$$\lim_{k \longrightarrow \infty} \sigma(k) = \infty$$

and, for all $k \geq k_0$, the following two estimates hold:

$$a_{\sigma(k)} \leq a_{\sigma(k)+1} \qquad (2.2)$$

$$a_k \leq a_{\sigma(k)+1} \qquad (2.3)$$

3. AN HYBRID EXTRAGRADIENT ALGORITHM FOR VIEP(C, F, G)

Algorithm 1. Pick $x^0 \in C$ and choose two parameters $\eta \in (0,1)$, $\rho > 0$.
At each iteration $k = 0, 1, ...$ having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min \Big\{ f(x^k, y) + \frac{1}{\rho}\Big[h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k\rangle\Big] :\ y \in C \Big\} \qquad CP(x^k)$$

to obtain its unique solution y^k.

If $y^k = x^k$, take $u^k = x^k$ and go to *Step 3.* Otherwise, do Step 2.

Step 2. (Armijo linesearch rule) Find m_k as the smallest positive integer number m satisfying

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^k : \\ \langle w^{k,m}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \\ \text{with } \ w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}). \end{cases} \tag{3.1}$$

Step 3. Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}$, $w^k := w^{k,m_k}$. Take

$$C_k := \{x \ \in C : \langle w^k, x - z^k\rangle \leq 0\},\ u^k := P_{C_k}(x^k). \tag{3.2}$$

Step 4. $x^{k+1} = P_C(u^k - \lambda_k G(u^k))$ and go to Step 1 with k is replaced by $k+1$.

Remark 3.1. (i) If $y^k = x^k$ then x^k is a solution to EP(C, f).

(ii) $w^k \neq 0\ \forall k$, indeed, at the begining of Step 2, $x^k \neq y^k$. By the Armijo linesearch rule and δ-strong convexity of h, we have

$$\langle w^k, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \geq$$
$$\geq \frac{\delta}{\rho}\|x^k - y^k\|^2 > 0.$$

Now we are going to analyze the validity and convergence of the algorithm. Some parts in our proofs are based on the proof scheme in [13].

Lemma 3.2. *Under Assumptions* $(A1)$, $(A2)$, $(A3)$, *and* $(A4)$, *the linesearch rule* (3.1) *is well-defined in the sense that, at each iteration* k, *there exists an integer number* $m > 0$ *satisfying the inequality in* (3.1) *for every* $w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$, *then for every solution* x^* *of EP*(C, f), *one has*

$$\begin{aligned} \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \big(\frac{\eta_k \delta}{\rho\|w^k\|}\big)^2\|x^k - y^k\|^4 \\ - 2\lambda_k\langle u^k - x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2\ \forall k. \end{aligned} \tag{3.3}$$

where $\bar{x}^k = P_{H_{z^k}}(x^k)$.

Proof. First we prove that there exists a positive integer m_0 such that

$$\langle w^{k,m_0}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big]$$
$$\forall w^{k,m_0} \in \partial_2 f(z^{k,m_0}, z^{k,m_0}).$$

Indeed, suppose by contradiction that, for every positive integer m and $z^{k,m} = (1-\eta^m)x^k + \eta^m y^k$ there exists $w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m})$ such that

$$\langle w^{k,m}, x^k - y^k\rangle < \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big].$$

Since $z^{k,m} \longrightarrow x^k$ as $m \longrightarrow \infty$, by Theorem 24.5 in [20], the sequence $\{w^{k,m}\}_{m=1}^{\infty}$ is bounded. Thus we may assume that $w^{k,m} \longrightarrow \bar{w}$ for some $\bar{w}$. Taking the limit as $m \longrightarrow \infty$, from $z^{k,m} \longrightarrow x^k$ and $w^{k,m} \longrightarrow \bar{w}$, by Lemma 2.7, it follows that $\bar{w} \in \partial_2 f(x^k, x^k)$ and

$$\langle \bar{w}, x^k - y^k \rangle \leq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big]. \tag{3.4}$$

Since $\bar{w} \in \partial_2 f(x^k, x^k)$, we have

$$f(x^k, y^k) \geq f(x^k, x^k) + \langle \bar{w}, y^k - x^k\rangle = \langle \bar{w}, y^k - x^k\rangle.$$

Combining with (3.4) yields

$$f(x^k, y^k) + \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \geq 0,$$

which contradicts to the fact that

$$f(x^k, y^k) + \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] < 0.$$

Thus, the linesearch is well defined.

Now we prove (3.3). For simplicity of notation, let $d^k := x^k - y^k$, $H_k := H_{z^k}$. Since $u^k = P_{C\cap H_k}(\bar{x}^k)$ and $x^* \in S_f$, by Lemma 2.6, $x^* \in C \cap H_k$, we have

$$\|u^k - \bar{x}^k\|^2 \leq \langle x^* - \bar{x}^k, u^k - \bar{x}^k\rangle$$

which together with

$$\|u^k - x^*\|^2 = \|\bar{x}^k - x^*\|^2 + \|u^k - \bar{x}^k\|^2 + 2\langle u^k - \bar{x}^k, \bar{x}^k - x^*\rangle$$

implies

$$\|u^k - x^*\|^2 \leq \|\bar{x}^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2. \tag{3.5}$$

Replacing

$$\bar{x}^k = P_{H_k}(x^k) = x^k - \frac{\langle w^k, x^k - z^k\rangle}{\|w^k\|^2} w^k$$

into (3.5) we obtain

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - 2\langle w^k, x^k - x^*\rangle \frac{\langle w^k, x^k - z^k\rangle}{\|w^k\|^2} + \frac{\langle w^k, x^k - z^k\rangle^2}{\|w\|^2}.$$

Substituting $x^k = z^k + \eta_k d^k$ into the last inequality we get

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 + \Big(\frac{\eta_k\langle w^k, d^k\rangle}{\|w^k\|}\Big)^2 - \frac{2\eta_k\langle w^k, d^k\rangle}{\|w^k\|^2}\langle w^k, x^k - x^*\rangle$$

$$= \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \Big(\frac{\eta_k\langle w^k, d^k\rangle}{\|w^k\|}\Big)^2 - \frac{2\eta_k\langle w^k, d^k\rangle}{\|w^k\|^2}\langle w^k, z^k - x^*\rangle.$$

In addition, by the Armijo linesearch rule, using the δ-strong convexity of h we have

$$\langle w^k, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \geq \frac{\delta}{\rho}\|x^k - y^k\|^2.$$

Note that $x^* \in H_k$ we can write

$$\|u^k - x^*\|^2 \leq \|x^k - x^*\|^2 - \|u^k - \bar{x}^k\|^2 - \Big(\frac{\eta_k \delta}{\rho\|w^k\|}\Big)^2 \|x^k - y^k\|^4. \tag{3.6}$$

We have

$$\|x^{k+1}-x^*\|^2 = \|P_C(u^k-\lambda_k G(u^k)) - P_C(x^*)\|^2 \le \|u^k - x^* - \lambda_k G(u^k))\|^2$$
$$= \|u^k-x^*\|^2 - 2\lambda_k\langle u^k - x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2,$$

which together with (3.6) implies

$$\|x^{k+1}-x^*\|^2 \le \|x^k-x^*\|^2 - \|u^k-\bar{x}^k\|^2 - \Big(\frac{\eta_k\delta}{\rho\|w^k\|}\Big)^2\|x^k-y^k\|^4$$
$$- 2\lambda_k\langle u^k-x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2 \ \forall k \tag{3.7}$$

as desired.

□

Lemma 3.3. *The sequences* $\{x^k\}, \{u^k\}$ *generated by the Algorithm 1, are bounded under Assumptions* $(A1)$, $(A2)$, $(A3)$, *and* $(A4)$.

Proof. We have

$$\|x^{k+1}-x^*\| = \|P_C(u^k-\lambda_k G(u^k)) - P_C(x^*)\| \le \|u^k - \lambda_k G(u^k) - x^*\|$$
$$\le \|(u^k-\lambda_k G(u^k)) - (x^*-\lambda_k G(x^*))\| + \lambda_k\|G(x^*)\|$$
$$= \|(1-L^2\frac{\lambda_k}{\beta})(u^k-x^*) - L^2\frac{\lambda_k}{\beta}\big[(\frac{\beta}{L^2}G-I)u^k - (\frac{\beta}{L^2}G-I)x^*\big]\|$$
$$+ \lambda_k\|G(x^*)\|$$
$$\le (1-L^2\frac{\lambda_k}{\beta})\|u^k-x^*\| + L^2\frac{\lambda_k}{\beta}T_k + \lambda_k\|G(x^*)\|, \tag{3.8}$$

where $T_k = \|(\frac{\beta}{L^2}G-I)u^k - (\frac{\beta}{L^2}G-I)x^*\|$.

Since G is L-Lipschitz and β-strongly monotone, we have

$$T_k^2 = \|\frac{\beta}{L^2}(G(u^k)-G(x^*)) - (u^k-x^*)\|^2$$
$$= \frac{\beta^2}{L^4}\|G(u^k)-G(x^*)\|^2 - 2\frac{\beta}{L^2}\langle G(u^k)-G(x^*), u^k-x^*\rangle + \|u^k-x^*\|^2$$
$$\le \frac{\beta^2}{L^2}\|u^k-x^*\|^2 - 2\frac{\beta^2}{L^2}\|u^k-x^*\|^2 + \|u^k-x^*\|^2$$
$$= (1-\frac{\beta^2}{L^2})\|u^k-x^*\|^2.$$

Hence $T_k \le \sqrt{1-\frac{\beta^2}{L^2}}\|u^k-x^*\|$. Then combining with (3.8) we get

$$\|x^{k+1}-x^*\| \le (1-\lambda_k\frac{L^2}{\beta}(1-\sqrt{1-\frac{\beta^2}{L^2}}\,))\|u^k-x^*\| + \lambda_k\|G(x^*)\|$$
$$= (1-\lambda_k\frac{L^2}{\beta}\gamma)\|u^k-x^*\| + \lambda_k\|G(x^*)\|$$
$$= (1-\gamma_k)\|u^k-x^*\| + \gamma_k(\frac{\beta}{L^2\gamma}\|G(x^*)\|)$$

where, $\gamma = 1-\sqrt{1-\frac{\beta^2}{L^2}}$ and $\gamma_k = \lambda_k\frac{L^2}{\beta}\gamma \in (0;1)$.

By induction we get

$\|x^{k+1}-x^*\| \le \max\{\|x^k-x^*\|, \frac{\beta}{L^2\gamma}\|F(x^*)\|\} \le ... \le \max\{\|x^0-x^*\|, \frac{\beta}{L^2\gamma}\|F(x^*)\|\}$.

Hence $\{x^k\}$ is bounded, which, from (3.6), implies that $\{u^k\}$ is bounded too. □

Lemma 3.4. *There exists a subsequence* $\{x^{k_i}\} \subset \{x^k\}$ *converges to some* $\bar{x} \in C$ *such that* $\{y^{k_i}\}, \{z^{k_i}\}, \{w^{k_i}\}$ *are bounded.*

Proof. First, we show that there exists $M > 0$ such that $\|x^{k_i} - y^{k_i}\| \leq M$ for all i large enough.

Indeed, from the δ-strong convexity of the function

$$f_{k_i}(.) = \rho f(x^{k_i}, .) + h(y) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), . - x^{k_i} \rangle$$

we have

$$\langle s(x^{k_i}) - s(y^{k_i}), x^{k_i} - y^{k_i} \rangle \geq \delta \|x^{k_i} - y^{k_i}\|^2, \ \forall s(x^{k_i}) \in \partial f_{k_i}(x^{k_i}), \forall s(y^{k_i}) \in \partial f_{k_i}(y^{k_i})$$

which implies

$$\langle s(x^{k_i}), x^{k_i} - y^{k_i} \rangle \geq \langle s(y^{k_i}), x^{k_i} - y^{k_i} \rangle + \delta \|x^{k_i} - y^{k_i}\|^2.$$

Since $y^{k_i} = \arg\min\{f_{k_i}(y) : y \in C\}$, we have $0 \in \partial f_{k_i}(y^{k_i}) + N_C(y^{k_i})$ which, by necessary and sufficient optimality condition for convex programming,is equivalent to $\langle s(y^{k_i}), y - y^{k_i} \rangle \geq 0 \ \forall y \in C$, in particular, $\langle s(y^{k_i}), x^{k_i} - y^{k_i} \rangle \geq 0$. Thus $\langle s(x^{k_i}), x^{k_i} - y^{k_i} \rangle \geq \delta \|x^{k_i} - y^{k_i}\|^2$, which implies

$$\|x^{k_i} - y^{k_i}\| \leq \frac{1}{\sqrt{\delta}} \|s(x^{k_i})\|, \ \forall s(x^{k_i}) \in \partial f_{k_i}(x^{k_i}). \tag{3.9}$$

Since $x^{k_i} \longrightarrow \bar{x}$ by Theorem 24.5 in [20] there exists an integer number $i_0 > 0$, large enough such that

$$\partial_2 f(x^{k_i}, x^{k_i}) \subset \partial_2 f(\bar{x}, \bar{x}) + B[0; 1], \ \forall i > i_0 \tag{3.10}$$

where $B[0;1]$ denotes the closed unit ball of $\mathbb{R}^n$.

In addition, $s(x^{k_i}) \in \partial f_{k_i}(x^{k_i}) = \rho \partial_2 f(x^{k_i}, x^{k_i}) \ \forall i$ and the set $\partial_2 f(\bar{x}, \bar{x})$ is bounded, we deduce from (3.9) and (3.10) that $\{\|x^{k_i} - y^{k_i}\|\}$ is bounded. So that, combining with Lemma 3.3 we get the boundedness of $\{y^{k_i}\}$. By definition of $z^{k_i} : z^{k_i} = (1 - \eta_{k_i}) x^{k_i} + \eta_{k_i} y^{k_i}$ it implies that $\{z^{k_i}\}$ is also bounded. Without loss of generality we may assume that z^{k_i} converges to some $\bar{z}$. Since $w^{k_i} \in \partial_2 f(z^{k_i}, z^{k_i})$, by again Theorem 24.5 in [20] we get the boundedness of the subsequence $\{w^{k_i}\}$. □

Lemma 3.5. *If the subsequence* $\{x^{k_i}\} \subset \{x^k\}$ *converges to some* $\bar{x}$ *and*

$$\|y^{k_i} - x^{k_i}\|^4 \left(\frac{\eta_{k_i}}{\|w^{k_i}\|}\right)^2 \longrightarrow 0 \ \text{ as } \ i \longrightarrow \infty \tag{3.11}$$

then $\bar{x} \in S_f$.

Proof. We will consider two distinct cases:

Case 1. $\text{Inf} \frac{\eta_{k_i}}{\|w^{k_i}\|} > 0$. Then by (3.11), one has $\lim_{i \longrightarrow \infty} \|y^{k_i} - x^{k_i}\| = 0$, thus $y^{k_i} \longrightarrow \bar{x}$ and $z^{k_i} \longrightarrow \bar{x}$.

From definition of y^{k_i} we have

$f(x^{k_i}, y) + \frac{1}{\rho}[h(y) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y - x^{k_i} \rangle]$

$$\geq f(x^{k_i}, y^{k_i}) + \frac{1}{\rho}[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i} \rangle], \ \forall y \in C$$

by the continuity of $h, \nabla h$, we get in the limit as $i \longrightarrow \infty$ that

$$f(\bar{x}, y) + \frac{1}{\rho}[h(y) - h(\bar{x}) - \langle \nabla h(\bar{x}), y - \bar{x} \rangle] \geq 0, \ \forall y \in C$$

this fact shows that $\bar{x} \in S_f$.

Case 2. $\text{Lim}\frac{\eta_{k_i}}{\|w^{k_i}\|} = 0$. By the linesearch rule and τ-strong convexity of h we have

$$\langle w^{k_i}, x^{k_i} - y^{k_i}\rangle \geq \frac{1}{\rho}\Big[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big]$$
$$\geq \frac{\tau}{\rho}\|x^{k_i} - y^{k_i}\|^2.$$

Thus $\|y^{k_i} - x^{k_i}\| \leq \sqrt{\frac{\rho}{\tau}}\|w^{k_i}\|$.

From the boundedness of $\{w^{k_i}\}$ and (3.11) it follows $\eta_{k_i} \longrightarrow 0$, so that $z^{k_i} = (1-\eta_{k_i})x^{k_i} + \eta_{k_i}y^{k_i} \longrightarrow \bar{x}$ as $i \longrightarrow \infty$. Without loss of generality, we suppose that $w^{k_i} \longrightarrow \bar{w} \in \partial_2 f(\bar{x}, \bar{x})$ and $y^{k_i} \longrightarrow \bar{y}$ as $i \longrightarrow \infty$.

We have

$$f(x^{k_i}, y) + \tfrac{1}{\rho}[h(y) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y - x^{k_i}\rangle]$$
$$\geq f(x^{k_i}, y^{k_i}) + \frac{1}{\rho}[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle],\ \forall y \in C$$

letting $i \longrightarrow \infty$, we obtain in the limit that

$$f(\bar{x}, y) + \tfrac{1}{\rho}[h(y) - h(\bar{x}) - \langle \nabla h(\bar{x}), y - \bar{x}\rangle]$$
$$\geq f(\bar{x}, \bar{y}) + \frac{1}{\rho}[h(\bar{y}) - h(\bar{x}) - \langle \nabla h(\bar{x}), \bar{y} - \bar{x}\rangle]\ \forall y \in C.$$

In the other hand, by the linesearch rule (3.1), for $m_{k_i}-1$ there exists $w^{k_i, m_{k_i}-1} \in \partial_2 f(z^{k_i, m_{k_i}-1}, z^{k_i, m_{k_i}-1})$ such that

$$\langle w^{m_{k_i}-1}, x^{k_i} - y^{k_i}\rangle < \frac{1}{\rho}\Big[h(y^{k_i}) - h(x^{k_i}) - \langle \nabla h(x^{k_i}), y^{k_i} - x^{k_i}\rangle\Big]. \tag{3.12}$$

Letting $i \longrightarrow \infty$ and combining with $z^{k_i, m_{k_i}-1} \longrightarrow \bar{x},\ w^{k_i, m_{k_i}-1} \longrightarrow \bar{w} \in \partial_2 f(\bar{x}, \bar{x})$ we obtain in the limit from (3.12) that

$$\langle \bar{w}, \bar{x} - \bar{y}\rangle \leq \frac{1}{\rho}\Big[h(\bar{y}) - h(\bar{x}) - \langle \nabla h(\bar{x}), \bar{y} - \bar{x}\rangle\Big].$$

Note that $\bar{w} \in \partial f(\bar{x}, \bar{y})$, it follows from the last inequality that,

$$f(\bar{x}, \bar{y}) + \frac{1}{\rho}\Big[h(\bar{y}) - h(\bar{x}) - \langle \nabla h(\bar{x}), \bar{y} - \bar{x}\rangle\Big] \geq 0.$$

Hence

$$f(\bar{x}, y) + \frac{1}{\rho}\Big[h(y) - h(\bar{x}) - \langle \nabla h(\bar{x}), y - \bar{x}\rangle\Big] \geq 0,\ \forall y \in C,$$

which shows that $\bar{x} \in S_f$. □

Now we are in a position to prove the convergence of the proposed algorithm.

Theorem 3.6. *Suppose that the solution set S_f of EP(C, f) is nonempty and that the function $h(.)$, the sequence $\{\lambda_k\}$ satisfying the conditions $(B1)$, $(B2)$ respectively. Then under Assumptions $(A1)$, $(A2)$, $(A3)$, and $(A4)$, the sequence $\{x^k\}$ generated by Algorithm 1 converges to the unique solution x^* of VIEP(C, f, G).*

Proof. By Lemma 3.2 we have

$$\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 + \Big(\frac{\eta_k \delta}{\rho\|w^k\|}\Big)^2\|x^k - y^k\|^4 \leq -2\lambda_k\langle u^k - x^*, G(u^k)\rangle + \lambda_k^2\|G(u^k)\|^2\ \forall k. \tag{3.13}$$

From the boundedness of $\{u^k\}$ and $\{G(u^k)\}$ it implies that, there exist positive numbers A, B such that

$$|\langle u^k - x^*, G(u^k)\rangle| \leq A,\ \|G(u^k)\|^2 \leq B\ \forall k.$$

By setting $a_k = \|x^k - x^*\|^2$, and combining with the last inequalities, (3.13) becomes

$$a_{k+1} - a_k + \Big(\frac{\eta_k \delta}{\rho\|w^k\|}\Big)^2 \|x^k - y^k\|^4 \leq 2\lambda_k A + \lambda_k^2 B. \tag{3.14}$$

We will consider two distinct cases:

Case 1. There exists k_0 such that $\{a_k\}$ is decreasing when $k \geq k_0$.

Then there exists $\lim_{k\longrightarrow\infty} a_k = a$, taking the limit on both sides of (3.14) we get

$$\lim_{k\longrightarrow\infty} \Big(\frac{\eta_k \delta}{\rho\|w^k\|}\Big)^2 \|x^k - y^k\|^4 = 0. \tag{3.15}$$

In addition,

$$\begin{aligned} \|x^{k+1} - u^k\| &= \|P_C(u^k - \lambda_k G(u^k)) - P_C(u^k)\| \\ &\leq \|u^k - \lambda_k G(u^k) - u^k\| \\ &= \lambda_k \|G(u^k)\| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \end{aligned} \tag{3.16}$$

From the boundedness of $\{u^k\}$ it implies that, there exists $\{u^{k_i}\} \subset \{u^k\}$ and $u^{k_i} \longrightarrow \bar{u} \in C$ such that $\liminf\langle u^k - x^*, G(x^*)\rangle = \lim_{i\longrightarrow\infty}\langle u^{k_i} - x^*, G(x^*)\rangle$.

Combining this fact with (3.15) and (3.16) we obtain

$$x^{k_i+1} \longrightarrow \bar{u} \text{ and } \Big(\frac{\eta_{k_i+1}\delta}{\rho\|w^{k_i+1}\|}\Big)^2 \|x^{k_i+1} - y^{k_i+1}\|^4 \longrightarrow 0 \text{ as } i \longrightarrow \infty.$$

By Lemma 3.5 we get $\bar{u} \in S_f$. Thus

$$\liminf_{k\longrightarrow\infty}\langle u^k - x^*, F(x^*)\rangle = \lim_{i\longrightarrow\infty}\langle u^{k_i} - x^*, G(x^*)\rangle = \langle \bar{u} - x^*, G(x^*)\rangle \geq 0.$$

Since F is β-strongly monotone, one has

$$\begin{aligned} \langle u^k - x^*, G(u^k)\rangle &= \langle u^k - x^*, G(u^k) - G(x^*)\rangle + \langle u^k - x^*, G(u^*)\rangle \\ &\geq \beta\|u^k - x^*\|^2 + \langle u^k - x^*, G(u^*)\rangle. \end{aligned}$$

Taking the limit as $k \longrightarrow \infty$ and remember that $a = \lim \|u^k - x^*\|^2$ we get

$$\liminf_{k\longrightarrow\infty}\langle u^k - x^*, G(u^k)\rangle \geq \beta a. \tag{3.17}$$

If $a > 0$, then by choosing $\epsilon = \frac{1}{2}\beta a$, from (3.17) it implies that, there exists $k_0 > 0$ such that

$$\langle u^k - x^*, G(u^k)\rangle \geq \frac{1}{2}\beta a, \ \forall k \geq k_0.$$

From (3.13) we get

$$a_{k+1} - a_k \leq -\lambda_k \beta a + \lambda_k^2 B, \ \forall k \geq k_0$$

and thus summing up from k_0 to k we have

$$a_{k+1} - a_{k_0} \leq -\sum_{j=k_0}^{k} \lambda_j \beta a + B\sum_{j=k_0}^{k} \lambda_j^2$$

combining this fact with $\sum_{k=0}^{\infty}\lambda_k = \infty$ and $\sum_{k=0}^{\infty}\lambda_k^2 < \infty$ we obtain

$\liminf a_k = -\infty$, which is a contradiction.

Thus we must have $a = 0$. i.e., $\lim_{k\longrightarrow\infty}\|x^k - x^*\| = 0$.

Case 2. There exists a subsequence $\{a_{k_i}\}_{i\geq 0} \subset \{a_k\}_{k\geq 0}$ such that $a_{k_i} < a_{k_i+1}$ for all $i \geq 0$. In this situation, we consider the sequence of indices $\{\sigma(k)\}$ defined as in Lemma 2.9. It follows that $a_{\sigma(k)+1} - a_{\sigma(k)} \geq 0$, which by (3.14) amounts to

$$\Big(\frac{\eta_{\sigma(k)}\delta}{\rho\|w^{\sigma(k)}\|}\Big)^2 \|x^{\sigma(k)} - y^{\sigma(k)}\|^4 \leq 2\lambda_{\sigma(k)} A + \lambda_{\sigma(k)}^2 B.$$

Therefore

$$\lim_{k\longrightarrow\infty}\Big(\frac{\eta_{\sigma(k)}\delta}{\rho\|w^{\sigma(k)}\|}\Big)^2\|x^{\sigma(k)}-y^{\sigma(k)}\|^4=0.$$

From the boundedness of $\{x^{\sigma(k)}\}$, without loss of generality we may assume that $x^{\sigma(k)}\longrightarrow\bar{x}$. By Lemma 3.5 we get $\bar{x}\in S_f$.

In addition, $u^{\sigma(k)}=P_{C\cap H_{\sigma(k)}}(x^{\sigma(k)})=P_{C_{\sigma(k)}}(x^{\sigma(k)})$.

Then combining with Lemma 2.6 we have

$$\|u^{\sigma(k)}-\bar{x}\|\le\|x^{\sigma(k)}-\bar{x}\|\longrightarrow 0 \text{ as } k\longrightarrow\infty$$

so that $\lim_{k\longrightarrow\infty}u^{\sigma(k)}=\bar{x}$.

By (3.13) we get

$$2\lambda_{\sigma(k)}\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})\rangle\le a_{\sigma(k)}-a_{\sigma(k)+1}-\Big(\frac{\eta_{\sigma(k)}\delta}{\rho\|g^{\sigma(k)}\|}\Big)^2\|x^{\sigma(k)}-y^{\sigma(k)}\|^4 + \lambda^2_{\sigma(k)}\|G(u^{\sigma(k)})\|^2\le\lambda^2_{\sigma(k)}B$$

which implies

$$\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})\rangle\le\frac{\lambda_{\sigma(k)}}{2}B. \tag{3.18}$$

Since G is β-strongly monotone, we have

$$\begin{aligned}\beta\|u^{\sigma(k)}-x^*\|^2&\le\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})-G(x^*)\rangle\\&=\langle u^{\sigma(k)}-x^*,G(u^{\sigma(k)})\rangle-\langle u^{\sigma(k)}-x^*,G(x^*)\rangle\end{aligned}$$

which combining with (3.18) we get

$$\|u^{\sigma(k)}-x^*\|^2\le\frac{1}{\beta}\Big[\frac{\lambda_{\sigma(k)}}{2}B-\langle u^{\sigma(k)}-x^*,G(x^*)\rangle\Big]$$

so that

$$\lim_{k\longrightarrow\infty}\|u^{\sigma(k)}-x^*\|^2\le-\langle u^{\sigma(k)}-x^*,G(x^*)\rangle\le 0$$

which amounts to

$$\lim_{k\longrightarrow\infty}\|u^{\sigma(k)}-x^*\|=0. \tag{3.19}$$

In addition,

$$\begin{aligned}\|x^{\sigma(k)+1}-u^{\sigma(k)}\|&=\|P_C(u^{\sigma(k)}-\lambda_{\sigma(k)}G(u^{\sigma(k)}))-P_(u^{\sigma(k)})\|\\&\le\lambda_{\sigma(k)}\|G(u^{\sigma(k)})\|\longrightarrow 0 \text{ as } k\longrightarrow\infty\end{aligned}$$

which together with (3.19), one has $\lim_{k\longrightarrow\infty}x^{\sigma(k)+1}=x^*$, which means that $\lim_{k\longrightarrow\infty}a_{\sigma(k)+1}=0$.

By (2.3) in Lemma 2.9 we have

$$0\le a_k\le a_{\sigma(k)+1}\longrightarrow 0 \text{ as } k\longrightarrow\infty.$$

Thus $\{x^k\}$ converges to x^*. □

4. Application to Minimizing the Euclidean Norm with Pseudomomotone Equilibrium Constraints

In this section, we consider the problem:

$$\min\{\|x-x^g\|^2 : x\in S_f\}, \qquad \text{MNEP}(C,f)$$

where $x^g\in C$ is given (plays the role of a guess-solution of EP(C,f)) and S_f is the solution set of problem EP(C,f). This problem arises in the Tikhonov regularization method for pseudomonotone equilibrium problems, see, e.g., [8]. In this case, by choosing $G(x) = x - x^g$, the problem MNEP(C, f) becomes to the one in the form of VIEP(C, f, G).

It is well known that, under Assumptions $(A1)$, $(A2)$ and $(A3)$, the solution set S_f of EP(C, f) is a closed convex set. As we have mentioned that the main difficulty in problem MNEP(C, f) is that its feasible domain S_f, although is convex, it is not given explicitly as in a standard mathematical programming problem. In the sequel, we always suppose that Assumptions (A1), (A2), and (A3) are satisfied. The algorithm for this case takes the form.

Algorithm 2. Take $x^1 := x^g \in C$ and choose parameters $\rho > 0$, $\eta, \in (0,1)$.
At each iteration $k = 1, 2, ..$ having x^k do the following steps:
Step 1. Solve the strongly convex program

$$\min \left\{ f(x^k, y) + \frac{1}{\rho}\Big[h(y) - h(x^k) - \langle \nabla h(x^k), y - x^k\rangle\Big] : \; y \in C\right\} \qquad CP(x^k)$$

to obtain its unique solution y^k. If $x^k = y^k$, take $u^k := x^k$ and go to Step 4.
Step 2. Find m_k as the smallest positive integer number m such that

$$\begin{cases} z^{k,m} = (1-\eta^m)x^k + \eta^m y^k : \\ \langle w^{k,m}, x^k - y^k\rangle \geq \frac{1}{\rho}\Big[h(y^k) - h(x^k) - \langle \nabla h(x^k), y^k - x^k\rangle\Big] \\ \text{with } \; w^{k,m} \in \partial_2 f(z^{k,m}, z^{k,m}). \end{cases} \tag{4.1}$$

Set $\eta_k := \eta^{m_k}$, $z^k := z^{k,m_k}, w^k = w^{k,m}$.
Step 3. Take $u^k := P_{C_k}(x^k)$, where

$$C_k := \{x \in C : \langle w^k, x - z^k\rangle \leq 0\}. \tag{4.2}$$

Step 4.

$$x^{k+1} := \lambda_k x^g + (1-\lambda_k)u^k \tag{4.3}$$

Repeat iteration k with k is replaced by $k+1$.

Similar to Theorem 3.1, we have the following theorem

Theorem 4.1. *Under Assumptions $(A1)$ $(A2)$, $(A3)$, and $(B1)$, $(B2)$, the sequence $\{x^k\}$ generated by Algorithm 2 converges to the unique solution x^* of MNEP(C, f).*

Conclusion. We have proposed an explicit hybrid extragradient algorithm for solving the variational inequality problems with equilibrium problems constraint, where the bifunction is pseudomonotone with respect to its solution set. The convergence of the algorithm is obtained, and a special case of this problem is considered.

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