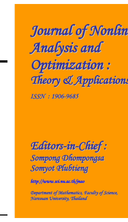
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 8, No. 1, (2017), 63-70
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



APPLICATIONS ON DIFFERENTIAL SUBORDINATION INVOLVING LINEAR OPERATOR

ABDUL RAHMAN SALMAN JUMA*

Department of Mathematics, University of Anbar, Ramadi, Iraq

ABSTRACT. In the present paper, we introduce and investigate some subclasses of strongly close-to-convex functions associated with the linear operator of meromorphic pvalently functions and study several inclusion relationships with some properties of this operator.

KEYWORDS linear operator , meromorphic functions , differential subordination, strongly close-to-convex functions, p-valently functions.

AMS Subject Classification:: Secondary 30C45.

1. INTRODUCTION

Let $\sum_p$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p}, \tag{1.1}$$

which are analytic and p-valently in the punctured unit disk $\mathcal{U}^* = \{z : z \in \mathbb{C} : 0 < |z| < 1\} = \mathcal{U} - 0$.

If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathcal{U}$ such that $f(z) = g(w(z))$.
A function $f(z) \in \sum_p$ is said to be p-valent meromorphic starlike of order $\alpha(0 \le \alpha \le p)$ if it satisfies

$$Re\Big\{ - \frac{zf'(z)}{f(z)} \Big\} > \alpha, \quad (z \in \mathcal{U}) \tag{1.2}$$

and the class of such functions is defined by $MS^*(\alpha)$.
Furthermore, a function $f(z) \in \sum_p$ is said to be p-valently meromorphic convex functions of order $\alpha(0 \le \alpha \le p)$ if it satisfies

$$Re\Big\{ - (1 + \frac{zf''(z)}{f'(z)}) \Big\} > \alpha, \quad (z \in \mathcal{U}) \tag{1.3}$$

* *Corresponding author.*
Email address : dr_juma@hotmail.com.
Article history : Received 4 November 2015 Accepted 26 January 2018.

and the class of such functions is defined by $MK(\alpha)$.

Let $f(z) \in \sum_p$ and $g(z) \in MS^*(\alpha)$. Then $f(z) \in MC(\alpha, \beta)$ if and only if

$$Re\Big\{ -\frac{zf'(z)}{g(z)} \Big\} > \beta, \quad (z \in \mathcal{U}), \tag{1.4}$$

where $0 \le \alpha < p$ and $0 \le \beta < p$. Such functions are called close-to-convex functions of order β and type α in $\mathcal{U}$, (see for details ,[[4], [9]].

Further, a function $f(z) \in \sum_p$ is called p-valently meromorphic strongly starlike of order $\gamma(0 < \gamma \le p)$ and type $\alpha(0 < \gamma \le p)$ in $\mathcal{U}$ if it satisfies

$$\Big| arg(-\frac{zf'(z)}{f(z)} - \alpha) \Big| < \frac{\pi}{2}\gamma, \quad (z \in \mathcal{U}), \tag{1.5}$$

and denoted by $MS^*(\gamma, \alpha)$.

If $f(z) \in \sum_p$ satisfies

$$\Big| arg(-(1 + \frac{zf'(z)}{f'(z)}) - \alpha) \Big| < \frac{\pi}{2}\gamma, \quad (z \in \mathcal{U}),$$

for some $\gamma(0 < \gamma \le p)$ and $\alpha(0 < \alpha \le p)$, then f is called p-valently meromorphic strongly convex of order γ and type α in $\mathcal{U}$ and denoted by $MC(\gamma, \alpha)$. We note that the classes mentioned above are the familiar classes which have been studied by many authors (see for example,([3],[6],[9],[10]).

For a function $f(z) \in \sum_p$ given by (1), we define a linear operator D^n by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z^{-p}(z^{p+1}f(z))' = z^{-p} + \sum_{k=0}^{\infty}(2p+k+1)a_{k+p}z^{k+p}$$

and

$$\begin{aligned} D^n f(z) &= D(D^{n-1}f(z)) = z^{-p}(z^{p+1}D^{n-1}f(z))' \\ &= z^{-p} + \sum_{k=0}^{\infty}(2p+k+1)^n a_{k+p}z^{k+p}. \quad (n \in \mathbb{N}) \end{aligned} \tag{1.6}$$

Using the relation (6), it is easy to verify that

$$z(D^n f(z))' = D^{n+1}f(z) - (p+1)D^n f(z). \tag{1.7}$$

Also, we note that $D^n f(z)$ of another form of function studied by Liu and Srivastava [7] ,Srivastava and Patel [13] who introduce several inclusion relationships by using various subclasses of meromorphic p-valent function. A special cases of linear operator D^n for $p = 1$ studied by Uralegaddi and Somanatha [14], Aouf and Hossen.[1], and got interesting results by using the operator D^n.

For $n \in \mathbb{N}$, let $MC_p^{n+1}(\alpha, \beta, \gamma, A, B)$ be the class of functions $f(z) \sum_p$ satisfying the condition:

$$\Big| -arg\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma \Big| < \frac{\pi}{2}\delta \quad (0 < \gamma \le p, 0 \le \delta < p; z \in \mathcal{U}), \tag{1.8}$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, where

$$S_p^{n+1}(\alpha, A, B) = \Big\{ g : \frac{1}{p+\alpha}(\frac{z(D^{n+1}g(z))'}{D^{n+1}g(z)} - \alpha) \prec \frac{1+Az}{1+Bz} \Big\} \tag{1.9}$$

$(0 \le \alpha < p, -1 \le B \le A \le 1, z \in \mathcal{U}$ and $g \in \sum_p)$ and the functions f belonging to this class is called strongly close-to-convex function. In this study and by using the technique of Cho[2],we find some argument properties of functions belonging to $\sum_p$ which include inclusion relationship and we obtain some interesting results for the functions class $MC_p^{n+1}(\alpha,\beta,\gamma,A,B)$ which we have defined here by the operator D^n.

To establish our main results, we shall need the following lemmas.

Lemma 1.1 [5] Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\{\varepsilon h(z) + \eta\} > 0(\varepsilon,\eta \in \mathbb{C})$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\varepsilon p(z) + \eta} \prec h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) \prec h(z) \quad (z \in \mathcal{U})$.

Lemma 1.2 [8]**:** Let $h(z)$ be convex univalent in $\mathcal{U}$ and $w(z)$ be analytic in $\mathcal{U}$ with $Re\{w(z) \geq 0$. If $p(z)$ is analytic in $\mathcal{U}$ with $p(0) = h(0)$,then

$$p(z) + w(z)zp'(z) - \frac{\pi}{2}(z) \prec h(z) \quad (z \in \mathcal{U}),$$

implies $p(z) \prec h(z) \quad (z \in \mathcal{U})$.

Lemma 1.3[9]**:** Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there exists two points z_1, z_2 in $\mathcal{U}$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \tag{1.10}$$

for some $\alpha_1, \alpha_2(\alpha_1, \alpha_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$, then we have

$$\frac{z_1p'(z_1)}{p(z_1)} = i\frac{\alpha_1 + \alpha_2}{2}m$$

and

$$\frac{z_2p'(z_2)}{p(z_2)} = i\frac{\alpha_1 + \alpha_2}{2}m, \tag{1.11}$$

where $m \geq \frac{1-|c|}{1+|c|}$ and

$$c = i\tan\frac{\pi}{4}(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}). \tag{1.12}$$

2. MAIN RESULTS

We first derive the following with use of Lemma 1.1.

Proposition 2.1. Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\{h(z)\} > 0$.
If a function $f(z) \in \sum_p$ satisfies the following condition:

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha) \prec h(z),$$

then

$$-\frac{1}{p+\alpha}(\frac{z(D^nf(z))'}{D^nf(z)} - \alpha) \prec h(z),$$

$$(0 \leq \alpha < p; z \in \mathcal{U})$$

Proof. Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha). \tag{2.1}$$

Then $p(z)$ is analytic function in $\mathcal{U}$ with $p(0) = 1$. By using (1.7), we obtain

$$p + 1 + \alpha + (p+\alpha)p(z) = -\frac{D^{n+1}f(z)}{D^nf(z)}. \tag{2.2}$$

Differentiating Logarithmically with resprct to z and multiplying by z, we get

$$p(z)+\frac{zp'(z)}{p+1+\alpha+(p+\alpha)p(z)}=-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha).$$

Now, by using Lemma 1.1,we obtain

$$-\frac{1}{p+\alpha}(\frac{z(D^{n}f(z))'}{D^{n}f(z)}-\alpha)\prec h(z),$$

deduce that $p(z)\prec h(z)$.
Setting $h(z)=\frac{1+Az}{1+Bz}\quad(-1\le B\le A\le 1)$, in Lemma 2.1, we obtain

Corollary 2.1: For $n\in N$ and $p\in\{1,2,...\}$, we have

$$S_p^{n+1}(\alpha,A,B)\subset S_p^{n}(\alpha,A,B).$$

Proposition 2.2: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0)=1$ and $Re\{h(z)\}>0$. If $f(z)\in\sum_p$ satisfies

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha)\prec h(z),$$

then

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta f(z)}-\alpha)\prec h(z),$$

$$(0\le\alpha<p;z\in\mathcal{U})$$

where

$$\mathbb{I}_\theta f(z)=\frac{\theta-p}{z^\theta}\int_0^z t^{\theta-1}f(t)dt\quad(\theta\ge 0)\tag{2.3}$$

Proof. From (2.3), we have

$$z(D^{n+1}\mathbb{I}_\theta f(z))'=(\theta-p)(D^{n+1}f(z))-\theta(D^{n+1}f(z)).\tag{2.4}$$

Let

$$p(z)=-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha),$$

$p(z)$ is analytic function in $\mathcal{U}$ with $p(0)=1$. Then from (2.4), we get

$$\theta+\alpha+(p+\alpha)p(z)=-(\theta-p)\frac{D^{n+1}f(z)}{D^{n+1}\mathbb{I}_\theta f(z)}.\tag{2.5}$$

By differentiating (2.5) logarthmically with respect to z and multiplying by z,we have

$$p(z)+\frac{zp'(z)}{\theta+\alpha+(p+\alpha)p(z)}=-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha).$$

Thus, by Lemma 1.1, we get

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha)\prec h(z).$$

Taking $h(z)=\frac{1+Az}{1+Bz}\quad(-1\le B\le A\le 1)$, in Proposition 2.2, we obtain

Corollary 2.2: If $f(z)\in S_p^{n+1}(\alpha,A,B)$, then $\mathbb{I}_\theta f(z)\in S_p^{n+1}(\alpha,A,B)$. Hence on Applying Proposition 2.2, we prove the following theorem

Theorem 2.1: Let $f(z)\in\sum_p$ and $(0<\delta_1,\delta_2\le p,0\le\alpha<p)$. If

$$-\frac{\pi}{2}\delta<\arg(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}-\gamma)<\frac{\pi}{2}\delta_2$$

for some $g(z)\in S_p^{n+1}(\alpha,A,B)$, then

$$-\frac{\pi}{2}\beta_1 < \arg(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma) < \frac{\pi}{2}\beta_2,$$

where β_1 and $\beta_2 (0 < \beta_1, \beta_2 \leq p)$ are the solution of the equations:

$$\delta_1 = \begin{cases} \beta_1 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2(\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} & B \neq -1 \\ \beta_1 & B = -1, \end{cases} \tag{2.6}$$

and

$$\delta_2 = \begin{cases} \beta_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2(\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1}\right\} & B \neq -1 \\ \beta_2 & B = -1, \end{cases} \tag{2.7}$$

where c is given by (1.12) and $t_1 = \frac{2}{\pi}\sin^{-1}(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(p+1+\alpha)(1-B^2)})$.

Proof. Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^n f(z))'}{D^n g(z)} - \gamma). \tag{2.8}$$

It follows from (1.7) that

$$[(p+\gamma)(p(z)-\gamma)]D^n g(z) = D^{n+1} f(z) - (p+1)D^n f(z). \tag{2.9}$$

Differentiating both sides of (2.9), and multiplying by z, we deduce that

$$(p+\gamma)zp'(z)D^n g(z) + [(p+\gamma)p(z)-\gamma]z(D^n g(z))' = z(D^{n+1} f(z))' - (p+1)z(D^n f(z))'. \tag{2.10}$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, by applying Corollary 2.1, we find that $g(z) \in S_p^n(\alpha, A, B)$. Thus, by using (1.7) and put $q(z) = -\frac{1}{p+\alpha}(\frac{z(D^n g(z))'}{D^n g(z)} - \alpha)$, we immediately have

$$(p+\alpha)q(z) + \alpha + p + 1 = -\frac{D^{n+1}g(z)}{D^n g(z)}. \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma) = p(z) + \frac{zp'(z)}{(p+\alpha)q(z)+\alpha+p+1}.$$

Making use the result of Silverman and Silvia [10] ,we obtain

$$|q(z) - \frac{1-AB}{1-B^2}| < \frac{A-B}{1-B^2} \quad (z \in \mathcal{U}; B \neq -1) \tag{2.12}$$

and

$$Re\{q(z)\} > \frac{1-A}{2} \quad (z \in \mathcal{U}; B = -1) \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$(p+\alpha)q(z) + p + \alpha + 1 = re^{i\frac{\pi\phi}{2}}.$$

Now, if $B \neq -1$, we have

$$\frac{(p+\alpha)(1-A)}{1-B} + \alpha + p + 1 < r < \frac{(p+\alpha)(1+A)}{1+B} + \alpha + p + 1, \quad -t_1 < \phi < t_1,$$

and if $B = -1$, we have

$$\frac{(p+\alpha)(1-A)}{2} + \alpha + p + 1 < r < \infty, \quad -1 < \phi < 1,$$

Applying Lemma 1.2 with $w = -\frac{1}{(p+\alpha)q(z)+p+\alpha+1}$, we note that $p(z)$ is analytic with $p(0) = 1$ and $Re\{p(z)\} > 0$ in $\mathcal{U}$.

Hence by Lemma 1.3 for $z_1, z_2 \in \mathcal{U}$, such that the condition (1.10) is satisfied ,then we obtain (1.11) under the restriction (1.12). On other hand, if $B \neq -1$, we readily get

$$\arg(-(p(z_1) + \frac{z_1 p'(z_1)}{(p+\alpha)q(z_1)+p+\alpha+1})) = -\frac{\pi}{2}\beta_1 + \arg(1 - i\frac{\beta_1+\beta_2}{2}m(re^{i\frac{\pi}{2}})^{-1})$$

$$\leq \frac{-\pi}{2}\beta_1 - \tan^{-1}(\frac{(\beta_1+\beta_2)m\sin\frac{\pi}{2}(1-\phi)}{2r+(\beta_1+\beta_2)m\cos\frac{\pi}{2}(1-\phi)})$$

$$\frac{-\pi}{2}\beta_1 - \tan^{-1}(\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\frac{(p+\alpha)(1+A)}{1+B}+p+\alpha+1)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1}) = \frac{-\pi}{2}\delta_1,$$

and

$$\arg(-(p(z_2)+\frac{z_2p'(z_2)}{(p+\alpha)q(z_2)+p+\alpha+1})) \geq \frac{-\pi}{2}\beta_2 - \tan^{-1}(\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_1}{2\frac{(p+\alpha)(1+A)}{1+B}+p+\alpha+1)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_1}) = \frac{-\pi}{2}\delta_2.$$

Also, if $B = -1$, we readily get

$$\arg(-(p(z_1) + \frac{z_1p'(z_1)}{(p+\alpha)q(z_1)+p+\alpha+1})) \leq \frac{-\pi}{2}\beta_1$$

and

$$\arg(-(p(z_2) + \frac{z_2p'(z_2)}{(p+\alpha)q(z_2)+p+\alpha+1})) \geq \frac{\pi}{2}\beta_2$$

There are contradiction with a assumption. This completes the proof of Theorem 2.1

Corollary 2.3:

$$MC_p^{n+1}(\alpha,\beta,\gamma,A,B) \subset MC_p^{n}(\alpha,\beta,\gamma,A,B).$$

Setting $n = 0, \delta_1 = \delta_2 = \delta$ in Theorem 2.1, we get:

Corollary 2.4: Let $f(z) \in \sum_p$. If

$$|\frac{z(z^{-p}(z^{p+1}f)')'}{z^{-p}(z^{p+1}g(z))'} - \gamma| < \frac{\pi}{2}\delta$$

for some $g(z) \in S_p^1$, then

where $\beta (0 < \beta \leq p)$ is the solution of equation:

$$\delta = \begin{cases} \beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\beta\cos\frac{\pi}{2}t_1}{\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha+\beta\sin\frac{\pi}{2}t_1}\right\} & B \neq -1 \\ \beta & B = -1, \end{cases}$$

and $t_1 = \frac{2}{\pi}\sin^{-1}(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(p+1+\alpha)(1-B^2)})$.

Theorem 2.2: Let $f(z) \in \sum_p$ and $(0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1)$. If

$$\frac{-\pi}{2}\delta_1 < \arg(-(\frac{z(D^{n+1}f(z))'}{D^{n+}g(z)} - \gamma)) < \frac{\pi}{2}\delta_2,$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

$$\frac{-\pi}{2}\beta_1 < \arg(-(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+}\mathbb{I}_\theta g(z)} - \gamma)) < \frac{\pi}{2}\beta_2,$$

where $\mathbb{I}_\theta$ is defined by (2.3), and β_1, β_2, are the solutions of

$$\delta_1 = \begin{cases} \beta_1 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_2}{2(\frac{(p+\alpha)(1+A)}{1+B}+\theta+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_2}\right\} & B \neq -1 \\ \beta_1 & B = -1, \end{cases} \quad (2.14)$$

and

$$\delta_2 = \begin{cases} \beta_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)\cos\frac{\pi}{2}t_2}{2(\frac{(p+\alpha)(1+A)}{1+B}+\theta+\alpha)(1+|c|)+(\beta_1+\beta_2)(1-|c|)\sin\frac{\pi}{2}t_2}\right\} & B \neq -1 \\ \beta_2 & B = -1, \end{cases} \quad (2.15)$$

here c is given by (1.12) and $t_2 = \frac{2}{\pi}\sin^{-1}(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(\theta+\alpha)(1-B^2)})$.

Proof. Let

$$p(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \gamma).$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, and by using Corollary 2.2, we obtain $\mathbb{I}_\theta g(z) \in S_p^{n+1}(\alpha, \beta, \gamma, A, B)$. By using (2.5), we get

$$[(p+\gamma)(p(z)-\gamma)]D^{n+1}\mathbb{I}_\theta g(z) = (\theta - p)(D^{n+1}f(z)) - \theta D^{n+1}\mathbb{I}_\theta f(z)$$

and simplifying, we obtain

$$(p+\gamma)zp'(z) + [(p+\gamma)p(z)+\gamma][(p+\alpha)q(z)+\theta+\alpha] = (\theta - p)\frac{z(D^{n+1}f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)},$$

where

$$q(z) = -\frac{1}{p+\alpha}(\frac{z(D^{n+1}\mathbb{I}_\theta g(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \alpha).$$

Therefore,

$$-\frac{1}{p+\alpha}(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \alpha) = p(z) + \frac{zp'(z)}{(p+\alpha)q(z)+\alpha+\theta}.$$

Applying a similar method as in the proof of Theorem 2.1 we get the required result and the proof is complete.

Setting $\delta_1 = \delta_2 = \delta$ in Theorem 2.2, we obtain

Corollary 2.5: Let $f(z) \in \sum_p$ and $0 \le \gamma < p, 0 < \delta \le 1$. If

$$\left|\arg(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma)\right| < \frac{\pi}{2}\delta$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

$$\left|\arg(-\frac{z(D^{n+1}\mathbb{I}_\theta(f)(z))'}{D^{n+1}\mathbb{I}_\theta(g)(z)} - \gamma)\right| < \frac{\pi}{2}\beta,$$

where $\mathbb{I}_\theta$ is given by (2.5), and $\beta(0 < \beta \le 1)$ is the solution of the equation

$$\delta = \begin{cases} \beta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\beta\cos\frac{\pi}{2}t_2}{(\frac{(p+\alpha)(1+A)}{1+B}+\theta+\alpha)+\beta\sin\frac{\pi}{2}t_2}\right\} & B \neq -1 \\ \beta & B = -1 \end{cases}$$

Corollary 2.6: If $f(z) \in MC_p^{n+1}(\gamma, \delta, \alpha, A, B)$, then $\mathbb{I}_\theta(f) \in MC_p^{n+1}(\gamma, \delta, \alpha, A, B)$.

References

1. M. K. Aouf and H. M. Hossen ,New criteria for meromorphic p-valent starlike functions,Tsukuba J. Math.,17(1993),481-486 .
2. N. Cho,The Noor integral operator and strongly close-t-convex functions,J.Math. Anal. Appl.,283(2003),202-212.
3. E. Deniz, H. Orhan, On The Univalence of Integral Operators Involving Meromorphic Functions, Math. Commun., 18 (2013), 1-9.
4. P.Enigenberg,S. S. Miller,P. T. Mocanu and M. O. Reade,On a Briot- Bouquet differential subordination,in:General Inequalities,Vol.3 Birkhauser,Basel(1983),339- 348.

5. R. Libra,Some radius of convexity problems,Duke Mat. J.,31(1964),143-153.
6. J. Liu and H.M. Srivastava,Class of meromorphic multivalent functions associated with generalized hypergeometric functions ,Math. Compute Mod.,39(2004),21-34.
7. J. Liu and H. M. Srivastava,Subclass of meromorphic multivalent functions associated with a certain linear operator ,Math. Compute Mod.,39(2004),35- 44.
8. S. S. Miller and P. T. Mocanu,differential subordinations and univalent functions,Michigan Math. J. ,28(1989),157-171.
9. M. Nunokawa,S. Owa,H. Saitoh,N. E. Cho and N. Takahashi,Some properties of analytic functions at extremal points for arguments,preprint,2003.
10. H. Orhan, D. Raducanu, E. Deniz, Subclasses Of Meromorphically Multivalent Functions De.ned By A Dierential Operator, Comput. Math.Appl., 61(4), (2011) 966-979.
11. H. Silverman ,On a Class of close-to-convex schilcht functions,Proc. Amer. Math. Soc. 36(1972),477-484.
12. H. Silverman and E. Silvia ,Subclasses of starlike functions subordinate to convex functions,Canad.J.Math. 37(1985),48-61.
13. H. M. Srivastava and J. Patel,Applications of differential subordination to certain subclasses of meromorphiclly multivalent functions,JIPAM.,6(3)(2005),1-15.
14. B. A. Uralegaddi and C. Somanatha, New criteria for meromorphic starlike univalent functions,Bull. Austral Math. Soc.,43(1991)137-140.