Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 8, No. 1, (2017), 49-61
ISSN : 1906-9685
http://www.math.sci.nu.ac.th

BALL CONVERGENCE FOR A TWO STEP METHOD WITH MEMORY AT LEAST OF ORDER $2+\sqrt{2}$

IOANNIS K. ARGYROS[1], RAMANDEEP BEHL[2]* AND S.S. MOTSA[3]

[1]Cameron University, Department of Mathematics Sciences Lawton, OK 73505, USA
[2]Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[3]Mathematics Department, University of Swaziland, Private Bag 4, Kwaluseni, M201, Swaziland

ABSTRACT. We present a local convergence analysis of at least $2+\sqrt{2}$ convergence order two-step method in order to approximate a locally unique solution of nonlinear equation in a Banach space setting. In the earlier study, [6, 15] the authors of these paper did not discuss that studies. Furthermore, the order of convergence was shown using Taylor series expansions and hypotheses up to the sixth order derivative or or even higher of the function involved which restrict the applicability of the proposed scheme. However, only first order derivative appears in the proposed scheme. In order to overcome this problem, we proposed the hypotheses up to only first order derivative. In this way, we not only expand the applicability of the methods but also propose convergence domain. Finally, we present some numerical experiments where earlier studies cannot apply to solve nonlinear equations but our study does not exhibit this type of problem/restriction.

KEYWORDS : Two-step method with memory; local convergence; convergence order.
AMS Subject Classification: 65D10; 65D99

1. INTRODUCTION

There are several problems of pure and applied science which can be studied in the unified frame work of the scalar or system of nonlinear equations. In this paper, we are concerned with one of the most important and challenging task in the field of numerical analysis, is to approximate the local unique solution x^* of the equation of the form

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet differentiable function defined on a subset $\mathbb{D}$ of $\mathbb{R}$ with values in $\mathbb{R}$.

We can say that either lack or intractability of their analytic solutions often forces researchers from the worldwide trying their best to resort to an iterative

* *Corresponding author: Ramandeep Behl.*
Email address : ramanbehl87@yahoo.in.
Article history : Received 30 August 2016 Accepted 19 January 2018.

method. While, using these iterative methods researchers face the problems of slow convergence, non-convergence, divergence, inefficiency or failure (for details please see Traub [14] and Petkovic et al. [12]).

The convergence analysis of iterative methods is usually divided into two categories: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of iteration procedures. A very important problem in the study of iterative procedures is the convergence domain. Therefore, it is very important to propose the radius of convergence of the iterative methods.

We study the local convergence analysis of two-step method with memory defined for each $n = 0,\ 1\ 2,\ \ldots$ by

$$\begin{aligned} y_n &= x_n - \left(F'(x_n) + \alpha_n F(x_n)\right)^{-1} F(x_n) \\ x_{n+1} &= y_n - \left(F(x_n) + (\beta-2)F(y_n)\right)^{-1}\left(F(x_n)+\beta F(y_n)\right)\left(F'(x_n)+2\alpha_n F(x_n)\right)^{-1} F(y_n), \end{aligned} \tag{1.2}$$

where $x_{-1},\ x_0$ are initial points, $\beta \in \mathbb{R}$, $\alpha_n = -\frac{1}{2}\frac{[x_{n-1},\, x_n;\, F']}{[x_{n-1},\, x_n;\, F]},\ n = 0,\ 1,\ 2,\ \ldots,$ $[\cdot,\ \cdot; F']$ and $[\cdot,\ \cdot; F]$ denote divided differences of order one for functions F' and F, respectively. Method (1.2) was introduced in [6] as an alternative to the King-like method

$$\begin{aligned} y_n &= x_n - \left(F'(x_n) + aF(x_n)\right)^{-1} F(x_n) \\ x_{n+1} &= y_n - \left(F(x_n) + (\beta-2)F(y_n)\right)^{-1}\left(F(x_n)+\beta F(y_n)\right)\left(F'(x_n)+2aF(x_n)\right)^{-1} F(y_n), \end{aligned} \tag{1.3}$$

where $a,\ \beta \in \mathbb{R}$. Method (1.2) was shown to be of order $2+\sqrt{2}$ using hypotheses up to the sixth derivative of function F [6]. Method (1.3) is of order four [15] and hypotheses up to the fourth derivative of the function. These hypotheses on the derivatives of F limit the applicability of method (1.2) and method (1.3). As a motivational example, define function F on $\mathbb{R}$, $\mathbb{D} = [-\frac{1}{\pi},\ \frac{2}{\pi}]$ by

$$F(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Then, we have that

$$F'(x) = 2x^2 - x^3\cos\left(\frac{1}{x}\right) + 3x^2\log(\pi^2x^2) + 5x^4\sin\left(\frac{1}{x}\right),$$

$$F''(x) = -8x^2\cos\left(\frac{1}{x}\right) + 2x(5+3\log(\pi^2x^2)) + x(20x^2-1)\sin\left(\frac{1}{x}\right)$$

and

$$F'''(x) = \frac{1}{x}\left[(1-36x^2)\cos\left(\frac{1}{x}\right) + x\left(22+6\log(\pi^2x^2)+(60x^2-9)\sin\left(\frac{1}{x}\right)\right)\right].$$

One can easily find that the function $F'''(x)$ is unbounded on $\mathbb{D}$ at the point $x = 0$. Hence, the results in [6, 15], cannot apply to show the convergence of method (1.2) and method (1.3) or its special cases requiring hypotheses on the fifth derivative of function F or higher. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations [1, 2, 3, 4, 5, 6, 15, 7, 9, 8, 10, 11, 12, 13, 14]. These results show that initial guess should be close to the required root for the convergence of the corresponding methods and same thing is also mentioned by the authors of papers [6, 15]. But, how close initial guess should be required for the convergence of the corresponding method? These local results give no information on the radius of the convergence ball for the corresponding method. The same technique can be used on other methods.

In the present study we expand the applicability of method (1.2) and method (1.3) using only hypotheses up to the second order derivative of function F. We also proposed the computable radii of convergence and error bounds based on the Lipschitz constants. We further present the range of initial guesses x_0 that tell us how close the initial guess should be required for granted convergence of the method (1.2) and method (1.3). This problem was not addressed in [6, 15]. The advantages of our approach are similar to the ones already mentioned for method (1.2) and method (1.3).

Definition 1.1. (Error Equation, Asymptotic Error Constant, Order of Convergence)

Let us consider a sequence $\{x_n\}$ converging to a root ξ of $f(x) = 0$. Let $e_n = x_n - \xi$ be the error at n^{th} iteration. If constants $p \geq 1$, $c \neq 0$ exist in such a way that $e_{n+1} = ce_n^p + O(e_n^{p+1})$ known as the error equation then p and $\eta = |c|$ are said to be the order of convergence and the asymptotic error constant, respectively. From this definition the asymptotic error constant is found to be $\eta = |c| = \lim\limits_{n \longrightarrow \infty} \dfrac{|e_{n+1}|}{|e_n^p|}$. However, some researchers call $c = \lim\limits_{n \longrightarrow \infty} \dfrac{e_{n+1}}{e_n^p}$ asymptotic error constant instead of $|c|$.

Definition 1.2. (Asymptotic Order of Convergence)

With the help of above definition 1.1, we can define the asymptotic order of convergence as follows:

$$p = \lim_{n \longrightarrow \infty} \frac{|e_{n+1}/\eta|}{|e_n^p|}.$$

But, the main drawback of calculating η according to the above formula is that it involves the exact root ξ and there are many real situations in which the exact root is not known in advance. To over come this problem, we can use $(x_{n+1} - x_n)$ instead of (e_{n+1}) in the above formula to calculate η.

2. Local convergence: One dimensional case

In this section, we shall define some scalar functions and parameters in order to present the local convergence of method (1.2) that follows.

Let $L_0 > 0,\ L > 0,\ M \geq 1$ and $\beta \in \mathbb{R}$ be given constants. Let us also assume some functions $p,\ h_p,\ p_1$ and h_{p_1}defined on the interval $\left[0,\ \frac{1}{L_0}\right)$ by

$$\begin{aligned} p(t) &= \left(L_0 + \frac{LM}{2(1 - L_0 t)}\right) t, \\ p_1(t) &= \left(L_0 + \frac{LM}{1 - L_0 t}\right) t, \end{aligned} \tag{2.1}$$

$h_p(t) = p(t) - 1$, and $h_{p_1}(t) = p_1(t) - 1$. We have $h_p(0) = h_{p_1}(0) = -1 < 0$ and $h_p(t) \longrightarrow +\infty,\ h_{p_1}(t) \longrightarrow +\infty$ as $t \longrightarrow \frac{1^{-1}}{L_0}$. Then, by the intermediate value theorem functions h_p and h_{p_1} have zeros in the interval $\left(0,\ \frac{1}{L_0}\right)$. Further, let r_p

and r_{p_1} respectively be the smallest such zeros. Then, we have that

$r_{p_1} < r_p, \ \ p(r_p) = p_1(r_{p_1}) = 1$

$0 \leq p(t) \leq 1,$

$0 \leq p_1(t) \leq 1$

and

$0 \leq p(t) \leq p_1(t)$ for each $t \in [0, r_{p_1})$.

Moreover, define functions $g_1, \ h_1, \ q$ and h_q in the interval $[0, \ r_{p_1})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}\left(1 + \frac{M^2}{1 - p_1(t)}\right),$$

$$h_1(t) = g_1(t) - 1$$

$$q(t) = \frac{L_0}{2}t + |\beta - 2|M g_1(t)$$

and

$$h_q(t) = q(t) - 1.$$

We get that $h_1(0) = h_q(0) = -1 < 0$ and $h_1(t) \longrightarrow +\infty$, $h_q(t) \longrightarrow +\infty$ as $t \longrightarrow r_{p_1}$. Then, it follows from the intermediate value theorem that functions h_1 and h_q have zeros in the interval $(0, \ r_{p_1})$. Denote by r_1 and r_q, respectively the smallest such zeros. Furthermore, define functions g_2 and h_2 on the interval $[0, \ r_q)$ by

$$g_2(t) = \left(1 + \frac{M^2\big(1 + |\beta| g_1(t)\big)}{\big(1 - q(t)\big)\big(1 - p_1(t)\big)}\right) g_1(t)$$

and

$$h_2 = g_2(t) - 1.$$

Then, we get $h_2(0) = -1$ and $h_2(t) \to +\infty$ as $t \to r_q^-$. Denote by r_2 the smallest zero of function h_2 on the interval $(0, \ r_q)$. Finally, define

$$r = \min\{r_1, \ r_2, \}. \tag{2.2}$$

Then, we have that for each $t \in [0, r)$

$$0 \leq p(t) < 1, \tag{2.3}$$

$$0 \leq p_1(t) < 1, \tag{2.4}$$

$$0 \leq p(t) < p_1(t), \tag{2.5}$$

$$0 \leq g_1(t) < 1, \tag{2.6}$$

$$0 \leq q(t) < 1 \tag{2.7}$$

and

$$0 \leq g_2(t) < 1. \tag{2.8}$$

Let $U(\gamma, \ \rho)$ and $\bar{U}(\gamma, \ \rho)$ stand, respectively for the open and closed balls in S with center $\gamma \in S$ and radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notations.

Theorem 2.1. *Let us consider $F : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Let us also assume $[\cdot, \cdot \,; F] : D^2 \to L(\mathbb{R})$ to be a divided difference of order one for function F. Suppose that there exist $x^* \in \mathbb{D}$ and $L_0 > 0$ such that for each $x \in \mathbb{D}$*

$$F(x^*) = 0, \quad F'(x^*) \neq 0 \tag{2.9}$$

and

$$\left|F(x^*)^{-1}\left(F'(x)-F'(x^*)\right)\right| \le L_0|x-x^*|. \tag{2.10}$$

Moreover, suppose that there exist $L>0,\ M\ge 1$ and $\beta\in S$ such that for each $x,\ y\in U\left(x^*,\ \frac{1}{L_0}\right)\cap\mathbb{D}$

$$\left|F'(x^*)^{-1}\left(F'(x)-F'(y)\right)\right| \le L|x-y|, \tag{2.11}$$

$$\left|F'(x^*)^{-1}F'(x)]\right| \le M, \tag{2.12}$$

and

$$\bar{U}\left(x^*,\ r\right)\subseteq\mathbb{D}, \tag{2.13}$$

where the radius of convergence r is defined by (2.2). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_{-1},\ x_0\in U(x^*,\ r)-\{x^*\}$ with $x_{-1}\neq x_0$ is well defined, remains in $U(x^*,\ r)$ for each $n=0,1,2,\ldots$ and converges to x^*. Moreover, the following estimates hold

$$|y_n-x^*|\le g_1(r)|x_n-x^*|<|x_n-x^*|<r \tag{2.14}$$

and

$$|x_{n+1}-x^*|\le g_2(r)|x_n-x^*|<|x_n-x^*|, \tag{2.15}$$

where the "g" functions are defined by previously. Furthermore, for $T\in\left[r,\ \frac{2}{L_0}\right)$, the limit point x^* is the only solution of equation $F(x)=0$ in $\bar{U}(x^*,\ r)$.

Proof. We shall show estimates (2.14) and (2.15) hold with the help of mathematical induction. First, we must show $\alpha_0\neq 0$. We can write

$$\begin{aligned}\alpha_0 &= -\frac{1}{2}\frac{\frac{F'(x_0)-F'(x_{-1})}{x_0-x_{-1}}}{\frac{F(x_0)-F(x_{-1})}{x_0-x_{-1}}}\\ &= -\frac{1}{2}\frac{\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta}{\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta},\quad\text{for } x_0\neq x_{-1}.\end{aligned} \tag{2.16}$$

Using (2.2) and (2.10) we have that

$$\begin{aligned}\left|F'(x^*)^{-1}\left[\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta - F'(x^*)\right]\right| &\le \frac{L_0}{2}\left(|x_{-1}-x^*|+|x_0-x^*|\right)\\ &< L_0r<1.\end{aligned} \tag{2.17}$$

Then, by (2.17) and the Banach Lemma on invertible functions [4, 13], we get that $\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta\neq 0$ and

$$\begin{aligned}\left|\left(\int_0^1 F'(x_{-1}+\theta(x_0-x_{-1}))d\theta\right)^{-1}F'(x^*)\right| &\le \frac{1}{1-\frac{L_0}{2}(|x_{-1}-x^*|+|x_0-x^*|)}\\ &\le \frac{1}{1-L_0r}.\end{aligned} \tag{2.18}$$

In view of (2.11), (2.16) and (2.18), we have that

$$\begin{aligned}|\alpha_0| &= \frac{1}{2}\frac{\left|\int_0^1 F'(x^*)^{-1}F''(x_{-1}+\theta(x_0-x_{-1}))d\theta\right|}{\left|\int_0^1 F'(x^*)^{-1}F'(x_{-1}+\theta(x_0-x_{-1}))d\theta\right|},\\ &\le \frac{1}{2}\frac{L}{\left(1-\frac{L_0}{2}(|x_{-1}-x^*|+|x_0-x^*|)\right)}\le\frac{L}{2(1-L_0r)}.\end{aligned} \tag{2.19}$$

We must show $F'(x_0) + \alpha_0 F(x_0) \neq 0$. We can write by (2.9) that

$$F(x_0 = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.20}$$

Notice that $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$. That is $x^* + \theta(x_0 - x^*) \in U(x^*,\ r)$. Then by (2.12) and (2.20), we get that

$$\left|F'(x^*)^{-1}F'(x_0)\right| \leq M|x_0 - x^*|. \tag{2.21}$$

Using (2.2), (2.3), (2.4), (2.19) and (2.21), we get in turn

$$\begin{aligned}
&\left|F'(x^*)^{-1}\left(F'(x_0) - F'(x^*) - \alpha_0 F(x_0)\right)\right| \\
&\leq \left|F'(x^*)^{-1}\left(F'(x_0) - F'(x^*)\right)\right| + |\alpha_0|\left|F'(x^*)^{-1}F(x_0)\right| \\
&\leq L_0|x_0 - x^*| + \frac{LM|x_0 - x^*|}{2\left(1 - \frac{L_0}{2}(|x_{-1} - x^*| + |x_0 - x^*|)\right)} \\
&\leq p(r) < p_1(r) < 1.
\end{aligned} \tag{2.22}$$

It follows from (2.22) that

$$\left|\left(F'(x_0) + \alpha_0 F(x_0)\right)^{-1} F'(x^*)\right| \leq \frac{1}{1 - p_1(r)} \tag{2.23}$$

and y_0 is well defined by the first sub step of method (1.2) for $n = 0$. As in (2.22) and (2.23), we obtain that

$$\left|\left(F'(x_0) + 2\alpha_0 F(x_0)\right)^{-1} F'(x^*)\right| \leq \frac{1}{1 - p_1(r)} \tag{2.24}$$

and

$$\left|F'(x_0)^{-1}F'(x^*)\right| \leq \frac{1}{1 - L_0(r)} \tag{2.25}$$

Using the first sub step of method (1.2) for $n = 0$, (2.2), (2.6), (2.9), (2.11), (2.21) and (2.23), we get in turn that

$$\begin{aligned}
|y_0 - x^*| &= \left|x_0 - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1}F(x_0) - \left(F'(x_0) + \alpha_0 F(x_0)\right)^{-1} F(x_0)\right| \\
&\leq \left|x_0 - x^* - F'(x_0)^{-1}F(x_0)\right| + |\alpha_0|\left|\left(F'(x_0) + \alpha_0 F(x_0)\right)^{-1} F(x_0)^2\right| \\
&\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{|\alpha_0|M^2|x_0 - x^*|^2}{1 - p(|x_0 - x^*|)} \\
&\leq \frac{L|x_0 - x^*|^2}{2(1 - L_0|x_0 - x^*|)} + \frac{LM^2|x_0 - x^*|^2}{2\left(1 - \frac{L_0}{2}(|x_{-1} - x^*| + |x_0 - x^*|)\right)(1 - p(|x_0 - x^*|))} \\
&\leq g_1(r)|x_0 - x^*| < |x_0 - x^*| < r,
\end{aligned} \tag{2.26}$$

which shows (2.14) for $n = 0$ and $y_0 \in U(x^*,\ r)$. Notice that (2.21) holds for y_0 replacing x_0, since $y_0 \in U(x^*,\ r)$. We must shows $F(x_0) + (\beta - 2)F(y_0) \neq 0$.

Using (2.2), (2.3), (2.7), (2.8), (2.9) and (2.21) (for $y_0 = x_0$), we get in turn that for $x_0 \neq x^*$

$$\begin{aligned}
&\left|\left((x_0 - x^*)F'(x^*)^{-1}\right)[F(x_0 - F(x^*) - F'(x^*)(x_0 - x^*) + (\beta-2)F(y_0)]\right| \\
&\leq |x_0 - x^*|^{-1}\left[\left|F'(x^*)^{-1}\left(F(x_0) - F(x^*) - F'(x^*)(x_0 - x^*)\right)\right| + |\beta - 2|\left|F'(x^*)^{-1}F(y_0)\right|\right] \\
&\leq |x_0 - x^*|^{-1}\left(\frac{L_0}{2}|x_0 - x^*|^2 + |\beta - 2|M|y_0 - x^*|\right) \\
&\leq \frac{L_0}{2}|x_0 - x^*| + M|\beta - 2|g_1(|x_0 - x^*|) \\
&= q(|x_0 - x^*|) < q(r) < 1.
\end{aligned} \tag{2.27}$$

Hence, we get from (2.27) that

$$\left|\left(F(x_0) + (\beta - 2)F(y_0)\right)^{-1}\right| \leq \frac{1}{|x_0 - x^*|(1 - q(|x_0 - x^*|))} \tag{2.28}$$

and x_1 is well defined by the second sub step of method (1.2) for $n = 0$. Then, in view of (2.2), (2.3), (2.8), (2.21)(for $x_0 = y_0$ and $x_0 = x_0$), (2.23), (2.26) and (2.28), we obtain in turn that

$$\begin{aligned}
|x_1 - x^*| \leq\ & |y_0 - x^*| + \left|\left(F(x_0) + (\beta - 2)F(y_0)\right)^{-1}F'(x^*)\right| \\
&\times \left[|F'(x^*)^{-1}F(x_0)| + |\beta||F'(x^*)^{-1}F(y_0)|\right] \\
&\times \left|\left(F(x_0) + 2\alpha_0 F(x_0)\right)^{-1}F'(x^*)\right| |F'(x^*)^{-1}F(y_0)| \\
\leq\ & \left(1 + \frac{M^2\left(1 + |\beta|g_1(|x_0 - x^*|)\right)|x_0 - x^*|}{|x_0 - x^*|(1 - q(|x_0 - x^*|))(1 - p_1(|x_0 - x^*|))}\right)|y_0 - x^*| \\
\leq\ & g_2(|x_0 - x^*|)|x_0 - x^*| < g_2(r)|x_0 - x^*| \\
<\ & |x_0 - x^*| < r,
\end{aligned} \tag{2.29}$$

which shows (2.15) and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0 by x_m, y_m, z_m in the preceding estimates we arrive at (2.17)-(2.19). Then, from the estimates $|x_{m+1} - x^*| < |x_m - x^*| < r$, we conclude that $\lim_{m \longrightarrow \infty} x_k = x^*$ and $x_{m+1} \in U(x^*, r)$. Finally, to show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ be such that $F(y^*) = 0$. Set $Q = \int_0^1 F'\left(x^* + \theta(y^* - x^*)\right)d\theta$. Then, using (2.12), we get that

$$\left|F'(x^*)^{-1}(Q - F'(x^*))\right| \leq L_0 \int_0^1 \theta|x^* - y^*|d\theta = \frac{L_0}{2}T < 1. \tag{2.30}$$

Hence, $Q^{-1} \in L(Y, X)$. Then, in view of the identity $F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$. □

Remark 2.1. (a) In view of (2.9) and the estimate

$$\begin{aligned}
\left|F'(x^*)^{-1}F'(x)\right| &= \left|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\right| \\
&\leq 1 + \left|F'(x^*)^{-1}(F'(x) - F'(x^*))\right| \\
&\leq 1 + L_0|x_0 - x^*|
\end{aligned}$$

condition (2.11) can be dropped and M can be replaced by

$$M = M(t) = 1 + L_0 t$$

or $M = 2$, since $t \in [0, \frac{1}{L_0})$.

(b) The results obtained here can be used for operators F satisfying the autonomous differential equation [4, 5] of the form

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x + 2$. Then, we can choose $P(x) = x - 2$.

(c) The radius $r_A = \frac{2}{2L_0+L_1}$ was shown by us in [4, 5] to be the convergence radius for Newton's method under conditions (2.9) - (2.11). Radius r_A is at least as large as the convergence ball given by Rheinboldt [13] and Traub [14]

$$r_R = \frac{2}{3L_1}.$$

Notice that for $L_0 < L_1$,

$$r_R < r_A.$$

Moreover,

$$\frac{r_R}{r_A} \longrightarrow \frac{1}{3} \quad \text{as} \quad \frac{L_0}{L_1} \longrightarrow 0.$$

Hence, r_A is at most three times larger than r_R. In the numerical examples, we compare r to $r_A^* = \frac{2}{2L_0+L} \geq r_A$ and r_R. Notice that L_1 satisfies $\left|F'(x_0)^{-1}\big(F'(x) - F'(y)\big)\right| \leq L_1|x-y|$ for each $x,\ y \in D$. Then, we have that $L < L_1$ since $U\left(x^*,\ \frac{1}{L_0}\right) \cap D \subset D$

(d) It is worth noticing that method (1.2) is not changing if we use the conditions of Theorem 2.1 instead of the stronger conditions given in [6, 15]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [8]

$$\xi = \frac{ln\frac{|x_{n+2}-x^*|}{|x_{n+1}-x^*|}}{ln\frac{|x_{n+1}-x^*|}{|x_n-x^*|}}, \quad \text{for each } n = 0, 1, 2, \ldots \tag{2.31}$$

or the approximate computational order of convergence (ACOC) [8]

$$\xi^* = \frac{ln\frac{|x_{n+2}-x_{n+1}|}{|x_{n+1}-x_n|}}{ln\frac{|x_{n+1}-x_n|}{|x_n-x_{n-1}|}}, \quad \text{for each } n = 1, 2, \ldots \tag{2.32}$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first Fréchet derivative. Notice also that the computation of ξ^* does not involve the solution x^*.

Remark 2.2. In order to obtain the corresponding results for method (1.3), simply replace functions $p_1,\ g_1$ and g_2 by $\bar{p}_1,\ \bar{g}_1$ and $\bar{g}_2$ defined by

$$\bar{p}_1(t) = (L_0 + |a|M)t,$$
$$\bar{h}_{p_1}(t) = \bar{p}_1(t) - 1$$
$$\bar{g_1}(t) = \left(\frac{L}{2(1-L_0t)} + \frac{|a|M^2}{1-p_1(t)}\right)t,$$
$$\bar{h}_1(t) = \bar{g}_1(t) - 1$$
$$\bar{q}(t) = \frac{L_0}{2}t + M|\beta - 2|\bar{g}_1(t),$$
$$\bar{h}_q(t) = \bar{q}(t) - 1$$

and

$$\bar{g}_2(t) = \left(1 + \frac{M^2(1+|\beta|\bar{g}_1(t))}{(1-\bar{q}(t))(1-\bar{p_1}(t))}\right),$$
$$\bar{h}_2(t) = \bar{g}_2(t) - 1,$$

respectively and follow the proof of Theorem 2.1 with these changes. Let us consider that $\bar{r}_{p_1}$, $\bar{r}_1$, $\bar{r}_q$ and $\bar{r}_2$ be the smallest zeros of the functions $\bar{h}_{p_1}(t)$, $\bar{h}_1(t)$, $\bar{h}_q(t)$ and $\bar{h}_2(t)$, respectively. Notice that we have

$$\bar{r} = \min\{\bar{r_1},\ \bar{r}_2\} < r_0 = \frac{1}{L_0 + |a|M} = \bar{r}_2 < \bar{r}_q.$$

Theorem 2.2. *Under the hypotheses of Theorem 2.1, the conclusions hold for method (1.3) replacing method (1.2) and functions $\bar{p_1}$, $\bar{g_1}$ and $\bar{g_2}$ replacing functions p_1, g_1 and g_2.*

3. Numerical example and applications

This section is fully devoted to verify the validity and effectiveness of our theoretical results which we have proposed earlier. In this regard, we will consider some numerical examples in order to demonstrate the convergence behavior of the scheme proposed in [6, 15]. We will also check the applicability of our study where earlier study did not work.

Now, we employ the three special cases of method (1.2) for $\beta = 0$, $\beta = \frac{1}{2}$ and $\beta = 1$ are denoted by (M_1), (M_2) and (M_3), respectively. In addition, we also consider three special cases of method (1.3) for $\beta = 0$, $\beta = \frac{1}{2}$ and $\beta = 1$ are called by (M_4) and (M_5), (M_6), respectively to check the effectiveness and validity of the theoretical results.

For every iterative method, we require an initial approximation x_0 close to the required root which gives the guarantee for convergence of the corresponding iterative method. In this regard, first of all, we shall calculate the values of r_A, r_R, r_p, r_{p_1}, r_1, r_q, r_2 and r which are defined in the section 2, to find the convergence domain. We displayed all theses values in the Tables 1 and 4 which are corrected up to 5 significant digits. However, we have the values of these constants up to several number of significant digits. Then, we will also verify the theoretical convergence behavior of these methods on the basis of computational order of convergence and $\left|\frac{e_{n+1}}{e_n^p}\right|$.

In the Tables 3 and 6, we presented the number of iteration indexes (n), approximated zeros (x_n), residual error of the corresponding function $(|F(x_n)|)$, errors $|e_n|$

(where $e_n = x_n - x^*$), $\left|\frac{e_{n+1}}{e_n^p}\right|$ and the asymptotic error constant $\eta = \lim_{n\longrightarrow\infty}\left|\frac{e_{n+1}}{e_n^p}\right|$. Moreover, we will also present the computational order of convergence which is calculated by using the above formulas proposed by Sánchez et al. in [8]. We calculate the computational order of convergence, asymptotic error constant and other constants up to several number of significant digits (minimum 1000 significant digits) to minimize the round off error.

As we mentioned in the above paragraph that we calculate the values of all the constants and functional residuals up to several number of significant digits but due to the limited paper space, we display the values of x_n up to 15 significant digits. In addition, the values of other constants namely, $\xi(COC)$ up to 5 significant digits and the values $\left|\frac{e_n}{e_{n-1}^p}\right|$ and η are up to 10 significant digits. Moreover, the residual error in the function ($|F(x_n)|$) and the error $|e_n|$ are display up to 2 significant digits with exponent power which are mentioned in the following Tables corresponding to the test function. However, minimum 1000 significant digits are available with us for every value.

During the current numerical experiments with programming language Mathematica (Version 9), all computations have been done with multiple precision arithmetic, which minimize round-off errors.

Further, we use $\alpha_n = -\frac{1}{2}\frac{[x_{n-1},\ x_n;\ F']}{[x_{n-1},\ x_n;\ F]}$, $n = 0,\ 1,\ 2,\ \dots$ in the method (1.2). and $a = \alpha_0$ in method (1.3).

Example 3.1. Let $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $\mathbb{D} = \bar{U}(0,\ 1)$. Define F on $\mathbb{D}$ by

$$F(x) = e^x - 1. \tag{3.1}$$

Then the derivative is given by

$$F'(x) = e^x.$$

Notice that $x^* = 0$, $L_0 = e - 1$, $L = M = e^{\frac{1}{L_0}}$ and $L_1 = e$. We obtain different radius of convergence, COC (ξ) and n in the following Table 1.

TABLE 1. (Different radius of convergence for different cases of method (1.2))

β	r_A	r_R	r_p	r_{p_1}	r_1	r_q	r_2	r
M_1	0.38269	0.24525	0.22932	0.16234	0.10455	0.050831	0.027969	0.027969
M_2	0.38269	0.24525	0.22932	0.16234	0.10455	0.061390	0.029590	0.029590
M_3	0.38269	0.24525	0.22932	0.16234	0.10455	0.077519	0.031207	0.031207

TABLE 2. (Different radius of convergence for different cases of method (1.3))

β	r_A^*	r_R	$\bar{r}_{p_1}$	$\bar{r}_1$	$\bar{r}_q$	$\bar{r}_2$	$\bar{r}$
M_4	0.38269	0.24525	0.38269	0.20602	0.083769	0.044751	0.044751
M_5	0.38269	0.24525	0.38269	0.20602	0.10341	0.047606	0.047606
M_6	0.38269	0.24525	0.38269	0.20602	0.135464	0.050500	0.050500

TABLE 3. (Convergence behavior of methods on example (3.1))

Methods; I. guesses	n	x_n	$\|f(x_n)\|$	$\|e_n\|$	ρ	$\left\|\frac{e_n}{e_{n-1}^p}\right\|$	η
M_1;	1	$-1.34790139256212e(-10)$	$1.3e(-10)^a$	$1.3e(-10)$		$3.975878133e(-5)$	$8.631045304e(-84)$
$x_0 = 0.026$;	2	$6.17956797140613e(-52)$	$6.2e(-52)$	$6.2e(-52)$	4.9997	$3.093237159e(-18)$	
$x_{-1} = 0.025$	3	$-1.25157843463683e(-258)$	$1.3e(-258)$	$1.3e(-258)$	5.0000	$8.631045304e(-84)$	
M_2;	1	$-2.37361265176050e(-10)$	$2.4e(-10)$	$2.4e(-10)$		$4.754928001e(-5)$	$7.666765294e(-82)$
$x_0 = 0.028$;	2	$1.04644581262418e(-50)$	$1.0e(-50)$	$1.0e(-50)$	4.9996	$7.587924013e(-18)$	
$x_{-1} = 0.027$	3	$-1.74281474660902e(-252)$	$1.7e(-252)$	$1.7e(-252)$	5.0000	$7.666765294e(-82)$	
M_2;	1	$-3.34959078958091e(-10)$	$3.3e(-10)$	$3.3e(-10)$		$5.301820294e(-5)$	$1.176621971e(-80)$
$x_0 = 0.030$;	2	$5.85633985970618e(-50)$	$5.9e(-50)$	$5.9e(-50)$	4.9996	$1.310170843e(-17)$	
$x_{-1} = 0.029$	3	$-9.56750866152990e(-249)$	$9.6e(-249)$	$9.6e(-249)$	5.0000	$1.176621971e(-80)$	
M_4;	1	$-1.83720876618666e(-4)$	$1.8e(-4)$	$1.8e(-4)$		$5.364665184e(-3)$	$4.037825064e(-23)$
$x_0 = 0.043$	2	$2.90723404620118e(-21)$	$2.9e(-21)$	$2.9e(-21)$	4.9861	$2.551796045e(-6)$	
$x_{-1} = 0.042$	3	$-2.88447967644456e(-105)$	$2.9e(-105)$	$2.9e(-105)$	5.0000	$4.037825064e(-23)$	
M_5;	1	$-2.53964035862797e(-4)$	$e(-)$	$e(-)$		$5.659552194e(-3)$	$2.038091513e(-22)$
$x_0 = 0.046$	2	$1.46742588925498e(-20)$	$1.5e(-20)$	$1.5e(-20)$	4.9838	$3.527502228e(-6)$	
$x_{-1} = 0.045$	3	$-9.45036076907927e(-102)$	$9.5e(-102)$	$9.5e(-102)$	5.0000	$2.038091513e(-22)$	
M_6;	1	$-3.43233270545234e(-4)$	$3.4e(-4)$	$3.4e(-4)$		$5.937295070e(-3)$	$9.190077267e(-22)$
$x_0 = 0.049$	2	$6.61685563211979e(-20)$	$6.6e(-20)$	$6.6e(-20)$	4.9811	$4.767537798e(-6)$	
$x_{-1} = 0.048$	3	$-1.76167503404114(-98)$	$1.8e(-98)$	$1.8e(-98)$	5.0000	$9.190077267e(-22)$	

[a] $1.3e(-10)$ denotes $1.3 \times 10^{(-10)}$ and [b] $4.6e(+1)$ denotes $4.6 \times 10^{(+1)}$.

Example 3.2. Returning back to the motivation example at the introduction on this paper, we have $L = L_0 = L_1 = 96.662907$, $M = 1.0631$ and our required zero is $x^* = \frac{1}{\pi}$. We obtain different radius of convergence, COC (ρ) and n in the following Table 4.

TABLE 4. (Different radius of convergence for different cases of method (1.2))

β	r_A^*	r_R	r_p	r_{p_1}	r_1	r_q	r_2	r
M_1	0.0068968	0.0068968	0.0050668	0.0038436	0.0029697	0.0021535	0.0014647	0.0014647
M_2	0.0068968	0.0068968	0.0050668	0.0038436	0.0029697	0.0024379	0.0015136	0.0015136
M_3	0.0068968	0.0068968	0.0050668	0.0038436	0.0029697	0.0028015	0.0015571	0.0015571

4. CONCLUSION

Most of time, whenever a researcher from the worldwide proposed a new or modified variant of Newton's method or Newton like method. He/she mentioned that initial guess should be very close to the required root for the granted convergence

TABLE 5. (Different radius of convergence for different cases of method (1.3))

β	r_A^*	r_R	$\bar{r}_{p_1}$	$\bar{r}_1$	$\bar{r}_q$	$\bar{r}_2$	$\bar{r}$
M_4	0.0068968	0.0068968	0.0094475	0.0063886	0.0039803	0.0026772	0.0026772
M_5	0.0068968	0.0068968	0.0094467	0.0063881	0.0045987	0.0027843	0.0027843
M_6	0.0068968	0.0068968	0.0094471	0.0063883	0.0054826	0.0028832	0.0028832

TABLE 6. (Convergence behavior of methods on example (3.2))

Methods; I. guesses	n	x_n	$\lvert f(x_n)\rvert$	$\lvert e_n\rvert$	ρ	$\left\lvert \frac{e_n}{e_{n-1}^p} \right\rvert$	η
M_1;	1	0.318309886198535	$3.5e(-12)$	$1.5e(-11)$		$5.072535235e(-2)$	$6.722422756e(-34)$
$x_0 = 0.3167$	2	0.318309886183791	$1.7e(-44)$	$7.1e(-44)$	4.0205	$6.793257776e(-7)$	
$x_{-1} = 0.3165$	3	0.318309886183791	$7.6e(-182)$	$3.2e(-181)$	4.2498	$6.722422756e(-34)$	
M_2;	1	0.318309886203752	$4.7e(-12)$	$2.0e(-11)$		$5.590350794e(-2)$	$1.922758031e(-33)$
$x_0 = 0.3166$	2	0.318309886183791	$6.0e(-40)$	$2.5e(-42)$	4.0206	$8.657309044e(-7)$	
$x_{-1} = 0.3164$	3	0.318309886183791	$1.7e(-179)$	$7.2e(-179)$	4.2499	$1.922758031e(-33)$	
M_3;	1	0.318309886202808	$4.5e(-12)$	$1.9e(-11)$		$5.325860168e(-2)$	$4.777713914e(-33)$
$x_0 = 0.3166$	2	0.318309886183791	$4.9e(-44)$	$2.1e(-43)$	4.0179	$8.413251931e(-7)$	
$x_{-1} = 0.3165$	3	0.318309886183791	$7.4e(-180)$	$3.2e(-179)$	4.2501	$1.634940044e(-33)$	
M_4;	1	0.318309886264393	$1.9e(-11)$	$1.8e(-11)$		1.292984230	3.498998586
$x_0 = 0.3155$	2	0.318309886183791	$3.5e(-40)$	$1.5e(-40)$	3.9427	3.498998654	
$x_{-1} = 0.3154$	3	0.318309886183791	$3.9e(-160)$	$1.7e(-159)$	4.0000	3.498998586	
M_5;	1	0.318309886285849	$2.4e(-11)$	$1.0e(-10)$		1.423451314	3.736693556
$x_0 = 0.3154$	2	0.318309886183791	$9.5e(-41)$	$4.1e(-41)$	3.9438	3.736693644	
$x_{-1} = 0.3153$	3	0.318309886183791	$2.4e(-158)$	$1.0e(-157)$	4.0000	3.736693556	
M_6;	1	0.318309886278453	$2.1e(-11)$	$9.5e(-11)$		1.423451314	3.736693556
$x_0 = 0.3153$	2	0.318309886183791	$6.8e(-41)$	$2.9e(-40)$	3.9438	3.736693644	
$x_{-1} = 0.3152$	3	0.318309886183791	$6.0e(-159)$	$2.6e(-158)$	4.0000	3.736693556	

of proposed scheme. But, they do not talk about the range or interval of the required root which give the grantee for the convergence of the proposed method. Therefore, we propose the computable radius of convergence and error bound by using Lipschitz conditions in this paper. Further, we also reduce the hypotheses from sixth order derivative of the involved function to only first order derivative. It is worth noticing that method (1.2) and method (1.3) are not changing if we use the conditions of Theorem 2.1 instead of the stronger conditions proposed by them. Moreover, to obtain the error bounds in practice and order of convergence, we can use the computational order of convergence which is defined in numerical section 3. Therefore,we obtain in practice the order of convergence in a way that avoids the bounds involving estimates higher than the first order derivative.

Finally, on accounts of the results obtained in section 3, it can be concluded that the proposed study not only expand the applicability but also given the computable radius of convergence and error bound of the scheme given by the authors of [6, 15], to solve nonlinear equations.

REFERENCES

1. S. Amat, S. Busquier, S. Plaza, Dynamics of the King and Jarratt iterations, Aequationes Math. **69**(3) (2005) 212–223.

2. S. Amat, S. Busquier, S. Plaza, Chaotic dynamics of a third-order Newton-type method, J. Math. Anal. Appl. **366**(1) (2010) 24–32.
3. S. Amat, M.A. Hernández, N. Romero, A modified Chebyshev's iterative method with at least sixth order of convergence, Appl. Math. Comput. **206** (1) (2008) 164–174.
4. I.K. Argyros, Convergence and Application of Newton-type Iterations, Springer, 2008.
5. I.K. Argyros, S. Hilout, Numerical methods in Nonlinear Analysis, World Scientific Publ. Comp. New Jersey, 2013.
6. B. Campos, A. Cordero, J.R. Torregrosa, P. Vindel, Stability of King's family of iterative methods with memory, J. Comput. Appl. Math. D-15-01428, Special issue: Conference CMMSE-2015.
7. Ramandeep Behl S.S. Motsa, Geometric construction of eighth-order optimal families of Ostrowski's method, Recent Theories and Applications in Approximation Theory, **2015** (2015) Article ID 614612, 11 pages.
8. J.A. Ezquerro, M.A. Hernández, New iterations of R-order four with reduced computational cost. BIT Numer. Math. **49** (2009) 325– 342.
9. V. Kanwar, Ramandeep Behl, Kapil K. Sharma, Simply constructed family of a Ostrowski's method with optimal order of convergence, Comput. Math. Appli. **62** (11) (2011) 4021–4027.
10. Á.A. Magreñán, Different anomalies in a Jarratt family of iterative root-finding methods, Appl. Math. Comput. **233** (2014) 29–38.
11. Á.A. Magreñán, A new tool to study real dynamics: The convergence plane, Appl. Math. Comput. **248** (2014) 215–224.
12. M.S. Petkovic, B. Neta, L. Petkovic, J. Džunič, Multipoint methods for solving nonlinear equations, Elsevier, 2013.
13. W.C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, Polish Academy of Science, Banach Ctr. Publ. **3** (1978) 129–142.
14. J.F. Traub, Iterative methods for the solution of equations, Prentice- Hall Series in Automatic Computation, Englewood Cliffs, N.J. (1964).
15. X. Wang, T. Zhang, A new family of Newton-type iterative methods with and without memory for solving nonlinear equations, Calcolo, **51**(1) (2014) 1–15.