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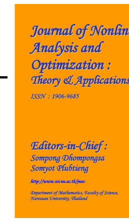

GENERAL ITERATIVE METHODS FOR A FAMILY OF NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper, we introduce implicit and explicit iterative methods for finding a common element of the set of solutions of a variational inequality and the set of common fixed points for a countable family of nonexpansive mappings in a Hilbert space. For these methods, we prove some strong convergence theorems. These theorems improve and extend some results of Yao et al. [21] and Xu [20].

KEYWORDS : Fixed point; Nonexpansive mapping; Weak contraction; Variational inequality.
AMS Subject Classification: 47H10, 47H09

1. INTRODUCTION

Let H be a real Hilbert space and A be a bounded operator on H. In this paper, we assume A is strongly positive; that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in H$. A typical problem is that of minimizing a quadratic function over the set of the fixed points of nonexpansive mapping on a real Hilbert space:

$$\min_{x\in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where b is a given point in H.

We recall a mapping T of H into itself is called nonexpansive, if $\|Tx - Ty\| \leq \|x-y\|$ for all $x, y \in H$. Let $F(T)$ denote the fixed points set of T, and a contraction on H is a self-mapping f of H such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in H$, where $\alpha \in [0, 1)$ is a constant.

Finding an optimal point in the intersection F of the fixed points set of a family of nonexpansive mappings is one that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed points set of a family of nonexpansive mappings; see, e.g., [3, 5]. The problem of finding an optimal point that minimizes a given cost function $\Theta : H \to \mathbb{R}$ over F is of wide

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interdisciplinary interest and practical importance see, e.g., [2, 4, 6, 23]. A simple algorithmic solution to the problem of minimizing a quadratic function over F is of extreme value in many applications including the set theoretic signal estimation, see, e.g., [23, 9]. The best approximation problem of finding the projection $P_F(a)$ (in the norm induced by inner product of H) from any given point a in H is the simplest case of our problem.

In 2006, Marino and Xu [10] considered an iterative method for a single nonexpansive mapping. Let f be a contraction on H and $A : H \to H$ be a strongly positive bounded linear operator. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{1.1}$$

where $\gamma > 0$ is a constant and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(III) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

Consequently, Marino and Xu [10] proved the sequence $\{x_n\}$ generated by (1.1) converges strongly to the unique solution of the following variational inequality:

$$\langle (A - \gamma f)x^*, x^* - x\rangle \leq 0, \text{ for all } x \in F(T),$$

which is the optimality condition for minimization problem

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf(i.e., $h'(x) = \gamma f(x)$ for $x \in H$).
In 2012, Razani and Yazdi [13] study convergence of a composite iterative scheme which generalizes iterative sequence (1.1).

In 2008, Yao et al. [21] introduced the iterative sequence

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)W_n x_n, \text{ for all } n \geq 0, \tag{1.2}$$

where W_n is the W-mapping generated by an infinite countable family of nonexpansive mappings $T_1, T_2, \ldots, T_n, \ldots$ and $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ such that the common fixed points set $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Under very mild conditions on the parameters, it was proved the sequence $\{x_n\}$ converges strongly to $p \in F$ where p is the unique solution in F of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^*\rangle \leq 0, \text{ for all } x^* \in F, \tag{1.3}$$

which is the optimality condition for minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x).$$

In this paper, motivated by Yao et al. [21] and Rhoades [14], we introduce an implicit and explicit iterative schemes for finding a common element of the set of solutions of a variational inequality and the set of common fixed points for a countable family of nonexpansive mappings in a Hilbert space. Then, we prove some strong convergence theorems which improve and extend some results of Yao et al. [21] and Xu [20].

Now, we collect some lemmas which will be used in the main result.

Lemma 1.1. [10] *Assume A is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 1.2. *Let H be a real Hilbert space. Then, for all $x, y \in H$*
(I) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(II) $\|x+y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 1.3. [17] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space X and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integer $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.4. [16] *Assume $\{s_n\}$ and $\{\gamma_n\}$ are two sequences of nonnegative real numbers such that*

$$s_{n+1} \leq s_n - r_n\Psi(s_n) + \gamma_n, \ n \geq 1,$$

where Ψ is a continuous and strict increasing function on $[0,\infty)$ with $\Psi(0) = 0$ and $\{r_n\}$ is a sequence of positive numbers satisfying the conditions:
(I) $\sum_{n=1}^{\infty} r_n = \infty$;
(II) $\limsup_{n\to\infty} \frac{\gamma_n}{r_n} = 0$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 1.5. [19] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(I) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(II) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

Let H be a real Hilbert space with inner product $\langle .,.\rangle$ and the norm $\|.\|$. We denote weak convergence and strong convergence by notation $\rightharpoonup$ and $\to$, respectively. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on H and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \geq 1$, define a mapping W_n of H into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{2.1}$$

Such a mapping W_n is called the $W-$mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Lemma 2.1. [15] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0,b]$ for some $b \in (0,1)$. Then, for every $x \in C$ and $k \geq 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.2. [22] It can be known from Lemma 2.1 that if D is a nonempty bounded subset of C, then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \varepsilon.$$

Remark 2.3. [22] Using Lemma 2.1, one can define mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for all $x \in C$. Such a W is called the $W-$mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$. Since W_n is nonexpansive, $W : C \to C$ is also nonexpansive.

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.2 that for an arbitrary $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$

$$\|W_n x_n - Wx_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \varepsilon.$$

This implies

$$\lim_{n\to\infty} \|W_n x_n - Wx_n\| = 0.$$

Throughout this paper, we assume $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$.

Lemma 2.4. [15] *Let C be a nonempty closed convex subset of a strictly convex Banach space X, $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers in $[0, b]$ for some $b \in (0, 1)$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Definition 2.5. [18] A self-mapping $f : C \to C$ is called weak contraction with the function Ψ if there exists a continuous and nondecreasing function $\Psi : [0, \infty) \to [0, \infty)$ such that $\Psi(s) > 0$, for all $s > 0$, $\Psi(0) = 0, \lim_{s\to\infty} \Psi(s) = +\infty$ and for any $x, y \in C, \|f(x) - f(y)\| \leq \|x - y\| - \Psi(\|x - y\|)$.

Remark 2.6. Clearly a contraction with constant k must be a weak contraction, where $\Psi(s) = (1 - k)s$, but the converse is not true.

Example 2.7. [1] The mapping $Ax = \sin x$ from $[0, 1]$ to $[0, 1]$ is a weak contraction with $\Psi(s) = \frac{s^3}{8}$. But A is not a contraction. Indeed, suppose that A is a contraction with constant $k \in (0, 1)$, i.e.,

$$|\sin x - \sin y| \leq k|x - y|, \text{ for all } x, y \in [0, 1]. \tag{2.2}$$

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$, taking $\varepsilon = 1 - k$, there exists $\delta > 0$ as $0 < x < \delta$, we have $|\frac{\sin x}{x} - 1| < 1 - k$. Therefore $k < |\frac{\sin x - \sin 0}{x - 0}|$, i.e., $k|x - 0| < |\sin x - \sin 0|$, which contradicts the assumption of (2.2). Thus A is not a contraction.

Lemma 2.8. [14] *Let (X, d) be a complete metric space and $f : X \to X$ be a weak contraction. Then f has a unique fixed point in X.*

Lemma 2.9. [7] *Let H be a real Hilbert space, C be a closed convex subset of H and $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges to y, then $(I - T)x = y$.*

Lemma 2.10. [8] *Let $\{T_n\}$ be a sequence of nonexpansive mapping on a closed convex subset C of H and A be a strongly positive bounded linear operator on H with coefficient $0 < \gamma \leq \overline{\gamma}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ with $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Define a sequence $\{y_n\}$ by $y_1 \in C$ and*

$$y_{n+1} = \alpha_n \gamma u + \beta_n y_n + ((1 - \beta_n)I - \alpha_n A)T_n y_n,$$

for all $n \in \mathbb{N}$. Suppose the sequence $\{y_n\}$ converges strongly. Set $Pu = \lim_{n\to\infty} y_n$, for each $u \in C$. Then, the following hold:

(I) Pu does not depend on the initial point y_1;
(II) P is a nonexpansive mapping on C.

Lemma 2.11. [16] *Let X be a Banach space , f be a weak contraction with a function Ψ on X and T be a nonexpansive mapping on X. Then, the composite mapping Tf is a weak contraction.*

It is easy to see the following lemma.

Lemma 2.12. *Let H be a real Hilbert space, $f : H \to H$ be a weak contraction and A be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}$*

$$\langle (A - \gamma f)x - (A - \gamma f)y, x - y\rangle \geq (\overline{\gamma} - \gamma)\|x - y\|^2, \text{ for all } x, y \in C.$$

That is, $A - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma$.

Let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma \leq \overline{\gamma}$ where γ is some constant. First, we give our implicit iterative scheme as follows: let $\{t_n\}$ be a sequence in $(0, 1)$ such that $t_n \leq \|A\|^{-1}$, for all $n \geq 1$ and $u \in H$. For each $n \geq 1$, define a mapping $S_{t_n} : H \to H$ by

$$S_{t_n}(x) = t_n\gamma u + (I - t_nA)W_nx, \ x \in H.$$

It is easy to see that for each $t_n \in (0, 1)$, $n \geq 1$, S_{t_n} is a weak contraction on H. Indeed, by Lemma 1.1,

$$\begin{aligned} \|S_{t_n}(x) - S_{t_n}(y)\| \leq& \ t_n\gamma\|u - u\| + \|(I - t_nA)(W_nx - W_ny)\| \\ \leq& \ (1 - t_n\overline{\gamma})\|x - y\|. \end{aligned}$$

By Banach contraction principle, for each $n \in \mathbb{N}$, there exists a unique element $z_n \in H$ of S_{t_n} such that

$$z_n = t_n\gamma u + (I - t_nA)W_nz_n, \text{ for all } n \geq 1. \tag{2.3}$$

Theorem 2.1. *Let H be a real Hilbert space and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{z_n\}$ be defined by (2.3) and $t_n \in (0, 1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{z_n\}$ converges strongly to $p \in F$ which is the unique solution of the following variational inequality:*

$$\langle Ap - \gamma u, p - x^*\rangle \leq 0, \text{ for all } x^* \in F. \tag{2.4}$$

Proof. First, we show the uniqueness of the solution of the variational inequality (2.4). In fact, if p, q are two distinct solutions of the variational inequality (2.4), then

$$\langle Ap - \gamma u, p - q\rangle \leq 0 \ \text{ and } \ \langle Aq - \gamma u, q - p\rangle \leq 0.$$

Adding up these two inequalities, we have

$$\langle (Ap - \gamma u) - (Aq - \gamma u), p - q\rangle \leq 0.$$

But the strong monotonicity of $A - \gamma u$ (Lemma 2.12) implies that $p = q$. We use $p \in F$ to denote the unique solution of variational inequality (2.4). Thus, for $p \in F$

$$z_n - p = t_n(\gamma u - Ap) + (I - t_nA)(W_nz_n - p). \tag{2.5}$$

From (2.5),

$$\begin{aligned} \|z_n - p\|^2 =& \ t_n\langle \gamma u - Ap, z_n - p\rangle + \langle (I - t_nA)(W_nz_n - p), z_n - p\rangle \\ \leq& \ t_n\langle \gamma u - Ap, z_n - p\rangle + (1 - t_n\overline{\gamma})\|z_n - p\|^2. \end{aligned} \tag{2.6}$$

Simplifying (2.6), we have

$$\|z_n - p\|^2 \le \frac{1}{\overline{\gamma}}\langle \gamma u - Ap, z_n - p\rangle. \tag{2.7}$$

Hence, $\{z_n\}$ is bounded, so are $\{AW_n z_n\}$. Therefore

$$\lim_{n\to\infty} \|z_n - W_n z_n\| = \lim_{n\to\infty} t_n\|\gamma u - AW_n z_n\| = 0. \tag{2.8}$$

Take a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma u - Ap, z_n - p\rangle = \lim_{k\to\infty}\langle \gamma u - Ap, z_{n_k} - p\rangle.$$

Since $\{z_{n_k}\}$ is bounded in H, without loss of generality, we assume $z_{n_k} \rightharpoonup z \in H$. It follows from (2.8) and Remark 2.3 that $z \in F(W)$. So

$$\limsup_{n\to\infty}\langle \gamma u - Ap, z_n - p\rangle = \langle \gamma u - Ap, z - p\rangle \le 0.$$

From (2.7), $\lim_{n\to\infty} z_{n_k} = z$. Next, we prove z solves the variational inequality (2.4). From (2.3),

$$Az_n - \gamma u = \frac{-1}{t_n}(I - t_n A)(z_n - W_n z_n).$$

Thus, for $q \in F$

$$\begin{aligned} \langle Az_n - \gamma u, z_n - q\rangle &= \tfrac{-1}{t_n}\langle (I - t_n A)(z_n - W_n z_n), z_n - q\rangle \\ &= \tfrac{-1}{t_n}\langle (I - W_n)z_n - (I - W_n)q, z_n - q\rangle + \\ &\quad \langle A(I - W_n)z_n, z_n - q\rangle \\ &\le \langle A(I - W_n)z_n, z_n - q\rangle, \end{aligned} \tag{2.9}$$

since $I - W_n$ is monotone (i.e.,$\langle (I - W_n)x - (I - W_n)y, x - y\rangle \ge 0$ for $x, y \in H$. This is due to the nonexpansivity of W_n). Now, replacing z_n in (2.9) with z_{n_k} and letting $k \to \infty$. Note that $\lim_{n\to\infty} z_{n_k} = z$ which implies

$$\langle Az - \gamma u, z - q\rangle \le 0.$$

That is, $z \in F$ is a solution of the variational inequality (2.4) and hence $z = p$ by uniqueness. Since each cluster point of $\{z_n\}$ equals p, $z_n \to p$ as $n \to \infty$. This completes the proof. □

Theorem 2.2. *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$. Let $0 < \gamma < \overline{\gamma}$ where γ is some constant. Let $\{z_n\}$ be defined by*

$$z_n = t_n\gamma f(z_n) + (I - t_n A)W_n z_n, \text{ for all } n \ge 1, \tag{2.10}$$

where $f : H \to H$ is a weak contraction with a function Ψ, $t_n \in (0, 1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{z_n\}$ converges strongly to $p \in F$ which is the unique solution of the following variational inequality:

$$\langle (A - \gamma f)p, p - x^*\rangle \le 0, \text{ for all } x^* \in F. \tag{2.11}$$

Proof. Define a sequence $\{u_n\}$ by

$$u_n = t_n\gamma u + (I - t_n A)W_n u_n, \text{ for all } n \ge 1,$$

for any $u \in H$. From Theorem 2.1, $\{u_n\}$ converges strongly. Set $Pu = \lim_{n\to\infty} u_n$, for each $u \in H$. It follows from Lemma 2.10 that P is nonexpansive. Then Pf is a weak contraction by Lemma 2.11. From Lemma 2.8, there exists a unique element $z \in H$ such that $z = P(f(z))$. Define a sequence $\{k_n\}$ by

$$k_n = t_n\gamma f(z) + (I - t_nA)W_nk_n, \text{ for all } n \geq 1. \tag{2.12}$$

Then, by Theorem 2.1, $\lim_{n\to\infty} k_n = P(f(z)) = z \in F(W)$. Therefore

$$\begin{aligned}\|z_n - k_n\| = \; & t_n\gamma\|f(z_n) - f(z)\| + (1 - t_n\overline{\gamma})\|W_nz_n - W_nk_n\| \\ \leq \; & t_n\gamma(\|f(z_n) - f(k_n)\| + \|f(k_n) - f(z)\|) + (1 - t_n\overline{\gamma})\|z_n - k_n\| \\ \leq \; & t_n\gamma(\|z_n - k_n\| - \psi(\|z_n - k_n\|) + \|k_n - z\| - \psi(\|k_n - z\|)) \\ & +(1 - t_n\overline{\gamma})\|z_n - k_n\| \\ \leq \; & (1 - t_n(\overline{\gamma} - \gamma))\|z_n - k_n\| + t_n\gamma\|k_n - z\|.\end{aligned}$$

Which implies

$$\|z_n - k_n\| \leq \frac{\gamma}{\overline{\gamma} - \gamma}\|k_n - z\|.$$

So $\lim_{n\to\infty} \|z_n - k_n\| = 0$ and hence $\lim_{n\to\infty} \|z_n - z\| = 0$. This complete the proof. □

Secondly, we give an explicit iterative scheme: for any given $x_0 \in H$, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n\gamma u + \beta_nx_n + ((1 - \beta_n)I - \alpha_nA)W_nx_n, \text{ for all } n \geq 0. \tag{2.13}$$

Now, we prove the following strong convergence theorem concerning the iterative scheme (2.13).

Theorem 2.3. *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $\|A\| \leq 1$. Let $0 < \gamma \leq \overline{\gamma}$ where γ i some constant. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $(0, 1)$ satisfying the following conditions:*
(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(III) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then, the sequence $\{x_n\}$ defined by (2.13) converges strongly to $p \in F$ which is the unique solution of the following variational inequality (2.4).

Proof. Let $Q = P_{\bigcap_{n=1}^{\infty} F(T_n)}$. So

$$\begin{aligned}\|Q((I - A)x + \gamma u) - Q((I - A)y + \gamma u)\| \leq \; & \|(I - A)x + \gamma u - \\ & ((I - A)y + \gamma u)\| \\ \leq \; & \|(I - A)x - (I - A)y\| \\ \leq \; & (1 - \overline{\gamma})\|x - y\|,\end{aligned}$$

for all $x, y \in H$. Therefore $Q = P_{\bigcap_{n=1}^{\infty} F(T_n)}$ is a contraction of H into itself. By Banach contraction principle there exists a unique element $p \in H$ such that $p = Q((I - A)p + \gamma u) = P_{\bigcap_{n=1}^{\infty} F(T_n)}((I - A)p + \gamma u)$ or equivalently

$$\langle Ap - \gamma u, p - x^*\rangle \leq 0, \text{ for all } x^* \in F.$$

From the condition (I), we may assume, without loss of generality, $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. Since A is strongly positive bounded linear operator on H,

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\}.$$

Observe

$$\begin{aligned}\langle((1 - \beta_n)I - \alpha_nA)x, x\rangle \; & = (1 - \beta_n) - \alpha_n\langle Ax, x\rangle \\ & \geq 1 - \beta_n - \alpha_n\|A\| \\ & \geq 0,\end{aligned}$$

that is to say $(1-\beta_n)I-\alpha_n A$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\{\langle((1-\beta_n)I-\alpha_n A)x,x\rangle : x\in H, \|x\|=1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle : x\in H, \|x\|=1\}\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

Next, we prove $\{x_n\}$ is bounded. Indeed, for $p\in F$

$$\begin{aligned}\|x_{n+1}-p\| = & \ \|\alpha_n(\gamma u-Ap)+\beta_n(x_n-p)+((1-\beta_n)I-\alpha_n A)(W_n x_n-p)\|\\ \le & \ (1-\beta_n-\alpha_n\overline{\gamma})\|x_n-p\|+\beta_n\|x_n-p\|+\alpha_n\|\gamma u-Ap\|\\ \le & \ (1-\alpha_n\overline{\gamma})\|x_n-p\|+\alpha_n\|\gamma u-Ap\|.\end{aligned} \tag{2.14}$$

It follows from (2.14) that

$$\|x_n-p\|\le \max\{\|x_0-p\|, \frac{\|\gamma u-Ap\|}{\overline{\gamma}}\}, \quad n\ge 1.$$

Hence $\{x_n\}$ is bounded, so are $\{W_n x_n\}$.

Define

$$x_{n+1}=\beta_n x_n+(1-\beta_n)y_n,\ n\ge 0.$$

Observe from the definition of y_n,

$$\begin{aligned}y_{n+1}-y_n = & \ \tfrac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\tfrac{x_{n+1}-\beta_n x_n}{1-\beta_n}\\ = & \ \tfrac{\alpha_{n+1}\gamma u+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}x_{n+1}}{1-\beta_{n+1}}\\ & -\tfrac{\alpha_n\gamma u+((1-\beta_n)I-\alpha_n A)W_n x_n}{1-\beta_n}\\ = & \ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma u-\tfrac{\alpha_n}{1-\beta_n}\gamma u+W_{n+1}x_{n+1}\\ & -W_n x_n+\tfrac{\alpha_n}{1-\beta_n}AW_n x_n-\tfrac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}x_{n+1}\\ = & \ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma u-AW_{n+1}x_{n+1}]+\tfrac{\alpha_n}{1-\beta_n}[AW_n x_n-\gamma u]\\ & +W_{n+1}x_{n+1}-W_{n+1}x_n+W_{n+1}x_n-W_n x_n.\end{aligned}$$

So

$$\begin{aligned}\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \le & \ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma u\|+\|AW_{n+1}x_{n+1}\|)\\ & +\tfrac{\alpha_n}{1-\beta_n}(\|AW_n x_n\|+\|\gamma u\|)\\ & +\|W_{n+1}x_{n+1}-W_{n+1}x_n\|+\|W_{n+1}x_n-W_n x_n\|\\ & -\|x_{n+1}-x_n\|\\ \le & \ \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma u\|+\|AW_{n+1}x_{n+1}\|)\\ & +\tfrac{\alpha_n}{1-\beta_n}(\|AW_n x_n\|+\|\gamma u\|)+\|W_{n+1}x_n-W_n x_n\|.\end{aligned} \tag{2.15}$$

From (2.1), Since T_i and $U_{n,i}$ are nonexpansive, we get

$$\begin{aligned}\|W_{n+1}x_n-W_n x_n\| = & \ \|\lambda_1 T_1 U_{n+1,2}x_n-\lambda_1 T_1 U_{n,2}x_n\|\\ \le & \ \lambda_1\|U_{n+1,2}x_n-U_{n,2}x_n\|\\ = & \ \lambda_1\|\lambda_2 T_2 U_{n+1,3}x_n-\lambda_2 T_2 U_{n,3}x_n\|\\ \le & \ \lambda_1\lambda_2\|U_{n+1,3}x_n-U_{n,3}x_n\|\\ \le & \ \dots\\ \le & \ \lambda_1\lambda_2\dots\lambda_n\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\|\\ \le & \ M\textstyle\prod_{i=1}^n\lambda_i,\end{aligned} \tag{2.16}$$

where $M\ge 0$ is a constant such that $\|U_{n+1,n+1}x_n-U_{n,n+1}x_n\|\le M$, for all $n\ge 0$. Substituting (2.16) into (2.15), we have

$$\begin{aligned}\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| & \le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma u\|+\|AW_{n+1}x_{n+1}\|)\\ & +\tfrac{\alpha_n}{1-\beta_n}(\|AW_n x_n\|+\|\gamma u\|)+M\textstyle\prod_{i=1}^n\lambda_i,\end{aligned}$$

which implies (noting that (I) and $0 < \lambda_i \le b < 1$, for all $i \ge 1$)

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 1.3,

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Consequently

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|y_n - x_n\| = 0. \tag{2.17}$$

Note

$$\begin{aligned} \|x_n - W_n x_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - W_n x_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_n\|\gamma u - AW_n x_n\| + \beta_n\|W_n x_n - x_n\|, \end{aligned} \tag{2.18}$$

which implies

$$\|x_n - W_n x_n\| \le \frac{\|x_{n+1} - x_n\| + \alpha_n\|\gamma u - AW_n x_n\|}{1 - \beta_n}.$$

It follows from (2.17) that

$$\lim_{n\to\infty} \|W_n x_n - x_n\| = 0. \tag{2.19}$$

By the same argument as in the proof of Theorem 2.1,

$$\limsup_{n\to\infty}\langle \gamma u - Ap, x_n - p\rangle \le 0, \tag{2.20}$$

where $p = P_{\bigcap_{n=1}^{\infty} F(T_n)}((I - A)p + \gamma u)$. From (2.19),

$$\limsup_{n\to\infty}\langle \gamma u - Ap, W_n x_n - p\rangle \le 0 \tag{2.21}$$

Finally, we prove $x_n \to p$ as $n \to \infty$. From (2.13),

$$\begin{aligned} \|x_{n+1} - p\|^2 = \ & \|\alpha_n(\gamma u - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(W_n x_n - p)\|^2 \\ = \ & \alpha_n^2\|\gamma u - Ap\|^2 + \|\beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(W_n x_n - p)\|^2 \\ & +2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle \\ & +2\alpha_n\langle \gamma u - Ap, ((1 - \beta_n)I - \alpha_n A)(W_n x_n - p)\rangle \\ \le \ & ((1 - \beta_n - \alpha_n\overline{\gamma})\|W_n x_n - p\| + \beta_n\|x_n - p\|)^2 + \alpha_n^2\|\gamma u - Ap\|^2 \\ & +2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle + 2(1 - \beta_n)\alpha_n\langle \gamma u - Ap, W_n x_n - p\rangle \\ & -2\alpha_n^2\langle \gamma u - Ap, A(W_n x_n - p)\rangle, \end{aligned}$$

which implies

$$\begin{aligned} \|x_{n+1} - p\|^2 \le \ & (1 - \alpha_n\overline{\gamma})^2\|x_n - p\|^2 + 2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle \\ & +\alpha_n^2\|\gamma u - Ap\|^2 + 2(1 - \beta_n)\alpha_n\langle \gamma u - Ap, W_n x_n - p\rangle \\ & -2\alpha_n^2\langle \gamma u - Ap, A(W_n x_n - p)\rangle \\ \le \ & [1 - 2\alpha_n\overline{\gamma}]\|x_n - p\|^2 + \overline{\gamma}^2\alpha_n^2\|x_n - p\|^2 + \alpha_n^2\|\gamma u - Ap\|^2 \\ & +2\beta_n\alpha_n\langle \gamma u - Ap, x_n - p\rangle + 2(1 - \beta_n)\alpha_n\langle \gamma u - Ap, W_n x_n - p\rangle \\ & +2\alpha_n^2\|\gamma u - Ap\|\|A(W_n x_n - p)\| \\ = \ & [1 - 2\alpha_n\overline{\gamma}]\|x_n - p\|^2 + \alpha_n\{\alpha_n(\overline{\gamma}^2\|x_n - p\|^2 \\ & +\|\gamma u - Ap\|^2 + 2\|\gamma u - Ap\|\|A(W_n x_n - p)\|) \\ & +2\beta_n\langle \gamma u - Ap, x_n - p\rangle + 2(1 - \beta_n)\langle \gamma u - Ap, W_n x_n - p\rangle\}. \end{aligned}$$

Since $\{x_n\}$ and $\{W_n x_n\}$ are bounded, we can take a constant $M_1 > 0$ such that

$$\overline{\gamma}^2\|x_n - p\|^2 + \|\gamma u - Ap\|^2 + 2\|\gamma u - Ap\|\|A(W_n x_n - p)\| \le M_1,$$

for all $n \ge 0$. So

$$\|x_{n+1} - p\|^2 \le [1 - 2\alpha_n\overline{\gamma}]\|x_n - p\|^2 + \alpha_n\xi_n, \tag{2.22}$$

where

$$\xi_n = 2\beta_n\langle \gamma u - Ap, x_n - p\rangle + 2(1-\beta_n)\langle \gamma u - Ap, W_n x_n - p\rangle + \alpha_n M_1.$$

By (I), (2.20) and (2.21), we get $\limsup_{n\to\infty} \xi_n \leq 0$. Now, applying Lemma 1.5 to (2.22) concludes $x_n \to p$ as $n \to \infty$. This completes the proof. □

Theorem 2.4. *Let H be a real Hilbert space, $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of H into itself which satisfies $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, $f : H \to H$ be a weak contraction with a function Ψ, A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $\|A\| \leq 1$. Let $0 < \gamma \leq \overline{\gamma}$ where γ is some constant. Let $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $(0,1)$ satisfying the following conditions:*
(I) $\lim_{n\to\infty} \alpha_n = 0$;
(II) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(III) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
For any given $x_0 \in H$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n x_n, \text{ for all } n \geq 1, \tag{2.23}$$

converges strongly to $p \in F$ which is the unique solution of the following variational inequality (2.11).

Proof. Define a sequence $\{u_n\}$ by

$$u_{n+1} = \alpha_n \gamma u + \beta_n u_n + ((1-\beta_n)I - \alpha_n A)W_n u_n, \text{ for all } n \geq 1,$$

for any $u \in C$. From Theorem 2.3, $\{u_n\}$ converges strongly. Set $Pu = \lim_{n\to\infty} u_n$, for each $u \in C$. By the same argument as in the proof of Theorem 2.2, there exists $z = P(f(z))$. Define a sequence $\{k_n\}$ by

$$k_{n+1} = \alpha_n \gamma f(z) + \beta_n k_n + ((1-\beta_n)I - \alpha_n A)W_n k_n, \text{ for all } n \geq 1. \tag{2.24}$$

Then, by Theorem 2.3, $\lim_{n\to\infty} k_n = P(f(z)) = z \in F(W)$. Therefore

$$\begin{aligned}
\|x_{n+1} - k_{n+1}\| = \;& \alpha_n\gamma\|f(x_n) - f(z)\| + \beta_n\|x_n - k_n\| \\
& +(1-\beta_n-\alpha_n\overline{\gamma})\|W_n x_n - W_n k_n\| \\
\leq \;& \alpha_n\gamma(\|f(x_n) - f(k_n)\| + \|f(k_n) - f(z)\|) + \beta_n\|x_n - k_n\| \\
& +(1-\beta_n-\alpha_n\overline{\gamma})\|x_n - k_n\| \\
\leq \;& \alpha_n\gamma(\|x_n - k_n\| - \psi(\|x_n - k_n\|) + \|k_n - z\| - \psi(\|k_n - z\|)) \\
& +(1-\alpha_n\overline{\gamma})\|x_n - k_n\| \\
\leq \;& (1-\alpha_n(\overline{\gamma}-\gamma))\|x_n - k_n\| - \alpha_n\gamma\psi(\|x_n - k_n\|) \\
& +\alpha_n\gamma(\|k_n - z\| - \psi(\|k_n - z\|)) \\
\leq \;& \|x_n - k_n\| - \alpha_n\gamma\psi(\|x_n - k_n\|) + \alpha_n\gamma(\|k_n - z\| - \psi(\|k_n - z\|)).
\end{aligned}$$

Set $s_n = \|x_n - k_n\|$, $\gamma_n = \alpha_n\gamma(\|k_n - z\| - \psi(\|k_n - z\|))$ and $r_n = \alpha_n\gamma$. Since

$$\lim_{n\to\infty} \frac{\gamma_n}{r_n} = \|k_n - z\| - \psi(\|k_n - z\|) = 0$$

and

$$\sum_{n=0}^{\infty} r_n = \sum_{n=0}^{\infty} \alpha_n\gamma = \infty,$$

by Lemma 1.4 , $\lim_{n\to\infty} \|x_n - k_n\| = 0$. Hence $\lim_{n\to\infty} \|x_n - z\| = 0$. This complete the proof. □

Remark 2.13. Theorem 2.2 is a generalization of [21, Theorem 3.1].

Remark 2.14. Theorem 2.4 is a generalization of [20, Theorem 3.2] and [21, Theorem 3.2] with assumption $\|A\| \leq 1$.

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