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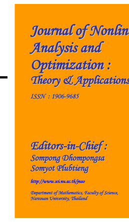

A NOTE ON A DUAL SCHEME OF A LINEAR FRACTIONAL PROGRAMMING PROBLEM

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ABSTRACT. We are interested in the duality scheme of a linear fractional programming problem proposed by Seshan [11]. The remarkable feature of the duality scheme is that the dual problem and the primal problem have the same linear fractional objective functions. Although the duality scheme is fascinating and has been introduced in literature, the steps how to build the scheme still be silent. The aim of this paper is to show that the dual problem can be obtained based upon the transformation forms given by Charnes-Cooper or by Dinkelbach with a simple change variable method.

KEYWORDS : Seshan duality; Charnes-Cooper transformation; Dinkelbach transformation.
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1. **Introduction**

Fractional programming problems were attracted by many authors early [4], [6], [7], [13]. As a generalization of linear programming problems, the following linear fractional programming problem was considered.

$$\text{(P) Max } F(x) = \frac{c^T x + c_0}{d^T x + d_0}$$
$$\text{s.t. } Ax \le b,$$
$$x \ge 0,$$

where $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, $d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$; c_0, d_0 are constants, A is an $m \times n$ real matrix ($m < n$), and $\text{rank}A = m$. There exist several methods for solving the problem (P) introduced in the books [2], [10]. Moreover, there are several dual schemes for (P) are proposed for years [1], [5], [6], [11], [12].

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It is well known that a dual problem of a linear programming problem is also a linear programming one. Naturally, it was expected that a dual problem of a linear fractional programming problem is also a linear fractional programming one. Among dual schemes in linear fractional programming, there exist some ones are linear fractional problems [3]. We are interested in the one proposed by Seshan [11]. In the paper the following dual problem for (P) is given.

$$\text{(D) Min } I(u,v) = \frac{c^T u + c_0}{d^T u + d_0} \tag{1.1}$$

$$\text{s.t. } c(d^T u) - d(c^T u) - A^T v \le c_0 d - d_0 c, \tag{1.2}$$

$$c_0 d^T u - d_0 c^T u + b^T v \le 0, \tag{1.3}$$

$$u \ge 0, v \ge 0. \tag{1.4}$$

The remarkable feature of the dual scheme is that the dual problem and the primal problem have the same linear fractional objective functions. For this duality scheme, the weak and strong duality theorems were established [11] and the results were also quoted in [2].

Although the dual scheme above was introduced since 1980 and was quoted in literature, as far as we know, the rule for building a dual problem (behind the construction) from the problem (P) was not introduced and the steps how to obtain the formulation (1.1)-(1.4) still be silent. In the paper [3], published in 2010, it was shown that there exist some duality schemes for (P) are equivalences. The paper only shows that the duality scheme for (P) proposed by Gol'stein [8] can derive the Seshan's scheme via the use of a Lagrange function associated with the Chanes-Cooper transformation [9].

Our aim of this paper is to clarify the way for building the Seshan's duality scheme proposed in [11]. For this purpose, firstly, we use Charnes - Cooper transformation [9] to change (P) to a linear problem. Next, by using a basic dual rule, we formulate its dual problem. Lastly, by using a simple change of variable method, we access to the Seshan's dual scheme. In addition, by using Dinkelbach transformation [2], we can see that the problem (P) is equivalent to the one in the form of linear problems. Then, we consider its dual problem. From this step, based on a simple change variable method, we can reach to the Seshan's dual problem.

The remains of the paper are organized as follows. The next section is devoted some basic results. In the last section, we introduce two way to obtain Seshan's dual scheme for (P).

2. Preliminaries and notations

Denote by $X = \{x \in \mathbb{R}^n \mid Ax \le b, x \ge 0\}$ the feasible set of (P). Suppose that $d^T x + d_0 > 0$ for all $x \in X$, X is bounded and the objective function F of (P) is not constant on X. We also denote the feasible set of (D) by Y and assume that $d^T u + d_0 > 0$ for all $(u,v) \in Y$. Using Charnes - Cooper transformation [2], the problem (P) can be changed to a linear programming (see [2, p.78]) as follows.

Let $t = \dfrac{1}{d^T x + d_0}$ and $y = tx$ we derive the following linear problem:

$$\begin{aligned} \text{(L1) Max } & G(y,t) = c^T y + c_0 t \\ \text{s.t. } & Ay - bt \le 0, \\ & d^T y + d_0 t = 1, \\ & y \ge 0,\ t > 0. \end{aligned}$$

Denote by F_1 the feasible set of (L_1). We also note that the problem (P) is solvable if and only if the problem (L1) is solvable also. Moreover, they have the same optimal values.

Setting $\alpha = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_0 \end{pmatrix}$, $z = \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ t \end{pmatrix}$, $\beta = \begin{pmatrix} d_1 \\ \vdots \\ d_n \\ d_0 \end{pmatrix}$, $\bar{A} = [A| - b]$, the problem (L_1) can be rewritten in the following formulation:

$$\begin{aligned} \text{(L1a)} \quad & \text{Max } \alpha^T z \\ & \text{s.t. } \bar{A} z \leq 0, \\ & \beta^T z = 1, \\ & z \geq 0 \ (z_{n+1} > 0). \end{aligned}$$

We need the following results. For the problem (P), set $f(x) = c^T x + c_0$ and $g(x) = d^T x + d_0$ where $d^T x + d_0 > 0$ for all $x \in X$.

Lemma 2.1. ([2, p. 88]) *The function F defined by $F(\lambda) = \max_{x \in X}[f(x) - \lambda g(x)], \lambda \in \mathbb{R}$, is strictly decreasing in λ.*

Lemma 2.2. ([2, Theorem 3.5.3, p. 87]) *The point $x_0 \in X$ is the optimal solution of* (P) *if and only if*

$$\max_{x \in X}[f(x) - \lambda_0 g(x)] = F(\lambda_0) = 0,$$

where $\lambda_0 = \dfrac{f(x_0)}{g(x_0)}$.

3. **Main results**

3.1. **Using Charnes - Cooper transformation to derive Seshan duality scheme.**

Since the problem (L1a) is a linear programming problem, by using the duality rule for (L1a), the dual problem of (L1a) is

$$\begin{aligned} \text{(DL1) Min } & H(\xi) = \xi^T s \\ \text{s.t. } & \xi_i \geq 0, i = \overline{1, m}, \\ & \xi_{m+1} \in \mathbb{R}, \\ & \xi^T B \geq \alpha^T, \end{aligned}$$

where $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \\ \xi_{m+1} \end{pmatrix}$, $s = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, $B = \left[\begin{array}{c|c} A & -b \\ d^T & d_0 \end{array} \right]$.

For (DL1), denote a_i the i-column of A, $i = \overline{1, n}$. Set $w = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix}$ and $\lambda = \xi_{m+1}$.

Then,

$$\xi^T B = (a_1^T w + \lambda d_n, ..., a_n^T w + \lambda d_n, -b^T w + \lambda d_0).$$

Hence,

$$\xi^T B \geq \alpha^T \Leftrightarrow A^T w + \lambda d \geq c \text{ v } - b^T w + \lambda d_0 \geq c_0.$$

Note that, $\xi^T s = \xi_{m+1}$ and $\lambda = \xi_{m+1}$. The problem (DL1) becomes

$$(\mathcal{Q}_\lambda)\ \text{Min}\ \lambda \tag{3.1}$$
$$\text{s.t. } A^T w \geq c - \lambda d, \tag{3.2}$$
$$b^T w + c_0 - \lambda d_0 \leq 0, \tag{3.3}$$
$$w \geq 0, w \in \mathbb{R}^m. \tag{3.4}$$

We will show that the problem (1.1)-(1.4) can be obtained by the problem (3.1)-(3.4) via the following transformation.

Proposition 3.1. *Assume that* $d^T u + d_0 > 0$ *for all* $u \geq 0$ *and set* $\lambda = \dfrac{c^T u + c_0}{d^T u + d_0}$, $v = (d^T u + d_0)w$. *Then, the constraint* (1.2) *is equivalent to* (3.2) *and the constraint* (1.3) *is equivalent to* (3.3).

Proof. We have

$$\begin{aligned}
& A^T w \geq c - \lambda d \\
\Leftrightarrow\ & -A^T w \leq \lambda d - c \\
\Leftrightarrow\ & -A^T v \leq (\lambda d - c)(d^T u + d_0) \\
\Leftrightarrow\ & (d^T u)c - (c^T u)d - A^T v \leq (d^T u)c - (c^T u)d - (d^T u + d_0)(c - \lambda d) \\
\Leftrightarrow\ & (d^T u)c - (c^T u)d - A^T v \leq (d^T u)c - (c^T u)d - (d^T u)c - d_0 c + (c^T u + c_0)d \\
\Leftrightarrow\ & (d^T u)c - (c^T u)d - A^T v \leq c_0 d - d_0 c.
\end{aligned}$$

Hence, the constraint (1.2) is equivalent to (3.2). Furthermore, we have

$$\begin{aligned}
& b^T w + c_0 - \lambda d_0 \leq 0 \\
\Leftrightarrow\ & (d^T u + d_0) b^T w + c_0(d^T u + d_0) - d_0(c^T u + c_0) \leq 0 \\
\Leftrightarrow\ & b^T v + c_0 d^T u - d_0 c^T u \leq 0.
\end{aligned}$$

Thus, the constraint (1.3) is equivalent to (3.3). □

Note that, since $d^T u + d_0 > 0$ and $w \geq 0$, $v \geq 0$.

Remark 3.2. From Proposition 3.1 and $w \geq 0$, we say that the problem (D) can be reached by $(\mathcal{Q}_\lambda)$.

3.2. **Using Dinkelbach transformation to derive Seshan duality scheme.**

Based on Lemma 2.2, an optimal solution of (P) is also an optimal solution of the following problem:

$$\begin{aligned}
\text{(L2) Max } & \{(c^T x + c_0) - \overline{\lambda}(d^T x + d_0)\} \\
\text{s.t. } & Ax \leq b, \\
& x \geq 0,
\end{aligned}$$

where $\overline{\lambda}$ is the optimal value of (P). Note that, the problem (L2) can be rewritten by the following formulation.

$$\begin{aligned}
\text{Max } & \{(c - \overline{\lambda} d)^T x + c_0 - \overline{\lambda} d_0\} \\
\text{s.t. } & Ax \leq b, \\
& x \geq 0.
\end{aligned}$$

By applying the basic duality rule for linear programming problem to (L2), we obtain the following problem

$$\begin{aligned}
\text{(DL2) Min } & \{b^T w + c_0 - \overline{\lambda} d_0\} \\
\text{s.t. } & A^T w \geq c - \overline{\lambda} d,
\end{aligned}$$

$$w \geq 0, w \in \mathbb{R}^m.$$

Denote by F_2 the feasible set of (DL2).

Remark 3.3. The optimal value of (L2) equals to 0 and the optimal value of (DL2) does also by the strong duality.

Lemma 3.4. *The function R defined by*

$$R(\lambda) = \min_w \{b^T w + c_0 - \lambda d_0 \mid A^T w \geq c - \lambda d, w \geq 0, w \in \mathbb{R}^m\}$$

is strictly decreasing.

Proof. Suppose that $\lambda_2 > \lambda_1$. We get

$$\begin{aligned} R(\lambda_2) &= \min_w \{b^T w + c_0 - \lambda_2 d_0 \mid A^T w \geq c - \lambda_2 d, w \geq 0, w \in \mathbb{R}^m\} \\ &= b^T \bar{w} + c_0 - \lambda_2 d_0 \end{aligned}$$

where $\bar{w}$ be a optimal solution of (DL2) according to λ_2. Based on the strong duality property of linear programming, we get

$$\begin{aligned} R(\lambda_2) &= \max_x \{(c - \lambda_2 d)^T x + c_0 - \lambda_2 d_0 \mid Ax \leq b, x \geq 0\} \\ &= (c - \lambda_2 d)^T \bar{x} + c_0 - \lambda_2 d_0 \\ &< (c - \lambda_1 d)^T \bar{x} + c_0 - \lambda_1 d_0 \\ &\leq \max_x \{(c - \lambda_1 d)^T x + c_0 - \lambda_1 d_0 \mid Ax \leq b, x \geq 0\} \\ &\leq \min_w \{b^T w + c_0 - \lambda_1 d_0 \mid A^T w \geq c - \lambda_1 d, w \geq 0, w \in \mathbb{R}^m\} = R(\lambda_1). \end{aligned}$$

Hence, the function R is strictly decreasing in λ. □

Proposition 3.5. *Suppose that $\bar{\lambda}$ is the optimal value of* (P). *Then, the vector $\bar{v}$ is an optimal solution of* (DL2) *if and only if $(\bar{v}, \bar{\lambda})$ is an optimal value of* (Q_λ).

Proof. Let $\bar{v}$ be an optimal solution of (DL2). Then, by Remark 3.3, we get

$$b^T \bar{v} + c_0 - \bar{\lambda} d_0 = 0.$$

Moreover

$$A^T \bar{v} \geq c - \bar{\lambda} d \text{ v } \bar{v} \geq 0.$$

Hence,

$$R(\bar{\lambda}) = 0 \text{ v } (\bar{v}, \bar{\lambda}) \in F_2.$$

On the other hand, for any $(w, \lambda) \in F_2$, we get

$$R(\lambda) \leq 0 = R(\bar{\lambda}).$$

Since the function R is strictly decreasing, it yields $\bar{\lambda} \leq \lambda$. Hence, $\bar{\lambda}$ is the optimal value of (Q_λ), i.e., $(\bar{v}, \bar{\lambda})$ is an optimal solution of (Q_λ).

Conversely, let $(\bar{v}, \bar{\lambda})$ be an optimal solution of (Q_λ). We have

$$b^T \bar{v} + c_0 - \bar{\lambda} d_0 \leq 0,$$

$$A^T \bar{v} \geq c - \bar{\lambda} d \text{ v } \bar{v} \geq 0.$$

Since $\bar{\lambda}$ is the optimal value of (P), $\min\limits_{w \in F_2} \{b^T w + c_0 - \bar{\lambda} d_0\} = 0$ by Remark 3.3. On the other hand, $b^T \bar{v} + c_0 - \bar{\lambda} d_0 \geq \min\limits_{w \in F_2} \{b^T w + c_0 - \bar{\lambda} d_0\} = 0$. We obtain

$$b^T \bar{v} + c_0 - \bar{\lambda} d_0 = 0.$$

This means that $\bar{v}$ is an optimal solution of (DL2). □

Remark 3.6. The problem (DL2) equals to (Q_λ). This together with Remark 3.2 imply that the problem (D) can be reached by (DL2).

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