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ON LOWER SEMICONTINUITY OF THE SOLUTION MAPPINGS OF THE VECTOR EQUILIBRIUM PROBLEMS

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ABSTRACT. By using a new assumption, the lower semicontinuity of the solution mapping of the vector equi- librium mappings in the setting of topological vector spaces without using a continuity assumption are established. Some examples in order to illustrate the main results are given. The results of the article can be viewed as improvements to the results published in this area.

KEYWORDS:Lower semicontinuity, Efficient solutions,vector equilibrium.
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1. Introduction

complementarity problem, the vector optimization problem and the vector saddle point problem, the vector equilibrium problem has been intensively studied in the literature. The stability analysis of the solution mappings for vector equilibrium problems is an important topic in optimization theory. Recently, the semicontinuity, especially the lower semicontinuity, of the solution mappings for parametric vector equilibrium problems has been of considerable interest.

Inspired by the pioneer work of Giannessi [14], the theory of vector equilibrium problems was started during the last decade of last century. The vector equilibrium problems (for short, VEP) are among the most interesting and intensively studied classes of nonlinear problems. They include fundamental mathematical problems, namely, vector optimization problems, vector variational inequality problems, Nash equilibrium problem for vector-valued mappings and fixed point problems as special cases. A large number of research papers have been published on different aspects of vector equilibrium problems, see, for example [1, 4, 5, 8, 9, 10, 11, 12, 13, 18, 19, 23, 26, 27, 28] and the references therein.

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There are several possible ways to generalize vector equilibrium problems for set-valued mappings, see, for example, [23, 28] and the references therein. Such generalizations are based on the concepts, namely, weak efficient solution, efficient solutions, strong efficient solutions, etc., of vector optimization problems.

In [2], Anh and Khanh studied the semicontinuity of the solution mappings of parametric multivalued vector quasi-equilibrium problems under the assumption C-inclusion property. In [17], Huang et. al. ,using local existence results for the models, considered the lower semicontinuity of solution mappings for parametric implicit vector equilibrium problems. Anh and Khanh [3] and Kimura and Yao [21] discussed the semicontinuity of solution mappings of parametric vector quasi-equilibrium problems by virtue of the closedness or openness assumptions for some certain sets, respectively. By using the ideas of Cheng and Zhu [7], Gong [15] discussed the continuity of the solution mappings for parametric weak vector equilibrium problems in topological vector spaces. In [6], by using a new proof which is different from the ones of [16, 7], Chen et al. established the lower semicontinuity and continuity of the solution mappings to a parametric generalized vector equilibrium problem involving set-valued mappings. In [22], Li and Fang investigated the lower semicontinuity of the solutions mapping to parametric weak vector equilibrium problems, called weak vector solutions to a generalized Ky Fan inequality, under a weaker assumption than C-strict monotonicity.

2. Preliminaries

Throughout this paper, let X, Y and Z be topological vector spaces and C be a pointed closed and convex cone in Y with nonempty interior $(, intC \neq \emptyset)$. Let B be a nonempty subset of X and $F : B \times B \longrightarrow Y$ be a vector valued mapping. The vector equilibrium problem (VEP) consists of finding $x \in B$ such that

$$f(x, y) \notin -C \backslash \{0\}, \ \forall y \in B.$$

When the subset B and the function f are perturbed by parameter μ which varies over a subset Λ of Z, we consider the following parametric vector equilibrium problem (PVEP)

$$\text{Finding } x \in A(\mu) \text{ such that } f(x, y, \mu) \notin -C\backslash\{0\}, \ \forall y \in A(\mu),$$

where $A : \Lambda \longrightarrow 2^B \setminus \{\emptyset\}$ is a set valued mapping and $f : B \times B \times \Lambda \longrightarrow Y$is a vector valued mapping. The solution set of PVEP is denoted by $S(\mu)$, i.e,

$$S(\mu) = \{x \in A(\mu) | f(x, y, \mu) \notin -C \setminus \{0\}, \ \forall y \in A(\mu)\} \tag{2.1}$$

Throughout this article we always assume $S(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. The main of this note is to investigate the continuity of the solution set map $S(\mu)$ as a set valued mapping from the set Λ into X.

The following definitions and results are needed in the next section.

Definition 2.1. Let X and Y be topological spaces and $F : X \longrightarrow 2^Y$ be a set valued mapping. The set valued mapping F is called:

- **lower semicontinuous** (lsc) at x if for any open set V satisfying $V \cap F(x) \neq \emptyset$, there exists open set U of X such that

$$V \cap F(y) \neq \emptyset, \ \ \forall y \in U.$$

- **upper semicontinuous** (usc) at x if for any open set V satisfying $F(x) \subset V$, there exists open set U of X such that
$$F(x) \subset V, \ \ \forall y \in U.$$
- **continuous** at x if it is both l.s.c and u.s.c at x.

Note F is called respectively lsc, usc and continuous on $A \subset X$ if it is respectively, lsc, usc and continuous at each $x \in A$.

Proposition 2.2. *([20]) Let X and Y be topological spaces and $F : X \longrightarrow 2^Y$ be a set valued mapping. Then*

- *F is lsc at $x \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \longrightarrow x$ and any $z \in F(x))$ there exists $z\alpha \in f(x_\alpha)$ such that $z_\alpha \longrightarrow x$.*
- *If F has compact values (i.e, $F(x)$ is compact set for each $x \in X$), then F is usc at x if and only if for any net $x_\alpha \subset X$ with $x_\alpha \longrightarrow x$and for any $z_\alpha \in F(x_\alpha)$, there exists $z \in F(x)$ and a subnet z_β of z_α such that $z_\beta \longrightarrow z$.*

We are going to recall the linear scalarization method. Let Y be topological vector space. The topological dual of Y is denoted by Y^* and it is consists of all continuous linear mappings from Y into the real line ($\mathbb{R}$). Let C be a subset of Y, The (positive) polar cone of C is defined by
$$C^* := \{c^* \in Y^* : \langle c^*, c\rangle \geq 0, \ \forall c \in C\},$$
and quasi interior of C^* is defined by
$$C^*_+ := \{c^* \in C^* : \langle c^*, c\rangle > 0, \ \forall x \in C\backslash\{0\}\}.$$
It follows from the bipolar theorem (see [4]) that if Y is a locally convex space and C is a closed convex cone with nonempty interior then the following assertions hold:
$$y \in C \Longleftrightarrow [\langle y^*, y\rangle \geq 0, \ \forall y^* \in C^*],$$
$$y \in intC \Longleftrightarrow [\langle y^*, y\rangle > 0, \forall y^* \in C^*_+].$$

Definition 2.3. Let X be a topological space and C be a convex cone with nonempty interior of the topological vector space Y. The vector valued mapping $g : X \longrightarrow Y$ is called:

- $C-$lower semicontinuous on X if for each fixed $x \in X$ and for any $y \in intC$, there exists a neighborhood $U(x)$ such that
$g(x) \in g(u) + y - intC, \ \forall u \in U(x)$.
- $C-$upper semicontinuous on X if for each fixed $x \in X$ and for any $y \in intC$, there exists a neighborhood $U(x)$ such that
$g(u) \in g(x) + y - intC, \ \forall u \in U(x)$.

Proposition 2.4. *If $g : K \longrightarrow Y$ is $C-$lower semicontinuous then the set $A := \{x \in K; g(x) \notin intC\}$ is closed in K.*

Proof. Suppose that $x \notin A$ then $g(x) \in intC$. Hence by Definition **??** (a) there exists $U(x)$ such that
$$g(x) \in g(u) + g(x) - intC, \ \forall u \in U(x),$$
which implies that $g(u) \in intC$. Therefore $U(x) \subset K \setminus A$. This shows that A is closed in K. □

Proposition 2.5. *([4]) If $g : X \longrightarrow Y$ is a vector valued mapping, $C-$lower semicontinuous ,and $c^* \in C^*$ then the mapping $\langle c^*, g(.)\rangle$ is lower semicontinuous.*

3. Main Results

In this section we present sufficient conditions in order to guarantee the lower semicontinuity of the solution set mapping PVEP.

Theorem 3.1. *Suppose that the following are satisfied:*
(i) $A(.)$ *is continuous with compact values on* Λ,
(ii) $f(.,.,.)$ *is* $C-$*lowe semicontinuous on* $B \times B \times \Lambda$,
(iii) For each $\mu \in \Lambda,\ x \in A(\mu) \setminus S(\mu)$, *there exists* $y \in S(\mu)$ *such that*

$$f(x, y, \mu) \in -C \setminus \{0\}.$$

Then $S(.)$ *is lower semicontinuous on* Λ.

Proof. Suppose to the contrary that there exists μ_0 such that $S(.)$ is not lower semicontinuous at μ_0. Then there exist a net $\{\mu_\alpha\}$ with $\mu_\alpha \longrightarrow \mu_0$ and $x_0 \in S(\mu_0)$, such that for any $x_\alpha \in S(\mu_\alpha),\ x_\alpha \nrightarrow x_0$. From $x_0 \in S(\mu_0)$ we have $x_0 \in A(\mu_0)$ and

$$f(x_0, y, \mu_0) \notin -C \setminus \{0\}\ \forall y \in A(\mu_0). \tag{3.1}$$

Since $A(.)$ is l.s.c at μ_0 there exists a net $\{\bar{x}_\alpha\} \subset A(\mu_\alpha)$ such that $\bar{x}_\alpha \longrightarrow x_0$. Obviously, $\bar{x}_\alpha \in A(\mu_\alpha) \setminus S(\mu_\alpha)$. By (iii) there exists $y_\alpha \in S(\mu_\alpha)$ such that $f(x_\alpha, y_\alpha, \mu_\alpha) \in -C$. Since $y_\alpha \in A(\mu_\alpha)$, it follows from the upper continuity and compactness of $A(.)$ at μ_0 that there exist $y_0 \in A(\mu_0)$ and subnet y_{α_β} of y_α such that $y_{\alpha_\beta} \longrightarrow y_0$, Suppose that $c^* \in C^*$ be arbitrary, since $f(x_{\alpha_\beta}, y_{\alpha_\beta}, \mu_{\alpha_\beta}) \in -C$ we have $\langle c^*, f(x_{\alpha_\beta}, y_{\alpha_\beta}, \mu_{\alpha_\beta})\rangle \le 0$ then by Proposition 2.5 we can obtain

$$\langle c^*, f(x_0, y_0, \mu_0)\rangle \le \liminf_{\beta} \langle c^*, f(x_{\alpha_\beta}, y_{\alpha_\beta}, \mu_{\alpha_\beta})\rangle \le 0.$$

Hence $f(x_0, y_0, \mu_0) \in -C$ and $y_0 \in A(\mu_0)$ which is contradicted by (3.1). This completes the proof. □

The following example indicates that assumption (iii) in Theorem (3.1) is essential.

Example 3.2. Let $X = Y = Z = \mathbb{R}$ and $C = \mathbb{R}_+, \Lambda = [0, 1], A(\mu) = B = [0, 1]$ and $f(x, y, \mu) = 2x - y + \mu$. It is easy to see that the assumptions (i) and (ii) of Theorem 3.1 are satisfied. It follows from a direct computation by Definition2.1 that

$$S(\mu) = \begin{cases} [\frac{1}{2}, 1],\ \mu = 0; \\ [\frac{1+\mu}{2}, 1],\ \mu \neq 0, \end{cases}$$

which is not lower semicotinuous at $\mu = 0$ and hence the condition (iii) of Theorem (3.1) is dropped.

Remark 3.3. Theorem 3.1 improves Theorem 3.1 of [29] by relaxing the continuity of the mapping f and mertizability of the topological vector space. Further, our approach can be also applied to study the lower semicontinuity of the following problem which is called weakly parametric vector equilibrium problem (WPVEP). Also one can consider Theorem 3.1 is an improvement of Theorem 3.6 of [24] for single valued mappings.

A vector $x \in A(\mu)$ is called a solution of WPVEP if,
$f(x, y, \mu) \notin -intC,\ \forall y \in A(\mu)$.
The set of WPVEP solutions is denoted by

$$S_1(\mu) = \{x \in A(\mu) | f(x, y, \mu) \notin -intC,\ \forall y \in A(\mu)\}.$$

By using a similar proof as given for Theorem 3.1 we can establish the following result about the semicontinuity of the solution mapping of WPVEP.

Theorem 3.4. *Suppose that the following condition are satisfied:*
(i) A(.) is continuous with compact values on Λ
(ii) f(.,.,.) is $C-l.s.c$ *on* $B \times B \times \Lambda$
(iii) $\mu \in \Lambda,\ x \in A(\mu) \setminus S_1(\mu)$, *there exists* $y \in S_1(\mu)$ *such that* $f(x, y, \mu) \in -intC \setminus \{0\}$.

Then $S_1(.)$ *is lower semicontinuous on* Λ

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