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A VISCOSITY NONLINEAR MIDPOINT ALGORITHM FOR NONEXPANSIVE SEMIGROUP

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ABSTRACT. In this paper, we propose a viscosity nonlinear midpoint algorithm (VNMA) for finding a solution of fixed point problem for a nonexpansive semigroup in real Hilbert spaces. Under certain conditions control on parameters, the iteration sequences generated by the proposed algorithm are proved to be strongly convergent to a solution of fixed point problem for a nonexpansive semigroup. Some numerical examples are presented to illustrate the convergence result. Our results improve and extend the corresponding results in the literature.

KEYWORDS:Nonexpansive semigroup, Equilibrium problem, Midpoint method, Strongly positive linear bounded operator, Fixed point, Hilbert space.
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1. Introduction

The explicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [2, 3, 4, 7, 8, 14, 17, 16] and the references cited therein. For instance, consider the initial value problem for the differential equation $y^{'}(t) = f(y(t))$ with the initial condition $y(0) = y_0$, where f is a continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$. The explicit midpoint rule which generates a sequence $\{y_n\}$ by following the recurrence relation

$$\frac{1}{h}(y_{n+1} - y_n) = f(\frac{y_{n+1} - y_n}{2}).$$

In 2015, Xu et al. [19] extended and generalized the results of Alghamdi et al. [1] and applied the viscosity method on the midpoint rule for nonexpansive mappings and they give the generalized viscosity explicit method:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}).$$

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In 2016, Rizvi [13] introduced the following iterative method for the explicit midpoint rule of nonexpansive mappings:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n B)T(\frac{x_n + x_{n+1}}{2}).$$

Motivated and inspired by the results mentioned and related literature in [1, 13, 19], we propose an iterative midpoint algorithm based on the viscosity method for finding a common element of the set of solutions of nonexpansive semigroup in Hilbert spaces. Then we prove strong convergence theorems that extend and improve the corresponding results of Rizvi [13], Xu [19], and others. Finally, we give examples and numerical result to illustrate our main result.

The rest of paper is organized as follows. The next section presents some preliminary results. Section 3 is devoted to introduce midpoint algorithm for solving it. The last section presents a numerical example to demonstrate the proposed algorithms.

2. Preliminaries

Let $\mathbb{R}$ denote the set of all real numbers, H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$ and C be a nonempty closed convex subset of H.
A mapping $T : C \to C$ is said to be a contraction if there exists a constant $\alpha \in (0, 1)$ such that $\|T(x) - T(y)\| \leq \alpha \|x - y\|$, for all $x, y \in C$. If $\alpha = 1$ then T is called nonexpansive on C.
The fixed point problem (FPP) for a nonexpansive mapping T is: To find $x \in C$ such that $x \in \text{Fix}(T)$, where $\text{Fix}(T)$ is the fixed point set of the nonexpansive mapping T.
In 2006, Marino and Xu [11] considered the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n$$

with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and prove that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality $\langle (B - \gamma f)z, x - z \rangle \geq 0$, $\forall x \in \text{Fix}(T)$ which is the optimality condition for minimization problem

$$\min_{x \in \text{Fix}(T)} \frac{1}{2} \langle Bx, x \rangle - h(x)$$

where h is the potential function for γf and $B : H \to H$ is a strongly positive linear bounded operator, i.e., if there exists a constant $\bar{\gamma} > 0$ such that $\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2$, $\forall x \in \text{Fix}(T)$.
A family $S := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$
(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$
(iv) For all $x \in C, s \to T(s)x$ is continuous.

Chen and Song [6] introduced and studied the following iterative method to prove a strong convergence theorem for FPP in a real Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) \tfrac{1}{s_n} \textstyle\int_0^{s_n} T(s)x_n ds.$$

where f is a contraction mapping. For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C

is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y\rangle \leq 0 \tag{2.1}$$

Further, it is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq (\frac{1}{2})\|(T(x) - x) - (T(y) - y)\|^2. \tag{2.2}$$

It is also known that H satisfies Opial's condition [12], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.3}$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. [5] *The following inequality holds in real space H:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \qquad \forall x, y \in H.$$

Definition 2.2. A mapping $M : C \to H$ is said to be monotone, if

$$\langle Mx - My, x - y\rangle \geq 0, \qquad \forall x, y \in C.$$

M is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Mx - My, x - y\rangle \geq \alpha\|Mx - My\|^2, \qquad \forall x, y \in C.$$

Lemma 2.3. [11] *Assume that B is a strong positive linear bounded self adjoint operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4. [15] *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $S := \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, for each $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(h)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 2.5. [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \delta_n$, $n \geq 0$ where α_n is a sequence in $(0, 1)$ and δ_n is a sequence in $\mathbb{R}$ such that*

(i) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$; *(ii)* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\Sigma_{n=1}^{\infty}\delta_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Viscosity Nonlinear Midpoint Algorithm

In this section, we prove a strong convergence theorem based on the explicit iterative for fixed point of nonexpansive semigroup. We firstly present the following unified algorithm.

Let C be a nonempty closed convex subset of real Hilbert space H. Let $S = \{T(s) : s \in [0, +\infty)\}$ be a nonexpansive semigroup on C such that $\mathrm{Fix}(S) \neq \emptyset$. Also $f : C \to H$ be a α-contraction mapping and A be a strongly positive bounded linear self adjoint operator on H with coefficient $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$ and $\bar{\gamma} \leq \|A\| \leq 1$.

Algorithm 3.1. For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds. \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{s_n\} \subset [s, \infty)$ with $s > 0$ satisfying conditions:

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0,\ \Sigma_{n=1}^{\infty} \alpha_n = \infty$;

$(C2)$ $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$;

$(C3)$ $\lim_{n\to\infty} s_n = \infty,\ \sup_{n\in\mathbb{N}} |s_{n+1} - s_n|$ *is bounded.*

In the next remark, we observe that the iterative Algorithm 3.1 is well defined for all n.

Remark 3.2. For all $t \in (0, \|A\|^{-1})$ and $u \in C$ fixed, the mapping

$$x \mapsto V_t x := t\gamma f(u) + (1 - tA)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{u + x}{2})ds$$

is a contraction with coefficient $\frac{1}{2}(1 - t\overline{\gamma}) \in (0,1)$. This is immediately clear, due to the nonexpansivity semigroup of $S = \{T(s) : s \in [0, +\infty)\}$ and the inequality (2.3). In fact, we have, for all $x, y \in H$,

$$\begin{aligned}
\|V_t x - V_t y\| &= \|t\gamma f(u) + (1 - tA)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{u+x}{2})ds - t\gamma f(u) - (1 - tA)\tfrac{1}{s_n}\int_0^{s_n} T(s)(\frac{u+y}{2})ds\| \\
&\le (1 - t\overline{\gamma})\tfrac{1}{s_n}\textstyle\int_0^{s_n} \|T(s)(\frac{u+x}{2})ds - T(s)(\frac{u+y}{2})ds\| \\
&\le \tfrac{1}{2}(1 - t\overline{\gamma})\|x - y\|.
\end{aligned}$$

Hence the Algorithm 3.1 is well defined. Moreover, V_t has a unique fixed point, denoted by x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma f(u) + (1 - tA)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{u + x_t}{2})ds. \tag{3.2}$$

Lemma 3.3. *Let $p \in Fix(S)$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $p \in \mathrm{Fix}(S)$, we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + (1 - \alpha_n A)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2})ds - p\| \\
&\le \alpha_n \|\gamma f(x_n) - Ap\| + (1 - \alpha_n \bar{\gamma})\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n + x_{n+1}}{2}) - T(s)p\|ds \\
&\le \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\|) + (1 - \alpha_n \bar{\gamma})\|\tfrac{x_n + x_{n+1}}{2} - p\| \\
&\le \alpha_n \gamma \alpha \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| + \tfrac{(1 - \alpha_n \bar{\gamma})}{2}(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned}$$

which implies that

$$\tfrac{1 + \alpha_n \bar{\gamma}}{2}\|x_{n+1} - p\| \le (\alpha_n \gamma \alpha + \tfrac{1 - \alpha_n \bar{\gamma}}{2})\|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|.$$

Then

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\tfrac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1+\alpha_n\bar{\gamma}})\|x_n-p\| + \tfrac{2\alpha_n(\bar{\gamma}-\gamma\alpha)}{1+\alpha_n\bar{\gamma}}\tfrac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha} \\ &\le \max\{\|x_n-p\|, \tfrac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\} \\ &\vdots \\ &\le \max\{\|x_0-p\|, \tfrac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\}.\end{aligned} \tag{3.3}$$

Hence $\{x_n\}$ is bounded. □

Now, set $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds$. Then $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.4. *The following properties are satisfying for the Algorithm 3.1*

$P1.$ $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$

$P2.$ $\lim_{n\to\infty}\|x_n-t_n\| = 0.$

$P3.$ $\lim_{n\to\infty}\|T(s)t_n-t_n\| = 0.$

Proof. P1: Let $p \in Fix(S)$, we have,

$$\begin{aligned}&\|t_{n+1}-t_n\| \\ &= \|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2})ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\| \\ &= \|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)(\tfrac{x_{n+1}+x_{n+2}}{2}) - T(s)(\tfrac{x_n+x_{n+1}}{2}))ds + (\tfrac{1}{s_{n+1}}-\tfrac{1}{s_n})\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds \\ &\quad + \tfrac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}} T(s)(\tfrac{x_n+x_{n+1}}{2})ds\| \\ &= \|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}(T(s)(\tfrac{x_{n+1}+x_{n+2}}{2}) - T(s)(\tfrac{x_n+x_{n+1}}{2}))ds \\ &\quad + (\tfrac{1}{s_{n+1}}-\tfrac{1}{s_n})\int_0^{s_n}(T(s)(\tfrac{x_n+x_{n+1}}{2}) - T(s)p)ds + \tfrac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}(T(s)(\tfrac{x_n+x_{n+1}}{2}) - T(s)p)ds\| \\ &\le \|\tfrac{x_{n+1}+x_{n+2}}{2} - \tfrac{x_n+x_{n+1}}{2}\| + \tfrac{|s_{n+1}-s_n|s_n}{s_{n+1}s_n}\|\tfrac{x_n+x_{n+1}}{2}-p\| + \tfrac{|s_{n+1}-s_n|}{s_{n+1}}\|\tfrac{x_n+x_{n+1}}{2}-p\| \\ &\le \tfrac{1}{2}(\|x_{n+1}-x_n\| + \|x_{n+2}-x_{n+1}\|) + \tfrac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\| + \|x_{n+1}-p\|).\end{aligned} \tag{3.4}$$

Next, we show that the sequence $\{x_n\}$ is asymptotically regular, i.e., $\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\| = 0$.
By (3.4) we estimate that

$$
\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&= \|(\alpha_{n+1}\gamma f(x_{n+1}) + (1-\alpha_{n+1}A)\tfrac{1}{s_{n+1}}\textstyle\int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2})ds) \\
&\quad -(\alpha_n\gamma f(x_n) + (1-\alpha_n A)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds)\| \\
&= \|(1-\alpha_{n+1}A)(\tfrac{1}{s_{n+1}}\textstyle\int_0^{s_{n+1}} T(s)(\tfrac{x_{n+1}+x_{n+2}}{2})ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds) \\
&\quad +(\alpha_n A - \alpha_{n+1}A)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\tfrac{x_n+x_{n+1}}{2})ds + (\alpha_{n+1}-\alpha_n)\gamma f(x_n) \\
&\quad +\alpha_{n+1}(\gamma f(x_{n+1}) - \gamma f(x_n))\| \\
&\le (1-\alpha_{n+1}\bar{\gamma})\|t_{n+1}-t_n\| + M|\alpha_n-\alpha_{n+1}| + \alpha_{n+1}\gamma\|f(x_{n+1})-f(x_n)\| \\
&\le (1-\alpha_{n+1}\bar{\gamma})\|t_{n+1}-t_n\| + M|\alpha_n-\alpha_{n+1}| + \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\| \\
&\le \tfrac{1-\alpha_{n+1}\bar{\gamma}}{2}(\|x_{n+1}-x_n\| + \|x_{n+2}-x_{n+1}\|) + (1-\alpha_{n+1}\bar{\gamma})\tfrac{|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\| \\
&\quad +\|x_{n+1}-p\|) + M|\alpha_n-\alpha_{n+1}| + \alpha_{n+1}\gamma\alpha\|x_{n+1}-x_n\|,
\end{aligned}
$$

where $M := \sup\{\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds + \gamma\|f(x_n)\|\}$.

Then

$$
\begin{aligned}
(1+\alpha_{n+1}\bar{\gamma})\|x_{n+2}-x_{n+1}\| &\le (1+(2\alpha\gamma-\bar{\gamma})\alpha_{n+1})\|x_{n+1}-x_n\| \\
&\quad +(1-\alpha_{n+1}\bar{\gamma})\tfrac{2|s_{n+1}-s_n|}{s_{n+1}}(\|x_n-p\| + \|x_{n+1}-p\|) \\
&\quad +2M|\alpha_n-\alpha_{n+1}|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\le (1-\tfrac{2(\bar{\gamma}-\alpha\gamma)\alpha_{n+1}}{1+\alpha_{n+1}\bar{\gamma}})\|x_{n+1}-x_n\| + (\tfrac{1-\alpha_{n+1}\bar{\gamma}}{1+\alpha_{n+1}\bar{\gamma}})(\tfrac{2|s_{n+1}-s_n|}{s_{n+1}})(\|x_n-p\| \\
&\quad +\|x_{n+1}-p\|) + \tfrac{2M}{1+\alpha_{n+1}\bar{\gamma}}|\alpha_n-\alpha_{n+1}|.
\end{aligned}
$$

Hence, it follows by Lemma 2.5 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.5}$$

And similarly, we have

$$\lim_{n\to\infty}\|x_{n+2}-x_{n+1}\| = 0. \tag{3.6}$$

Also by (3.4), (3.5),(3.6) and (C3) we have $\lim_{n\to\infty}\|t_{n+1}-t_n\| = 0$.

P2: We can write

$$
\begin{aligned}
\|x_n-t_n\| &\le \|x_{n+1}-x_n\| + \|\alpha_n\gamma f(x_n) + (1-\alpha_n A)t_n - t_n\| \\
&\le \|x_n-x_{n+1}\| + \alpha_n\|\gamma f(x_n) - At_n\|.
\end{aligned}
$$

By (C1) and (3.5), we obtain

$$\lim_{n\to\infty}\|x_n-t_n\| = 0. \tag{3.7}$$

P3: Let $K := \{w \in C : \|w - p\| \leq \|x_0 - p\|, \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(p) - Bp\|\}$. Then K is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s \in [0, +\infty)$ and contains $\{x_n\}$. So, without loss of generality, we may assume that $S := \{T(s) : s \in [0, +\infty)\}$ is a nonexpansive semigroup on K.

$$\begin{aligned}
\|T(s)x_n - x_n\| &= \|T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds + T(s)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds \\
&\quad -\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds + \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\leq \|T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\leq \|x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\| \\
&\quad +\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&= 2\|\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - x_n\| \\
&\quad +\|T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds - \frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds\|
\end{aligned}$$

Since $\frac{x_n+x_{n+1}}{2} \in C$, from (3.7) and Lemma 2.4, we obtain $\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0$.
Therefore

$$\begin{aligned}
\|T(s)t_n - t_n\| &\leq \|T(s)t_n - T(s)x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\| \\
&\leq \|t_n - x_n\| + \|T(s)x_n - x_n\| + \|x_n - t_n\|.
\end{aligned}$$

Then we have $\lim_{n\to\infty} \|T(s)t_n - t_n\| = 0$. □

4. **Convergence Algorithm**

Theorem 4.1. *The Algorithm defined by (3.1) convergence strongly to $z \in Fix(S)$, which is a unique solution in of the variational inequality* $\langle(\gamma f - A)z, y - z\rangle \leq 0, \quad \forall y \in Fix(S)$.

Proof. Let $s = P_{\mathrm{Fix}(S)}$. We get

$$\begin{aligned}
\|s(I - A + \gamma f)(x) - s(I - A + \gamma f)(y)\| &\leq \|(I - A + \gamma f)(x) - (I - A + \gamma f)(y)\| \\
&\leq \|I - A\|\|x - y\| + \gamma\|f(x) - f(y)\| \\
&\leq (1 - \bar{\gamma})\|x - y\| + \gamma\alpha\|x - y\| \\
&= (1 - (\bar{\gamma} - \gamma\alpha))\|x - y\|.
\end{aligned}$$

Then $s(I - A + \gamma f)$ is a contraction mapping from H into itself. Therefore by Banach contraction principle, there exists $z \in H$ such that $z = s(I - A + \gamma f)z =$

$P_{\mathrm{Fix}(S)}(I - A + \gamma f)z$.
We show that $\langle(\gamma f - A)z, x_n - z\rangle \leq 0$. To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)z, t_n - z\rangle = \lim_{i\to\infty}\langle(\gamma f - A)z, t_{n_i} - z\rangle. \tag{4.1}$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\} \subseteq K$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $t_{n_i} \rightharpoonup w$. Now, we prove that $w \in \mathrm{Fix}(S)$. Assume that $w \notin \mathrm{Fix}(S)$. Since $t_{n_i} \rightharpoonup w$ and $T(s)w \neq w$, from Opial's conditions (2.3) and Lemma 3.4 (P3), we have

$$\begin{aligned}
\liminf_{i\to\infty}\|t_{n_i} - w\| &< \liminf_{i\to\infty}\|t_{n_i} - T(s)w\| \\
&\leq \liminf_{i\to\infty}(\|t_{n_i} - T(s)t_{n_i}\| + \|T(s)t_{n_i} - T(s)w\|) \\
&\leq \liminf_{i\to\infty}\|t_{n_i} - w\|,
\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in \mathrm{Fix}(S)$. Now from (2.1), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle &= \limsup_{n\to\infty}\langle(\gamma f - A)z, t_n - z\rangle \\
&\leq \limsup_{i\to\infty}\langle(\gamma f - A)z, t_{n_i} - z\rangle \\
&= \langle(\gamma f - A)z, w - z\rangle \\
&\leq 0.
\end{aligned} \tag{4.2}$$

Now we prove that x_n is strongly convergence to z.

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \alpha_n\langle\gamma f(x_n) - Az, x_{n+1} - z\rangle + \langle(1 - \alpha_n A)(t_n - z), x_{n+1} - z\rangle \\
&\leq \alpha_n(\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + \langle\gamma f(z) - Az, x_{n+1} - z\rangle) \\
&\quad +\|1 - \alpha_n A\|\|t_n - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\alpha\gamma\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\quad +(1 - \alpha_n\bar{\gamma})\|\tfrac{x_n + x_{n+1}}{2} - z\|\|x_{n+1} - z\| \\
&\leq \alpha_n\alpha\gamma\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\quad +\tfrac{1-\alpha_n\bar{\gamma}}{2}(\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - z\| \\
&= \tfrac{1-\alpha_n\bar{\gamma}+2\alpha_n\alpha\gamma}{2}\|x_n - z\|\|x_{n+1} - z\| + \tfrac{1-\alpha_n\bar{\gamma}}{2}\|x_{n+1} - z\|^2 \\
&\quad +\alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\leq \tfrac{1-\alpha_n(\bar{\gamma}-2\alpha\gamma)}{4}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \tfrac{1-\alpha_n\bar{\gamma}}{2}\|x_{n+1} - z\|^2 \\
&\quad +\alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\
&\leq \tfrac{1-\alpha_n(\bar{\gamma}-2\alpha\gamma)}{4}\|x_n - z\|^2 + \tfrac{3}{4}\|x_{n+1} - z\|^2 + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle.
\end{aligned}$$

This implies that

$$4\|x_{n+1} - z\|^2 \leq (1 - \alpha_n(\bar{\gamma} - 2\alpha\gamma))\|x_n - z\|^2 + 3\|x_{n+1} - z\|^2 + 4\alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle.$$

Then

$$\begin{aligned} \|x_{n+1}-z\|^2 &\leq (1-\alpha_n(\bar{\gamma}-2\alpha\gamma))\|x_n-z\|^2+4\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle \\ &= (1-l_n)\|x_n-z\|^2+4\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle, \end{aligned} \tag{4.3}$$

where $l_n = \alpha_n(\bar{\gamma}-2\alpha\gamma)$.
Since $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty} l_n = 0$, $\Sigma_{n=0}^{\infty} l_n = \infty$. Hence, from (4.2) and (4.3) and Lemma 2.5, we deduce that $x_n \to z$, where $z = P_\Theta(I-A+\gamma f)z$. □

5. Numerical Examples

In this section, we give some examples and numerical results for supporting our main theorem. All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory

Example 5.1. *Consider a Fredholm integral equation of the following form*

$$x(t) = g(t) + \int_0^t F(t,k,x(k))dk, t \in [0,1], \tag{5.1}$$

where g *is a continuous function on* $[0,1]$ *and* $F : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous and satisfies the following condition*

$$|F(t,k,x) - F(t,k,y)| \leq |x-y|, \qquad \forall t,s \in [0,1], \quad x,y \in \mathbb{R},$$

then equation (5.1) has at least one solution in $L^2[0,1]$ *(see [9]).*
Define a mapping $T(s) : L^2[0,1] \to L^2[0,1]$ *by*

$$(T(s)x)(t) = e^{-2s}(g(t) + \int_0^t F(t,k,x(k))\ dk, \qquad t \in [0,1].$$

It is easy to observe that $S = \{T(s) : s \in [0,+\infty)\}$ *is nonexpansive semigroup. In fact, we have, for* $x, y \in L^2[0,1]$,

$$\begin{aligned} \|T(s)x - T(s)y\|^2 &= \textstyle\int_0^1 |(T(s)x)(t) - (T(s)y)(t)|^2\ dt \\ &= \textstyle\int_0^1 |e^{-2s}\int_0^1 (F(t,k,x(k)) - F(t,k,y(k)))\ dk|^2\ dt \\ &\leq \textstyle\int_0^1 (\int_0^1 |x(k)-y(k)|^2\ dk)\ dt \\ &= \textstyle\int_0^1 |x(k)-y(k)|^2\ dk \\ &= \|x-y\|^2. \end{aligned}$$

This means that to find the solution of integral equation (5.1) is reduced to find a fixed point of the nonexpansive semigroup S *in* $L^2[0,1]$.
For any given function $x_0 \in L^2[0,1]$, *define a sequence of functions* x_n *in* $L^2[0,1]$ *by*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1-\alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)(\frac{x_n+x_{n+1}}{2})ds$$

satisfying the conditions of Algorithm 3.1. Then the sequence $\{x_n\}$ *converges strongly in* $L^2[0,1]$ *to the solution of integral equation (5.1) which is also a solution of the following variational inequality*

$$\langle(\gamma f - A)z, y-z\rangle \leq 0, \quad \forall y \in Fix(S).$$

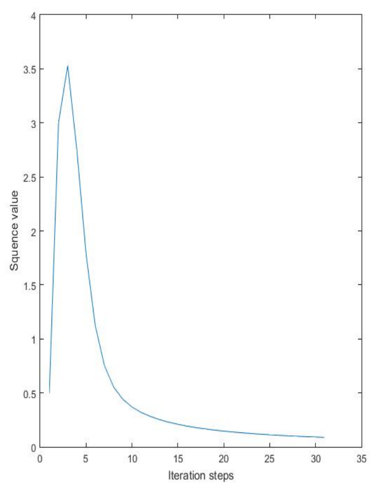
Example 5.2. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [-3, 0]$; Let $f(x) = \frac{1}{5}(x+2)$, $A(x) = \frac{1}{3}x$ and let, for each $x \in C$, $T(s)x = \frac{1}{1+3s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ generated by the iterative scheme*

$$x_{n+1} = \frac{6}{5}(x_n + 2) + (1 - \frac{3}{n}A)\frac{1}{s_n}\int_0^{s_n} \frac{1}{1+3s}(\frac{x_n + x_{n+1}}{2})ds \tag{5.2}$$

where $\alpha_n = \frac{3}{n}$ and $s_n = n$. Then $\{x_n\}$ converges to $\{0\} \in Fix(S)$. f is contraction mapping with constant $\alpha = \frac{1}{3}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma} = 1$ on C. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $Fix(S) = \{0\} \neq \emptyset$. After simplification, scheme (5.2) reduce to

$$x_{n+1} = \frac{\frac{12}{5n} + \frac{1}{n}(\frac{6}{5} + \frac{1}{6}(1 - \frac{1}{n})\ln(1+3n))x_n}{1 - \frac{1}{6n}(1 - \frac{1}{n})\ln(1+3n)}.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$ converges strongly to $w = \{0\} \in Fix(S)$.



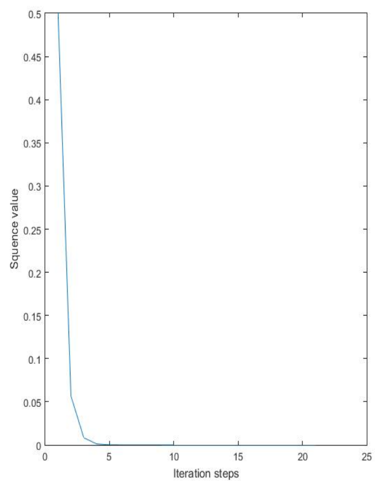
Example 5.3. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [0, 2]$; Let $f(x) = \frac{1}{8}x$, $A(x) = 2x$ and let, for each $x \in C$, $T(s)x = e^{-2s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$ generated by the iterative scheme*

$$x_{n+1} = \frac{1}{4\sqrt{n}}x_n + (1 - \frac{1}{\sqrt{n}}A)\frac{1}{s_n}\int_0^{s_n} e^{-2s}(\frac{x_n + x_{n+1}}{2})ds \tag{5.3}$$

where $\alpha_n = \frac{1}{\sqrt{n}}$ and $s_n = 2n$. Then $\{x_n\}$ converges to $\{0\} \in Fix(S)$. f is contraction mapping with constant $\alpha = \frac{1}{5}$ and A is a strongly positive bounded linear operator with constant $\bar{\gamma} = 1$ on C. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $Fix(S) = \{0\} \neq \emptyset$. After simplification, scheme (5.3) reduce to

$$x_{n+1} = \frac{\frac{1}{4\sqrt{n}} - \frac{1}{8n}(1 - \frac{2}{\sqrt{n}})(e^{-4n} - 1)}{1 + \frac{1}{8n}(1 - \frac{2}{\sqrt{n}})(e^{-4n} - 1)}x_n.$$

Following the proof of Theorem 4.1, we obtain that $\{x_n\}$ *converges strongly to* $w = \{0\} \in Fix(S)$.



6. Conculsion

We have proposed a viscosity nonlinear midpoint algorithm (VNMA) in real Hilbert spaces. The strong convergence of iteration sequence generated by the algorithm to a solution of VNMA is obtained. Some numerical examples are also provided to illustrate the convergence of proposed algorithm.

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