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SHRINKING PROJECTION METHOD WITH ALLOWABLE RANGES

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ABSTRACT. In this note, we introduce an iterative method with allowable ranges which is a revised version of the shrinking projection method. Allowable ranges associated with the method are related to errors caused by a corresponding numerical calculation method.

KEYWORDS: Shrinking projection method, allowable ranges, errors.
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1. Introduction

In 2008, Takahashi, Takeuchi and Kubota [10] introduced an iterative method finding a common fixed point of some families of nonlinear mappings in Hilbert spaces. In 2009, Kimura and Takahashi [5] improved the method. Before them, there were some results in Banach spaces; see [5] and its references. Nevertheless, by reviewing the structure, they improved the method itself. In their direction, we can deal with wider classes of mappings in wider spaces. Typically, we can apply the method to find a common fixed point of a family of mappings of Type P in suitable Banach spaces. This iterative method is called shrinking projection method. Also, in 2014, Kimura [4] considered the method with non–summable errors.

Let C, Q be closed convex subsets of a Banach space with $Q \subset C$. For simplicity, consider a method which generate $v = n_{xt}(w) \in C$ from $w \in C$ theoretically. So, for $x_1 \in C$, we can generate $\{x_n\}$ in theory, where $x_{n+1} = n_{xt}(x_n)$. Then, $\{x_n\}$ is required to converge strongly to a point of Q. Also, consider a corresponding numerical calculation procedure. Let $x_1 = z_1 = y_1$ and generate $z_2 = n_{xt}(y_1) \in C$. By actual restrictions, usually we can only have $y_2 \in C$ which is slightly different from z_2. Generate $z_3 = n_{xt}(y_2)$. In this way, practically, we can only have $\{y_n\}$; $\{z_n\}$ is also in theory. For step n, we call $\|z_n - y_n\|$ error. We may consider that there are $\{b_n\} \subset (0, \infty)$ and $M \in (0, \infty)$ satisfying $\|z_n - y_n\| \leq b_n \leq M$ for $n \in N$.

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As a matter of course we face a difficulty: We do not know the size of $\|x_n - y_n\|$, that is, we do not know whether $\{y_n\}$ converges. Some researchers studied the cases as below: Assuming $\sum_{n\in N} b_n < \infty$, "Strong convergence of $\{y_n\}$ to a point of Q" is guaranteed. However, errors can satisfy neither $\sum_{n\in N} b_n < \infty$ nor $\lim_n b_n = 0$. Recently, avoiding the conditions, some replacements of "Strong convergence of $\{y_n\}$ to a point of Q" are studied. So, maybe handling of errors is unsatisfactory.

On the other hand, an allowable range A_n for step n is a subset of C associated with such a method. Then, $\{y_n\}$ is required to converge strongly to some $u \in Q$ theoretically if $y_n \in A_n$ for $n \in N$. Suppose, by actual restrictions, we cannot get a point $y_{n_0+1} \in A_{n_0+1}$. Then, our procedure has to be stopped. Nevertheless, for the method, maybe y_{n_0} is a best approximate point of u even if $\|y_{n_0} - u\|$ is unknown.

In this note, motivated by the works as above, we present a shrinking projection method which has an allowable range for each step. In a sense, we give another interpretation of Kimura's idea [4]. To clarify basic structures of our method, we only deal with mappings related to Type P, and present only typical and basic applications. The concept of allowable ranges is not bounded by an iterative method.

2. Preliminaries

For details of this section, consult Takahashi [9] and Aoyama and co–authors [2]. In the sequel, without notice, sometimes we use the facts and symbols below.

N and R denote sets of positive integers and real numbers, respectively. For $k \in N$, N_k denotes $\{j \in N : 1 \le j \le k\}$. E denotes a real Banach space with norm $\|\cdot\|$, and E^* denotes the dual of E. C always denotes a non–empty set; in this note, normally "non–empty" is omitted.

Let E be a Banach space. The normalized duality mapping J is the set valued mapping from E into E^* as below:

$$Jx = \{x^* \in E : \langle x, x^*\rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|\} \quad \text{for} \quad x \in E.$$

Let C be a subset of E. Then, C is weakly closed if C is closed and convex. Let T be a mapping from C into E. $F(T)$ denotes $\{x \in C : x = Tx\}$, that is, $F(T)$ is the fixed point set of T. T is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for $x, y \in C$.

In the canonical way, E is embedded in E^{**}; we may consider E as a subset of E^{**}. E is called reflexive if the embedding of E is E^{**}. In this case, we may consider $E = E^{**}$. So, weak topology and weak* topology of E^* are coincide; we only use "weak topology". E is called strictly convex if $\|\cdot\|^2$ is strictly convex, that is, for $x, y \in E$ with $x \ne y$ and $a \in (0,1)$, $\|(1-a)x + ay\|^2 < (1-a)\|x\|^2 + a\|y\|^2$. E is called smooth if $\lim_{t\longrightarrow 0}(\|x+ty\| - \|x\|)/t$ exists for $x, y \in E$ with $\|x\| = \|y\| = 1$. E is said to have the Kadec–Klee property if a sequence $\{x_n\}$ in E converges strongly to $x \in E$ whenever $\{x_n\}$ converges weakly to x and $\{\|x_n\|\}$ converges to $\|x\|$.

Let E be reflexive. Then, any bounded sequence $\{x_n\}$ in E has a weakly convergent subsequence. A sequence $\{x_n\}$ in E converges weakly to $z \in E$ if every weak cluster point of $\{x_n\}$ and z are the same. Let E be a strictly convex reflexive Banach space and let C be a closed convex subset of E. Then, for $x \in E$, there is a unique $z_x \in C$ satisfying $\|x - z_x\| = \inf_{z\in C}\|x - z\|$. Define a mapping P_C from E onto C by $P_C x = z_x$ for $x \in E$. P_C is called the metric projection from E onto C. We know the following: $z = P_C x$ if and only if $z \in C$ and $\inf_{y\in C}\langle y - z, J(z-x)\rangle \ge 0$. So, $\inf_{y\in C}\langle y - P_C x, J(P_C x - x)\rangle \ge 0$ holds.

In this note, we mainly deal with smooth strictly convex reflexive Banach spaces. Let E be such a Banach space. Then, we refer to some basic concepts and facts needed in the sequel; of course, some assertions hold under more weak conditions.

In the setting, Jx is singleton for $x \in E$. So, we can regard J as a mapping from E into E^*. Of course, for $x \in E$, $y^* \in E^*$ and $z^{**} \in E^{**}$, $\langle x, y^* \rangle$ and $\langle x^{**}, y^* \rangle$ denote $y^*(x)$ and $z^{**}(y^*)$. Define a mapping ϕ by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$. ϕ is called Alber's bi–function [1] from $E \times E$ into R. We denote by ϕ^* Alber's bi–function from $E^* \times E^*$ into R. Let A be a mapping from a subset C of E into E^*. A is called monotone if $\langle x - y, Ax - Ay\rangle \geq 0$ for $x, y \in C$. A is called strictly monotone if A is monotone and $\langle x - y, Ax - Ay\rangle = 0$ implies $x = y$.

In the setting, the following hold:

(1) E^* is smooth, strictly convex and reflexive.
(2) J is a bijection from E onto E^*.
(3) J is norm to weak continuous.
(4) The normalized duality mapping J^* from E^* onto E and J^{-1} are coincide.
(5) For $y \in E$, $\langle \cdot, Jy\rangle$ is continuous and linear.
(6) For $x, y \in E$, the following hold: $\phi(x,y) = \phi^*(Jy, Jx) \geq (\|x\| - \|y\|)^2 \geq 0$, $\langle x - y, Jy\rangle \leq \frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2 \leq \langle x - y, Jx\rangle$, $\langle x - y, Jx - Jy\rangle = \frac{1}{2}\phi(y,x) + \frac{1}{2}\phi(x,y) \geq (\|x\| - \|y\|)^2 \geq 0$.
(7) For $y \in E$, $\phi(\cdot, y)$ is weakly lower semi–continuous and strictly convex.
(8) For $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.
(9) Suppose a sequence $\{x_n\}$ in E satisfies $\lim_n \langle x_n - y, Jx_n - Jy\rangle = 0$. Then, $\lim_n \|x_n\| = \|y\| = \lim_n \|Jx_n\| = \|Jy\|$, $\lim_n \phi(y, x_n) = \lim_n \phi(x_n, y) = \lim_n \phi^*(Jy, Jx_n) = \lim_n \phi^*(Jx_n, Jy) = 0$.
(10) J is strictly monotone.

We give short explanations of (6)–(10). Fix any $x, y \in E$. By the definitions of J, ϕ and ϕ^*, obviously $\|x\| = \|Jx\|$ and $\phi(x,y) = \phi^*(Jy, Jx)$ hold. Also, we see $\phi(x,y) \geq (\|x\| - \|y\|)^2 \geq 0$ by $-\langle x, Jy\rangle \geq -\|x\|\|y\|$. Then,

$$\langle x - y, Jx\rangle - \tfrac{1}{2}(\|x\|^2 - \|y\|^2) = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \langle y, Jx\rangle = \tfrac{1}{2}\phi(y,x).$$

From these, the following immediately follow: For $x, y \in E$,

$$\langle x - y, Jy\rangle \leq \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}\|y\|^2 \leq \langle x - y, Jx\rangle,$$
$$\langle x - y, Jx - Jy\rangle = \tfrac{1}{2}\phi(y,x) + \tfrac{1}{2}\phi(x,y) \geq (\|x\| - \|y\|)^2 \geq 0.$$

Since $\|\cdot\|^2$ is weakly lower semi–continuous and strictly convex, by (5), so is $\phi(\cdot, y)$. Suppose $\phi(x,y) = 0$ and $x \neq y$. Then, $0 \leq \phi(\frac{1}{2}(x+y), y) < \frac{1}{2}\phi(x,y) + \frac{1}{2}\phi(y,y) = 0$. So, $\phi(x,y) = 0$ implies $x = y$. We confirmed that (6)–(8) hold. By (8) and the last inequality in (6), we immediately see that (9) and (10) hold.

Some classes of mappings.

Let C be a subset of a smooth strictly convex reflexive Banach space E. We denote by $\mathcal{F}^C_{E^*}$ the class of all mappings from C into E^*, by $\mathcal{F}^C$ the class of all mappings from C into E. Consider the following:

$$\mathcal{M}^C_{E^*} = \{A \in \mathcal{F}^C_{E^*} : A \text{ is norm to weak continuous and monotone}\},$$
$$\mathcal{M}^C_P = \{S \in \mathcal{F}^C : J(I - S) \text{ is norm to weak continuous and monotone}\},$$
$$\mathcal{M}^C_R = \{U \in \mathcal{F}^C : JU \text{ is norm to weak continuous and monotone}\},$$
$$\mathcal{T}^C_P = \{S \in \mathcal{F}^C : \langle Sx - Sy, J(x - Sx) - J(y - Sy)\rangle \geq 0 \quad \text{for} \quad x, y \in C\},$$
$$\mathcal{T}^C_R = \{U \in \mathcal{F}^C : \langle (x - Ux) - (y - Uy), JUx - JUy\rangle \geq 0 \quad \text{for} \quad x, y \in C\}.$$

$\mathcal{T}^C_P$ and $\mathcal{T}^C_R$ are called Type P and Type R, respectively. For details of Type P, Type Q, and Type R, see Aoyama and co–authors [2]. In a Hilbert space, T is called firmly nonexpansive if $\langle (x-y)-(Tx-Ty), Tx-Ty\rangle \geq 0$ for $x, y \in C$. In this setting, these four classes are coincide. However, in our setting, the difference in mathematical properties between Type P and Type Q is not small. In a sense, Type P and Type R are dual each other; see (11). Then, we do not deal with Type Q.

Following [2], by considering (1)–(10), we can confirm (11)–(14) below.

(11) $S \in \mathcal{T}_P^C$ if and only if $U = I - S \in \mathcal{T}_R^C$.
$S \in \mathcal{M}_P^C$ if and only if $U = I - S \in \mathcal{M}_R^C$.

(12) $\mathcal{T}_R^C \subset \mathcal{M}_R^C$; JU is norm to weak continuous and monotone for $U \in \mathcal{T}_R^C$.
$\mathcal{T}_P^C \subset \mathcal{M}_P^C$; $J(I-S)$ is norm to weak continuous and monotone for $S \in \mathcal{T}_P^C$.

Suppose further that E has the Kadec–Klee property. Then, the following holds:

(13) U is continuous if $U \in \mathcal{T}_R^C$, and S is continuous if $S \in \mathcal{T}_P^C$.

Suppose E^* has the Kadec–Klee property. Then, the following holds:

(14) JU is norm to norm continuous if $U \in \mathcal{T}_R^C$.
$J(I-S)$ is norm to norm continuous if $S \in \mathcal{T}_P^C$.

For (11), we only show the following: Let $x, y \in C$, $S \in \mathcal{F}^C$ and $U = I - S$. Then,

$$\langle Sx - Sy, J(I-S)x - J(I-S)y\rangle = \langle (I-U)x - (I-U)y, JUx - JUy\rangle.$$

To reduce the burden of readers, we confirm that (12)–(14) hold.

Let $U \in \mathcal{T}_R^C$. Fix any $x, y \in C$. Since J is monotone, by (6), we see

$$\begin{aligned} &\langle x-y, JUx - JUy\rangle - \langle (x-Ux)-(y-Uy), JUx-JUy\rangle \\ &\quad = \langle Ux - Uy, JUx - JUy\rangle \geq 0. \end{aligned} \tag{2.1}$$

Then, by $U \in \mathcal{T}_R^C$, we see $\langle x-y, JUx-JUy\rangle \geq 0$. So, JU is monotone.

Let $\{x_n\}$ be a sequence in C converging strongly to $u \in C$. For $n \in N$, set $a_n = \|Ux_n\| + \|Uu\| = \|JUx_n\| + \|JUu\|$. By $U \in \mathcal{T}_R^C$, (6) and (2.1), we see

$$\begin{aligned} \|x_n - u\|(\|Ux_n\| + \|Uu\|) &\geq \|x_n - u\|\|JUx_n - JUu\| \geq \langle x_n - u, JUx_n - JUu\rangle \\ &\geq \langle Ux_n - Uu, JUx_n - JUu\rangle \geq (\|Ux_n\| - \|Uu\|)^2 \geq 0. \end{aligned}$$

In the case of $Uu = 0$, we immediately see $\|Ux_n\|^2 \leq \|x_n - u\|\|Ux_n\|$. Then $\{Ux_n\}$ converges strongly to $0 = Uu$ and $\{JUx_n\}$ converges strongly to $0 = JUu$.

In the case of $Uu \neq 0$, by $a_n \geq \|Uu\| > 0$ and the inequality as above, we see

$$\begin{aligned} \|x_n - u\| &\geq \tfrac{1}{a_n}\langle Ux_n - Uu, JUx_n - JUu\rangle \\ &\geq \tfrac{1}{a_n}(\|Ux_n\| - \|Uu\|)^2 = \tfrac{1}{a_n}(\|Ux_n\| + \|Uu\| - 2\|Uu\|)^2 \\ &= a_n\left(1 - \tfrac{2\|Uu\|}{a_n}\right)^2 \geq \|Uu\|\left(1 - \tfrac{2\|Uu\|}{a_n}\right)^2 \geq 0. \end{aligned}$$

So, $\{a_n\}$ must converge to $2\|Uu\| > 0$, and $\lim_n\langle Ux_n - Uu, JUx_n - JUu\rangle = 0$. Then, by (9), we also see $\lim_n \|Ux_n\| = \|Uu\| = \lim_n \|JUx_n\| = \|JUu\|$, and

$$\lim_n \phi(Ux_n, Uu) = \lim_n \phi^*(JUx_n, JUu) = 0.$$

Since $\{Ux_n\}$ is bounded, $\{Ux_n\}$ has a weakly convergent subsequence. Let $\{Ux_{n_j}\}$ be a subsequence of $\{Ux_n\}$ which converges weakly to some $v \in E$. Then, since $\phi(\,\cdot, Uu)$ is weakly lower semi–continuous, we have

$$0 = \lim_j \phi(Ux_{n_j}, Uu) = \liminf_j \phi(Ux_{n_j}, Uu) \geq \phi(v, Uu).$$

Thus $\phi(v, Uu) = 0$, that is, $v = Uu$. From these, any weakly convergent subsequence of $\{Ux_n\}$ converges weakly to Uu. Then, $\{Ux_n\}$ itself converges weakly to Uu. We confirmed that U is norm to weak continuous. By replacing $\{Ux_n\}$, Uu and ϕ by $\{JUx_n\}$, JUu and ϕ^*, we also see that JU is norm to weak continuous.

Suppose further that E has the Kadec–Klee property. By the argument as above, it is immediate that $\{Ux_n\}$ converges strongly to Uu. Suppose E^* has the Kadec–Klee property. Similarly, we see that $\{JUx_n\}$ converges strongly to JUu. Finally, from the argument so far, by (11), we see that (12)–(14) hold.

Let C be a subset of a smooth strictly convex reflexive Banach space E. For $V \in \mathcal{F}^C$, define A^V and D_x^V as below:

$$A^V = J(I - V), \quad D_x^V = \{y \in C : \langle Vx - y, A^V x\rangle \geq 0\} \text{ for } x \in C.$$

For simplicity, we use A and D_x instead of A^V and D_x^V if it causes no confusion.

Let $S \in \mathcal{T}_P^C$. By the definition of $\mathcal{T}_P^C$, for $x \in C$ and $u \in F(S)$, we easily see $0 \leq \langle Sx - Su, J(x - Sx) - J(u - Su)\rangle = \langle Sx - u, J(x - Sx)\rangle$. Then, $F(S) \subset \cap_{x\in C} D_x$. Suppose $z \in \cap_{x\in C} D_x$. Then, $z \in D_z$ and $-\|z - Sz\|^2 = \langle Sz - z, J(z - Sz)\rangle \geq 0$. So, we see $z \in F(S)$, that is, $\cap_{x\in C} D_x \subset F(S)$ holds.

(15) $F(S) = \cap_{x\in C} D_x$ for $S \in \mathcal{T}_P^C$.

In the case that C is closed and convex, so is D_x. By (15), the following follows:

(16) $F(S)$ is closed and convex for $S \in \mathcal{T}_P^C$.

We use the expression $B \in \mathcal{C}^C$ if $B \in \mathcal{F}^C$ is continuous and $\cap_{x\in C} D_x \neq \emptyset$. Let $B \in \mathcal{C}^C$. Then, since J is norm to weak continuous, $A = J(I - B)$ is also norm to weak continuous. Obviously, $v \in \cap_{x\in C} D_x$ implies $v \in F(B)$. However, in general, $u \in F(B)$ does not imply $u \in \cap_{x\in C} D_x$. Suppose further that E has the Kadec–Klee property. In this case, by (13) and (15), $S \in \mathcal{T}_P^C$ and $F(S) \neq \emptyset$ imply $S \in \mathcal{C}^C$. On the other hand, it is easy to find C and $B \in \mathcal{C}^C$ satisfying $B \notin \mathcal{T}_P^C$.

Let $C = [0, 1]$. Consider B such that $Bx = x^2$ for $x \in C$. Then, B is continuous and $F(B) = \{0, 1\}$. We know $0 \in \cap_{x\in C} D_x$; $\langle Bx - 0, Ax\rangle = (x^2 - 0)(x - x^2) \geq 0$ for $x \in C$. So, $B \in \mathcal{C}^C$. However, $\langle By - Bz, Ay - Az\rangle = (\frac{1}{4} - 1)((\frac{1}{2} - \frac{1}{4}) - (1 - 1)) < 0$, where $y = 1/2$ and $z = 1$. We see $B \notin \mathcal{T}_P^C$. Similarly, $B \notin \mathcal{M}_P^C$. Confirm that $F(B)$ is not convex, $F(B) \not\subset \cap_{x\in C} D_x$ ($1 \notin \cap_{x\in C} D_x$), and $A = J(I - B)$ is not monotone.

In later sections, we deal with $B \in \mathcal{C}^C$. Then, we refer to the following: Let $B \in \mathcal{C}^C$. Even if $I - B$ is demiclosed at 0, maybe it plays no important roll to find a fixed point of B. Let $a \in [0, 1)$ and $T = aI + (1 - a)B$. Then,

(17) $\langle Tx - u, J(x - Bx)\rangle \geq 0$ for $x \in C$ and $u \in \cap_{x\in C} D_x \subset F(B)$.

It follows from $\langle x - u, J(x - Bx)\rangle - \langle Bx - u, J(x - Bx)\rangle = \|x - Bx\|^2 \geq 0$.

3. Basic structures

The following lemma is the origin of shrinking projection method; see section 5.

Lemma 3.1. *Let E be a strictly convex reflexive Banach space. Let $x_0 \in E$ and let D be a non-empty closed convex subset of E. Let $\{x_n\}$ be a sequence in E satisfying*

$$\limsup_n \|x_0 - x_n\| \leq \|x_0 - P_D x_0\|. \tag{3.1}$$

Then the following hold:

1. *Suppose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to $u \in D$.*
   Then, $u = P_D x_0$; $\{x_{n_j}\}$ converges weakly to $P_D x_0$.
   When E has the Kadec–Klee property, $\{x_{n_j}\}$ converges strongly to $P_D x_0$.
2. *Suppose every weak cluster point of $\{x_n\}$ is a point of D.*
   Then, $\{x_n\}$ converges weakly to $P_D x_0$.
   When E has the Kadec–Klee property, $\{x_n\}$ converges strongly to $P_D x_0$.

Remark. *Of course, we can replace (3.1) by the following:*

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \leq \|x_0 - P_D x_0\| \quad \text{for } n \in N. \tag{3.2}$$

Proof. By (3.1), $\{x_n\}$ is bounded, that is, $\{x_n\}$ has a weakly convergent subsequence. Obviously, every subsequence of $\{x_n\}$ satisfies (3.1).

We show (1). Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in D$, that is, $\{x_0 - x_{n_j}\}$ converges weakly to $x_0 - u$. Since $\{x_{n_j}\}$ satisfies (3.1) and $\|\cdot\|$ is weakly lower semi–continuous, by $u \in D$ and (3.1),

$$\|x_0 - u\| \le \liminf_j \|x_0 - x_{n_j}\| \le \limsup_j \|x_0 - x_{n_j}\| \le \|x_0 - P_D x_0\| \le \|x_0 - u\|.$$

Then, $\|x_0 - u\| = \|x_0 - P_D x_0\| = \lim_j \|x_0 - x_{n_j}\|$. Since $P_D x_0$ is unique, we see $u = P_D x_0$. So, $\{x_{n_j}\}$ converges weakly to $P_D x_0$. Suppose E has the Kadec–Klee property. Then, by the argument as above, we immediately see that $\{x_0 - x_{n_j}\}$ converges strongly to $x_0 - u$. Thus, $\{x_{n_j}\}$ converges strongly to $P_D x_0$.

We show (2). Suppose every weak cluster point of $\{x_n\}$ is a point of D, that is, every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of D. By (1), every weakly convergent subsequence of $\{x_n\}$ converges weakly to $P_D x_0$. Then, $\{x_n\}$ itself converges weakly to $P_D x_0$. Thus, by (1), we see that (2) holds. □

The following lemma expresses a basic structure of our method; it follows from Lemma 3.1. This lemma is closely connected with Tsukada's lemma [11].

Lemma 3.2. *Let E be a strictly convex reflexive Banach space. Let $x_0 \in E$ and let $\{D_n\}$ be a sequence of closed convex subsets of E satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \neq \emptyset$. Let $x_1 = P_{D_1} x_0$. For $n \in N$, define x_{n+1}, K_n and z_n by*

$$x_{n+1} = P_{D_{n+1}} x_0, \quad K_n = \{y \in D_n : \|x_0 - y\| \le \|x_0 - x_{n+1}\|\}, \quad z_n \in K_n.$$

Then $\{x_n\}$ and $\{z_n\}$ converge weakly to $P_D x_0$. Furthermore, when E has the Kadec–Klee property, $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_D x_0$.

Proof. Since $\emptyset \neq D \subset D_{n+1} \subset D_n$ for $n \in N$ and properties of metric projection, we know that $\{x_n\}$ satisfies (3.2). Then, $\{x_n\}$ has a weakly convergent subsequence. Also, every subsequence of $\{x_n\}$ satisfies (3.2). Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $u \in E$. For $m \in N$, since $\{x_{n_j}\}_{n_j \ge m} \subset D_m$ and D_m is weakly closed, $u \in D_m$ is immediate. So, $u \in D$. We confirmed that every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of D.

By $K_n \neq \emptyset$ $(x_n, x_{n+1} \in K_n)$, such $\{z_n\}$ exists. We easily see that, for $n \in N$,

$$\|x_0 - z_n\| \le \|x_0 - x_{n+1}\| \le \|x_0 - z_{n+1}\| \le \|x_0 - x_{n+2}\| \le \|x_0 - P_D x_0\|.$$

Then, $\{z_n\}$ also satisfies (3.2). Note $z_k \in K_k \subset D_k \subset D_m$ for $k \ge m$. So, every weakly convergent subsequence of $\{z_n\}$ converges weakly to a point of D.

From these, by Lemma 3.1 (2), we immediately have the desired results. □

The following lemma expresses another basic structure of our method. Recall, in the setting, $A = J(I - V)$, $D_x = \{y \in C : \langle Vx - y, Ax\rangle \ge 0\}$ for $V \in \mathcal{F}^C$, $x \in C$.

Lemma 3.3. *Let E be a smooth strictly convex reflexive Banach space. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Let $\{y_n\}$ be a sequence in C. Set $D_n = \cap_{j \in N_n} D_{y_j}$ for $n \in N$ and $D = \cap_{n \in N} D_n$. Then, the following hold:*

(1) *For $x \in C$, D_x is non–empty, closed and convex.*
(2) *Each D_n and D are non–empty, closed and convex.*
(3) *Suppose $\{y_n\}$ converges strongly to some $v \in D$. Then, $v \in F(B)$.*

Proof. By $B \in \mathcal{C}^C$, we know $\emptyset \neq \cap_{y \in C} D_y \subset D_x$ for $x \in C$. Since C is closed and convex, by the definition of D_x and properties of dual pair, D_x is closed and convex. From these, (1) and (2) are immediate. Note $D = \cap_{n \in N} D_n = \cap_{n \in N} D_{y_n}$.

We show (3). In this setting, J is norm to weak continuous. Then, since B is continuous, $A = J(I - B)$ is also norm to weak continuous. Since $\{y_n\}$ converges strongly to $v \in D$, we see that $\{Ay_n\}$ converges weakly to Av and $\{Ay_n\}$ is bounded. For $n \in N$, by $v \in D = \cap_{n \in N} D_{y_n} \subset D_{y_n}$, we see $0 \leq \langle By_n - v, Ay_n \rangle$,

$$\langle By_n - v, Ay_n \rangle = \langle By_n - v, Ay_n - Av \rangle + \langle By_n - v, Av \rangle,$$
$$\langle By_n - v, Ay_n - Av \rangle \leq \|By_n - Bv\| \|Ay_n - Av\| + \langle Bv - v, Ay_n - Av \rangle.$$

From these, since $\{y_n\}$ converges strongly to $v \in D$, the following hold:

$$\lim_n \langle By_n - v, Ay_n - Av \rangle = 0, \quad \lim_n \langle By_n - v, Av \rangle = \langle Bv - v, Av \rangle,$$
$$0 \leq \lim_n \langle By_n - v, Ay_n \rangle = \langle Bv - v, Av \rangle.$$

By $A = J(I - B)$, we see $0 \leq \langle Bv - v, J(v - Bv) \rangle = -\|v - Bv\|^2$. Thus $v \in F(B)$. □

Remark 3.4. In Lemma 3.2, for $\{D_n\}$, we only require that $\{D_n\}$ is a sequence of closed convex subsets of E satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $\emptyset \neq D = \cap_n D_n$. Then, the lemma has no relation with method of generating $\{D_n\}$. Also, any subsequence $\{D_{n_k}\}$ of $\{D_n\}$ satisfies $D_{n_{k+1}} \subset D_{n_k}$ for $k \in N$ and $D = \cap_{k \in N} D_{n_k}$. In Lemma 3.3 (3), for $\{y_n\}$, we only require that $\{y_n\}$ converges strongly to some $v \in D$. So, (3) has no relation with method of generating $\{y_n\}$. The importance of these facts were suggested in Kimura and Takahashi [5]. For example, from these properties of $\{D_n\}$ and $\{y_n\}$, we can present Theorem 4.4. Also, for our method, we can confirm that the difference between to find a fixed point of a mapping and to find a common fixed point of a family of such mappings is so slight.

4. Applications

In this section, we present some strong convergence theorems as typical and basic applications of shrinking projection method with allowable ranges.

Theorem 4.1. *Let E be a smooth strictly convex reflexive Banach space which has the Kadec–Klee property. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Consider an iterative procedure as below: Let $x_0 \in E$, $w_1 \in C$, $D_1 = D_{w_1}$ and $x_1 = P_{D_1} x_0$. Let $A_1 = C \backslash (D_1 \cup \{w_1\})$ and let $y_1 \in A_1$. For $n \in N$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$D_{n+1} = D_n \cap D_{y_n}, \quad x_{n+1} = P_{D_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_n : \|x_0 - y\| \leq \|x_0 - x_{n+1}\|, y \neq y_n\}, \quad y_{n+1} \in A_{n+1}.$$

Then, either of the following holds:

(1) *$A_n \neq \emptyset$ for $n \in N$; the procedure is not stopped. In this case, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0 \in F(B)$, where $D = \cap_{n \in N} D_n$.*
(2) *$A_k = \emptyset$ for some $k \in N$; the procedure is stopped. In this case, either $y_{k-1} \in F(B)$ or $w_1 \in F(B)$ holds.*

Proof. Recall Lemma 3.3 (1)–(2). For $x \in C$, by $B \in \mathcal{C}^C$, $\emptyset \neq \cap_{y \in C} D_y \subset D_x$ and D_x is closed and convex. So, C, $\{w_1\}$ and $D_1 = D_{w_1}$ are non–empty closed and convex. Then $x_1 = P_{D_1} x_0$ exists. A_1 may be empty. In the case of $A_1 \neq \emptyset$, we can find $y_1 \in A_1$ and generate D_2, x_2 and A_2; D_2 is nonempty closed and convex. A_2 may be empty. In the case of $A_2 \neq \emptyset$. we can find $y_2 \in A_2$ and continue this process. So, the procedure is stopped when we meet $k \in N$ satisfying $A_k = \emptyset$.

In the case of (1), we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$ and $\{y_n\}$ inductively. By our generating method, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \neq \emptyset$. For $n \in N$, let

K_n be as in Lemma 3.2, that is, $K_n = \{y \in D_n : \|x_0 - y\| \le \|x_0 - x_{n+1}\|\}$. Then, $y_{n+1} \in A_{n+1} \subset K_n$ for $n \in N$. By Lemma 3.2, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0$. Since $\{y_n\}$ converges strongly to $P_D x_0$, by Lemma 3.3 (3), we see $P_D x_0 \in F(B)$. These complete the proof of (1).

We show (2). Assume $A_1 = C \backslash (D_1 \cup \{w_1\}) = \emptyset$. We know that $\{w_1\}$ and D_1 are non–empty closed and convex. Then, since C is connected, by $C = D_1 \cup \{w_1\}$, we see $w_1 \in D_{w_1}$; $-\|Bw_1 - w_1\|^2 = \langle Bw_1 - w_1, Aw_1\rangle \ge 0$. Thus, $w_1 \in F(B)$. Suppose we generated D_{k+1}, x_{k+1} and $A_{k+1} = \emptyset$ for some $k \in N$. Then, D_k and D_{k+1} are non–empty closed and convex. By $K_k = \{y \in D_k : \|x_0 - y\| \le \|x_0 - x_{k+1}\|\}$, we see $x_k, x_{k+1} \in K_k$ and $K_k \ne \emptyset$. By $\emptyset = A_{k+1} = K_k \backslash \{y_k\}$, we see that $y_k \in K_k$ and K_k is singleton. From these, $y_k = x_k = x_{k+1}$ holds. So, by $y_k = x_{k+1} \in D_{k+1} \subset D_{y_k}$, we see $-\|By_k - y_k\|^2 = \langle By_k - y_k, Ay_k\rangle \ge 0$. Thus, $y_k \in F(B)$. □

Remark 4.2. In the setting of Theorem 4.1, neither $F(B) \subset D$ nor the convexity of $F(B)$ are guaranteed. Suppose $B \in \mathcal{T}_P^C$ and $F(B) \ne \emptyset$. Then, by section 2 (15)–(16), $F(B) = \cap_{x \in C} D_x \subset D$, and $F(B)$ is closed and convex. In this case, we can see that $P_D x_0 \in F(B)$ implies $P_D x_0 = P_{F(B)} x_0$. Refer to section 2 (17) and Remark 3.4. So, we can apply the method to have a conventional expression of Theorem 4.1 and to find a common fixed point of some families of mappings.

Consider a corresponding numerical calculation procedure. By considering actual restrictions, we should think that the boundary of D_n and the position of $P_{D_n} x_0$ are obscure. So, $P_{D_n} x_0$ exists only in theory; we cannot generate D_{n+1} from $P_{D_n} x_0$ practically. Then, we considered an allowable range A_n for each $n \in N$. The boundary of A_n is also obscure. However, we may get a practical $y_n \in A_n$ if A_n has a certain size; we can generate D_{n+1} from y_n even if its boundary is obscure.

Recall section 1. We repeat the following: Consider a method finding a point of $Q \subset C$. Let $b_n \in (0, \infty)$ for $n \in N$. It is strange to assume either $\sum_{n \in N} b_n < \infty$ or $\lim_n b_n = 0$ if we regard b_n as an upper bound of error for $n \in N$. Because errors cannot satisfy such conditions. For example, suppose we can consider an allowable range A_n for $n \in N$ such that $\|y - z_n\| \le b_n$ for $y \in A_n$, where z_n is our target. In this case, $\{b_n\}$ may satisfy either $\sum_{n \in N} b_n < \infty$ or $\lim_n b_n = 0$. However, maybe the condition of $\{b_n\}$ is related to effectiveness of the method. The procedure has to be stopped when we cannot have $y_{n_0+1} \in A_{n_0+1}$. Even if $\{y_n\}$ converges strongly to $u \in Q$ theoretically, by actual restrictions, it may be stopped at some step.

The following is a variant of Ibaraki and Kimura's theorem [3]. They considered b_n as an upper bound of error; they mainly studied the case of $\limsup_n b_n < \infty$. With respect to their method, we select the allowable range A_n for $n \in N$.

Theorem 4.3. *Let E be a smooth strictly convex reflexive Banach space which has the Kadec–Klee property. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Let $\{b_n\}$ be a sequence in $(0, \infty)$ satisfying $\lim_n b_n = 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $D_1 = C$ and $x_1 = P_{D_1} x_0$. Let $A_1 = C$ and let $y_1 \in A_1$. For $n \in N$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$D_{n+1} = D_n \cap D_{y_n}, \quad x_{n+1} = P_{D_{n+1}} x_0,$$
$$A_{n+1} = \{y \in D_{n+1} : \|x_{n+1} - y\| \le b_n\}, \quad y_{n+1} \in A_{n+1}.$$

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0 \in F(B)$, where $D = \cap_{n \in N} D_n$.

Proof. Recall Lemma 3.3 (1)–(2). By $B \in \mathcal{C}^C$, inductively, we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$ and $\{y_n\}$. By our generating method, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \ne \emptyset$.

Note $\lim_n \|x_{n+1} - y_{n+1}\| \le \lim_n b_n = 0$. Then, by Lemma 3.2, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0$. Since $\{y_n\}$ converges strongly to $P_D x_0 \in D$, by Lemma 3.3 (3), we see $P_D x_0 \in F(B)$. This completes the proof. $\square$

By taking notice of Remarks 3.4, we can combine Theorems 4.1 and 4.3.

Theorem 4.4. *Let E be a smooth strictly convex reflexive Banach space which has the Kadec–Klee property. Let C be a closed convex subset of E and let $B \in \mathcal{C}^C$. Let $\{b_n\}$ be a sequence in $(0, \infty)$ satisfying $\lim_n b_n = 0$. Consider an iterative procedure as below: Let $x_0 \in E$, $w_1 \in C$, $D_1 = D_{w_1}$ and $x_1 = P_{D_1} x_0$. Let $A_1 = C \backslash (D_1 \cup \{w_1\})$ and let $y_1 \in A_1$. For $n \in N$, generate $D_{n+1}, x_{n+1}, A_{n+1}$ and y_{n+1} by*

$$\begin{aligned} D_{n+1} &= D_n \cap D_{y_n}, \quad x_{n+1} = P_{D_{n+1}} x_0, \\ A^1_{n+1} &= \{y \in D_{n+1} : \|x_{n+1} - y\| \le b_n\}, \\ A^2_{n+1} &= \{y \in D_n : \|x_0 - y\| \le \|x_0 - x_{n+1}\|, y \ne y_n\}, \\ A_{n+1} &= A^1_{n+1} \cup A^2_{n+1}, \quad y_{n+1} \in A_{n+1}. \end{aligned}$$

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_D x_0 \in F(B)$, where $D = \cap_{n \in N} D_n$.
Remark. *We may ignore the case of $A_1 = \emptyset$ because $w_1 \in F(B)$ if $A_1 = \emptyset$.*

Proof. Recall Lemma 3.3 (1)–(2). By $B \in \mathcal{C}^C$, inductively, we can generate sequences $\{D_n\}$, $\{x_n\}$, $\{A_n\}$ and $\{y_n\}$. By our generating method, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$ and $D = \cap_n D_n \ne \emptyset$.

By Lemma 3.2, $\{x_n\}$ converges strongly to $P_D x_0$. By collecting $n \in N$ satisfying $y_{n+1} \in A^1_{n+1}$, we have a subsequence $\{y_{n_i+1}\}$ of $\{y_n\}$. By collecting $n \in N$ satisfying $y_{n+1} \in A^2_{n+1} \backslash A^1_{n+1}$, we have another subsequence $\{y_{n_j+1}\}$ of $\{y_n\}$. Then we easily see that $N = \{n_i\} \cup \{n_j\}$, and either $\{n_i\}$ or $\{n_j\}$ has infinite terms.

Suppose $\{n_i\}$ has infinite terms. Then, since $\{x_{n_i+1}\}$ converges strongly to $P_D x_0$, by $\lim_i b_{n_i} = 0$, $\{y_{n_i+1}\}$ also converges strongly to $P_D x_0$.

Suppose $\{n_j\}$ has infinite terms. Then, $\{D_{n_j}\}$ is a sequence of closed convex subsets of C satisfying $D_{n_{j+1}} \subset D_{n_j}$ for $j \in N$ and $D = \cap_{j \in N} D_{n_j} \ne \emptyset$. For $j \in N$, we know $D_{n_{j+1}} \subset D_{n_j+1} \subset D_{n_j}$, and $y_{n_j+1} \in A^2_{n_j+1}$; $y_{n_j+1} \in D_{n_j}$. Then, we see $y_{n_j+1} \in K'_{n_j} = \{y \in D_{n_j} : \|x_0 - y\| \le \|x_0 - x_{n_{j+1}}\|\}$ because

$$\|x_0 - y_{n_j+1}\| \le \|x_0 - x_{n_j+1}\| \le \|x_0 - x_{n_{j+1}}\| \quad \text{for} \quad j \in N.$$

By Lemma 3.2, $\{y_{n_j+1}\}$ converges strongly to $P_D x_0$.

From these, we see that $\{y_n\}$ converges strongly to $P_D x_0 \in D$. By considering Lemma 3.3 (3), this implies that $\{y_n\}$ converges strongly to $P_D x_0 \in F(B)$. $\square$

5. Additional explanations

In this section, by taking account of historical viewpoints, we give a summary of shrinking projection method to support the main issue. For simplicity, let C be a closed convex subset of a real Hilbert space H. Of course, H is a smooth strictly convex reflexive Banach space having the Kadec–Klee property. In this setting, $S \in \mathcal{T}^C_P$ is nonexpansive, however, the reverse is not always true. Nevertheless, for a nonexpansive mapping $S' \in \mathcal{F}^C$, $S = \frac{1}{2}(I + S') \in \mathcal{T}^C_P$ and $F(S) = F(S')$ hold.

Let S be a mapping from C into H such that $F(S)$ is non–empty closed and convex. Let $x_0 \in H$, $w = P_{F(S)} x_0$ and $\{u_n\}$ be a sequence in C. Here we present a sufficient condition to guarantee that $\{u_n\}$ converges strongly to $w = P_{F(S)} x_0$.

- $(*)$ Suppose $\limsup_n \|x_0 - u_n\| \le \|x_0 - w\|$ and $\lim_n \|Su_n - u_n\| = 0$.
  Suppose further that $I - S$ is demiclosed at 0.
  Then, $\{u_n\}$ converges strongly to $w = P_{F(S)} x_0$.

Takahashi, Takeuchi and Kubota focused on $(*)$. Correctly, they studied properties of Browder's sequence and reach $(*)$. In a sense, Browder's fixed point theorem and the prototype of shrinking projection method are related through the fact that $I-S$ is demiclosed at 0. Even if $\limsup_n \|x_0-u_n\| \le \|x_0-u\|$, $\{\|x_0-u_n\|\}$ need not be non–decreasing. Nevertheless, to simplify their assignment, they placed importance on the case that $\{\|x_0-u_n\|\}$ is non–decreasing.

From now on, C denotes a closed convex subset of H and S denotes a nonexpansive mapping from C into H which satisfies $F(S) \neq \emptyset$. Also $\{a_n\}$ denotes a sequence in $(0,1)$ satisfying $\lim_n a_n = 0$. Then, $F(S)$ is closed and convex. Also $I-S$ is demiclosed at 0: $u \in F(S)$ holds if there is a sequence $\{u_n\}$ in C such that $\{u_n\}$ converges weakly to $u \in C$ and $\{\|Su_n-u_n\|\}$ converges to 0.

For reference, we show some typical properties of a Browder's sequence. For our purpose, we have to assume that $x_0 \in C$ and S is a self–mapping on C. In advance, we confirm the following: Suppose $x_0 \neq w = P_{F(S)}x_0$ and there is $k \in N$ satisfying $a_k < a_{k+1}$. For $n \in N$, let $u_n = a_n x_0 + (1-a_n)w \in C$. Then, for $n \in N$,

$$\|x_0-u_n\| = \|x_0-(a_nx_0+(1-a_n)w)\| = (1-a_n)\|x_0-w\| < \|x_0-w\|.$$

So, for $\{u_n\}$, we see $\limsup_n \|x_0-u_n\| \le \|x_0-w\|$ and $\|x_0-u_{k+1}\| < \|x_0-u_k\|$.

For $n \in N$, let S_n be the contraction on C defined by $S_nx = a_nx_0+(1-a_n)Sx$ for $x \in C$ and let $x_n \in C$ be the unique fixed point of S_n. So, we call $\{x_n\}$ a Browder's sequence. Fix any $u \in F(S)$ and $n \in N$. Then,

$$\begin{aligned}\|x_n-u\|^2 &= \langle a_n(x_0-u)+(1-a_n)(Sx_n-Su), x_n-u\rangle \\ &= a_n\langle x_0-u, x_n-u\rangle + (1-a_n)\langle Sx_n-Su, x_n-u\rangle \\ &\le a_n\langle x_0-u, x_n-u\rangle + (1-a_n)\|x_n-u\|^2.\end{aligned}$$

By $a_n > 0$, we see $\langle x_0-u, x_n-u\rangle \ge \|x_n-u\|^2 \ge 0$. Also, we see

$$a_n\langle x_0-u, x_n-u\rangle + (1-a_n)\|x_n-u\|^2 = a_n\langle x_0-x_n, x_n-u\rangle + \|x_n-u\|^2.$$

So, very interesting inequalities $\langle x_0-x_n, x_n-u\rangle \ge 0$ and $\langle x_0-u, x_n-u\rangle \ge 0$ hold.

Set $C_{x_n} = \{y \in C : \langle x_0-x_n, x_n-y\rangle \ge 0\}$. By $\langle x_0-x_n, x_n-u\rangle \ge 0$, we see $u \in C_{x_n}$. Then C_{x_n} is closed and convex, and $P_{C_{x_n}}x_0 = x_n \in C_{x_n}$. Set $D = \cap_n C_{x_n}$. So, we confirmed $F(S) \subset D \subset C_{x_n}$. For $y \in C_{x_n}$, by $P_{C_{x_n}}x_0 = x_n$, we know

$$\|x_n-y\|^2 \le \|x_0-y\|^2 - \|x_0-x_n\|^2, \quad \text{that is,} \quad \|x_0-x_n\| \le \|x_0-y\|.$$

Also, set $v = P_Dx_0$ and $w = P_{F(S)}x_0$. From these, $\{x_n\}$ satisfies the following:

$$F(S) \subset D \subset C_{x_n}, \quad \|x_0-x_n\| \le \|x_0-v\| \le \|x_0-w\| \quad \text{for} \quad n \in N. \tag{5.1}$$

Of course, $\limsup_n \|x_0-x_n\| \le \|x_0-w\|$ follows. At present, we do not know whether there is $k > m$ satisfying $x_k \in C_{x_m}$ for $m \in N$. Also, we do not know whether $\{\|x_0-x_n\|\}$ has a non–decreasing subsequence. Nevertheless, by (5.1), we can have a result which contains the assertion of Browder's fixed point theorem.

Since S is nonexpansive and $F(S) \neq \emptyset$, by (5.1), $\{x_n\}$ and $\{Sx_n\}$ are bounded. Then, $\{x_n\}$ has a weakly convergent subsequence. By $\lim_n a_n = 0$ and $x_n = a_nx_0+(1-a_n)Sx_n$, we see $\lim_n \|x_n-Sx_n\| = \lim_n a_n\|x_0-Sx_n\| = 0$.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ which converges weakly to $z \in C$. By $\lim_n \|x_n-Sx_n\| = 0$, we see $z \in F(S)$. Obviously, $\{x_0-x_{n_j}\}$ converges weakly to x_0-z. By $w = P_{F(S)}x_0$, $z \in F(S)$ and (5.1), we have

$$\begin{aligned}\|x_0-z\| &\le \liminf_j \|x_0-x_{n_j}\| \\ &\le \limsup_j \|x_0-x_{n_j}\| \le \|x_0-v\| \le \|x_0-w\| \le \|x_0-z\|.\end{aligned}$$

Then, $\lim_j \|x_0-x_{n_j}\| = \|x_0-z\| = \|x_0-v\| = \|x_0-w\|$.

Since H has the Kadec–Klee property, $\{x_0 - x_{n_j}\}$ converges strongly to $x_0 - z$, that is, $\{x_{n_j}\}$ converges strongly to z. Since $v = P_D x_0$ is unique, by $z, w \in D$, $z = w = v$ holds. So, $\{x_{n_j}\}$ converges strongly to $P_{F(S)}x_0 = P_D x_0$. Since $\{x_{n_j}\}$ is arbitrary, $\{x_n\}$ converges strongly to $P_{F(S)}x_0 = P_D x_0$.

Takahashi, Takeuchi and Kubota considered as below: For such S, maybe we can choose a non–empty closed convex subset D of C with $F(S) \subset D$ and a sequence $\{u_n\}$ which satisfy $\|x_0 - u_n\| \le \|x_0 - u_{n+1}\| \le \|x_0 - P_D x_0\|$ for $n \in N$. Then, maybe $\{u_n\}$ converges strongly to $P_{F(S)}x_0$ if we choose suitable D and $\{u_n\}$.

Details of their way of thinking is presented below. For $\{x_n\}$ as above, $\{C_{x_n}\}$ is a sequence of non–empty closed convex subsets of C satisfying $F(S) \subset D = \cap_{n\in N} C_{x_n}$. By setting $D_n = \cap_{i=1}^n C_{x_i}$ and $u_n = P_{D_n}x_0$, we have $\{u_n\}$ satisfying

$$\|x_0 - u_n\| \le \|x_0 - u_{n+1}\| \le \|x_0 - P_D x_0\| \le \|x_0 - P_{F(S)}x_0\| \quad \text{for} \quad n \in N. \tag{5.2}$$

By $u_n \in D_n$, (5.2) and uniqueness of $P_D x_0$, $\{u_n\}$ converges strongly to $P_D x_0$. By $P_{F(S)}x_0 = P_D x_0$, $\{u_n\}$ has to converge strongly to $P_{F(S)}x_0 = P_D x_0$.

This is the prototype of shrinking projection method. Also, as an abstract of the argument, they considered Lemma 3.1 which is the origin of their method. Here, we must refer to a lemma in Martinez–Yanes and Xu [6]. In a Hilbert space, they are almost the same. The author did not know about it until he reads Saejung [8].

For the prototype, $\{D_n\}$ is made from $\{C_{x_n}\}$. Maybe there are some generating methods of $\{D_n\}$ such that $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$, and $F(S) \subset D = \cap_n D_n$. For such $\{D_n\}$, $\{u_n\} = \{P_{D_n}x_0\}$ converges strongly to $P_D x_0$. Then, there remain to find a suitable $\{D_n\}$ and to confirm $P_D x_0 \in F(S)$. From this viewpoint, their method appeared.

So far, to prove $P_D x_0 = P_{F(S)}x_0 \in F(S)$, we used the facts that $I - S$ is demiclosed at 0 and $F(S)$ is closed and convex. However, observing proofs in Kimura and Takahashi [5] and Ibaraki and Kimura [3], we notice the following: To prove $P_D x_0 \in F(S)$, we need not know whether these hold if we can choose suitable $\{D_n\}$. So, we use neither of them to show $P_D x_0 \in F(S)$ in the argument below.

Let C be a closed convex subset of H and let S be a nonexpansive mapping from C into H with $F(S) \neq \emptyset$. Let $x_0 \in H$. Following Nakajo and Takahashi [7], they took notice of the following equality: For $x, y, z \in H$,

$$\|y - z\|^2 + 2\langle y - x, z - y\rangle = \|z - x\|^2 - \|y - x\|^2.$$

Let $x, y, z \in H$. By $y - z = y - x + x - z$, we easily have the equality by

$$\begin{aligned}\|y - z\|^2 &= \|y - x\|^2 + \|x - z\|^2 + 2\langle y - x, (x - y) + (y - z)\rangle \\ &= \|z - x\|^2 - \|y - x\|^2 - 2\langle y - x, z - y\rangle.\end{aligned}$$

For $y, z \in C$, define a function $f_{y,z}$ from C into R by

$$f_{y,z}(x) = \|y - z\|^2 + 2\langle y - x, z - y\rangle = \|z - x\|^2 - \|y - x\|^2 \quad \text{for} \quad x \in C.$$

It follows from only properties of inner product that $f_{y,Sy}$ is continuous and convex. Set $L_y = \{x \in C : f_{y,Sy}(x) \le 0\}$. Then, L_y is a closed convex subset of C.

For $y \in C$ and $u \in F(S)$, we see the following:

$$f_{y,Sy}(u) = \|Sy - u\|^2 - \|y - u\|^2 \le 0, \quad \text{that is,} \quad u \in L_y.$$

According to [10], we may replace $f_{y,Sy}$ by $f_{y,ay+(1-a)Sy}$, where $a \in [0, 1)$.

Let $D_1 = C$ and $u_1 = P_{D_1}x_0$. Inductively, generate D_{n+1} and u_{n+1} as below: $D_{n+1} = D_n \cap L_{u_n}$ and $u_{n+1} = P_{D_{n+1}}x_0$ for $n \in N$. Set $D = \cap_{n\in N} D_n$. Then, $\{D_n\}$ is a sequence of closed convex subsets of C satisfying $D_{n+1} \subset D_n$ for $n \in N$, and $\emptyset \neq F(S) \subset D = \cap_n D_n$. So, $\{u_n\} = \{P_{D_n}x_0\}$ converges strongly to $P_D x_0$.

At present, we know that $\{u_n\}$ converges strongly to $v = P_D x_0 \in D$ and S is continuous. Then, by $u_{n+1} \in D_{n+1} = D_n \cap L_{u_n} \subset L_{u_n}$ for $n \in N$, we see

$$\|Sv - v\|^2 = \lim_n \|Su_n - u_{n+1}\|^2 \leq \lim_n \|u_n - u_{n+1}\|^2 = 0, \text{ that is, } v \in F(S).$$

In this argument, the continuity of S and the fact $D = \cap_{n \in N} D_n \neq \emptyset$ play important rolls. Referring $f_{y,z}(x) = \|y - z\|^2 + 2\langle y - x, z - y\rangle$, consider $h_{y,z}$ such that

$$h_{y,z}(x) = \|y - z\|^2 + \langle y - x, z - y\rangle = \langle x - z, y - z\rangle \quad \text{for} \quad x \in C.$$

Let B be a continuous mapping from C into R. For $y \in C$ and $z \in H$, let $h_{y,z}(x) = \langle x - z, y - z\rangle$ for $x \in C$ and let $D_y = \{x \in C : h_{y,By}(x) \leq 0\}$. Then, D_y is closed and convex. Assume $\cap_{y \in C} D_y \neq \emptyset$. Let $D_1 = C$ and $u_1 = P_{D_1} x_0$. Inductively, generate D_{n+1} and u_{n+1} by $D_{n+1} = D_n \cap D_{u_n} \supset \cap_{y \in C} D_y \neq \emptyset$ and $u_{n+1} = P_{D_{n+1}} x_0$ for $n \in N$. Set $D = \cap_{n \in N} D_n \supset \cap_{y \in C} D_y \neq \emptyset$. In this setting, we know that $\{u_n\}$ converges strongly to $v = P_D x_0$. For $n \in N$, by $v = P_D x_0 \in D \subset D_{u_n}$, we see

$$\begin{aligned} 0 &\geq \langle v - Bu_n, u_n - Bu_n\rangle = \langle v - Bv + Bv - Bu_n, u_n - Bu_n\rangle \\ &= \langle v - Bv, u_n - Bu_n\rangle - \|Bv - Bu_n\|\|u_n - Bu_n\|. \end{aligned}$$

So, since $\{u_n\}$ converges strongly to $v = P_D x_0$ and B is continuous, we easily see $0 \geq \lim_n \langle v - Bv, u_n - Bu_n\rangle = \|v - Bv\|^2$, that is, $v \in F(B)$.

Essentially, this is a revised version of shrinking projection method which is used in this note. For a nonexpansive mapping $S \in \mathcal{F}^C$ with $F(S) \neq \emptyset$, set $B = \frac{1}{2}(I + S)$. Then, by $B \in \mathcal{T}_P^C$, B is continuous and $\emptyset \neq F(S) = F(B) = \cap_{y \in C} D_y$.

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References

1. Y. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996.
2. K. Aoyama, F. Kohsaka, and W. Takahashi, Three generalizations of firmly nonexpansive mappings: their relations and continuity properties, J. Nonlinear Convex Anal. 10 (2009), 131 –- 147.
3. T. Ibaraki and Y. Kimura, Approximation of a fixed point of generalized firmly nonexpansive mappings with nonsummable errors, Linear Nonlinear Anal. 2 (2016), 301 – 310.
4. Y. Kimura, Approximation of a common fixed point of a finite family of nonexpansive mappings with nonsummable errors in a Hilbert space, J. Nonlinear Convex Anal. 15 (2014), 429 – 436.
5. Y. Kimura and W. Takahashi, On a hybrid method for a family of relatively nonexpansive mappings in a Banach space, J. Math. Anal. Appl. 357 (2009), 356 – 363.
6. C. Martinez-Yanes and H-K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006), 2400 – 2411.
7. K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, J. Math. Anal. Appl. 279 (2003), 372 –- 379.
8. S. Saejung, Fixed Point Algorithms and Related Topics, Yokohama Publishers, Yokohama, 2018.
9. W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
10. W. Takahashi, Y. Takeuchi and R. Kubota, Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 341 (2008), 276 – 286.
11. M. Tsukada, Convergence of best approximations in a smooth Banach space, J. Approx. Theory 40 (1984), 301 – 309.