Salahuddin, J. Nonlinear Anal. Optim. Vol. 10(2) (2019), 69-82

Journal of Nonlinear Analysis and Optimization
Volume 10(2) (2019)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

SYSTEM OF GENERALIZED HIERARCHICAL VARIATIONAL INEQUALITY PROBLEMS

SALAHUDDIN*[1]

[1] Department of Mathematics, Jazan University, Jazan-45142, Kingdom of Saudi Arabia

ABSTRACT. In this study, we discussed the solution of a system of generalized hierarchical variational inequality problems in Hilbert spaces by using the concepts of Mainge′s. We also discussed some applications.

KEYWORDS:System of generalized hierarchical variational inequality problems; fixed point problems; r-strongly monotone mappings; Hilbert spaces.
AMS Subject Classification: 49J40, 47H06.

1. Introduction

In various practical problems arising in decision theory, economics theory, game theory portfolio selection, etc. it is required to optimize the ratio of several linear or nonlinear functions to achieve the goal efficiently. The optimization problems are called mathematical functional programming problems or optimal control problems. The study of mathematical functional programming problem has been of great interest in the recent past due to its diversified applications. The variational inequality theory is well known and well developed because of its applications in the diversified area of science, social science, engineering, and commercial management. The variational inequality problems provide a convenient framework for the unified study of the optimal solution in many optimization related fields. Several numerical methods has been developed for solving variational inequality and related optimization problems. Hierarchical optimization was first defined by Bracken and McGill [2, 3] as a generalization of mathematical programming. In this context, the constraint region is implicitly determined by a series of optimization problems which must be solved in a predetermined sequence.
Inspired and motivated by the recent works [1, 4, 5, 6, 9, 14, 16, 17], we introduced the system of generalized hierarchical variational inequality problems and

* *Corresponding author.*
Email address : drsalah12@hotmail.com, smohammad@jazanu.edu.sa.
Article history : Received 2 December 2018; Accepted 5 June 2019.

investigate a more general form of the schemes to solve the system of generalized hierarchical variational inequality problems.

2. Preliminaries

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let C be a nonempty closed convex subsets of H. $F(T)$ denotes the set of fixed points of $T : C \longrightarrow C$, *that is,* $F(T) = \{x \in C : Tx = x\}$ and a variational inequality problems [8] is the problem of finding a point $x \in C$ such that

$$\langle Qx, y - x \rangle \geq 0, \forall y \in C, \tag{2.1}$$

where $Q : C \longrightarrow C$ is a nonlinear mapping and solution set of (2.1) is denoted by $\Upsilon(Q, C)$.
The hierarchical fixed point problems [11, 12, 13, 18, 19] is the problem of finding a point $x^* \in F(T)$ such that

$$\langle Qx^*, x - x^* \rangle \geq 0, \ \forall x \in F(T). \tag{2.2}$$

When the set $F(T)$ is replaced by the solution set of variational inequality (2.1), then (2.2) is known as hierarchical variational inequality problems.

In this paper, we define the system of generalized hierarchical variational inequality problems for finding $x_i^* \in \Upsilon(Q_i, C)$ such that for given positive real number $\eta_i, \ (i = 1, 2, \cdots, N)$ the following inequalities are hold:

$$\begin{aligned}
\langle \eta_1 F(x_2^*) + x_1^* - x_2^*, \ x_1 - x_1^* \rangle &\geq 0, \ \forall \ x_1 \in \Upsilon(Q_1, C), \\
\langle \eta_2 F(x_3^*) + x_2^* - x_3^*, \ x_2 - x_2^* \rangle &\geq 0, \ \forall \ x_2 \in \Upsilon(Q_2, C), \\
&\vdots \\
\langle \eta_{N-1} F(x_N^*) + x_{N-1}^* - x_N^*, \ x_{N-1} - x_{N-1}^* \rangle &\geq 0, \ \forall \ x_{N-1} \in \Upsilon(Q_{N-1}, C), \\
\langle \eta_N F(x_1^*) + x_N^* - x_1^*, \ x_N - x_N^* \rangle &\geq 0, \ \forall \ x_N \in \Upsilon(Q_N, C),
\end{aligned} \tag{2.3}$$

where $F, Q_i : H \longrightarrow H \ (i = 1, 2, \cdots, N)$ are mappings.

Definition 2.1. Let $T, F : H \longrightarrow H$ be the single valued mappings. Then

(i) T is said to be nonexpansive if
$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in H;$$
(ii) T is said to be quasi nonexpansive if $F(T) \neq \emptyset$ and
$$\|Tx - p\| \leq \|x - p\|, \ \forall x \in H, p \in F(T);$$
(iii) T is quasi nonexpansive if and only if for all $x \in H, p \in F(T)$
$$\langle x - Tx, \ x - p \rangle \geq \frac{1}{2}\|x - Tx\|^2;$$
(iv) T is said to be strongly quasi nonexpansive if T is quasi nonexpansive and
$$x_n - Tx_n \longrightarrow 0$$
whenever $\{x_n\}$ is a bounded sequence in H and
$$\|x_n - p\| - \|Tx_n - p\| \longrightarrow 0, \ \text{for some } p \in F(T);$$
(v) F is said to be μ-Lipschitzian if there exists $\mu > 0$ such that
$$\|F(x) - F(y)\| \leq \mu\|x - y\|, \ \forall x, y \in H;$$

(vi) F is said to be r-strongly monotone if there exists $r > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq r\|x - y\|^2, \ \forall x, y \in H;$$

(vii) If F is a μ-Lipschitzian and r-strongly monotone mapping and $\rho \in (0, \frac{2r}{\mu^2})$, then

$$I - \rho F$$

is a contraction mapping;

(viii) F is said to be α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \alpha\|F(x) - F(y)\|^2, \ \forall x, y \in H.$$

Lemma 2.2. [20] *Let $Q : H \longrightarrow H$ be an α-inverse strongly monotone mapping. Then*

(i) *Q is an $\frac{1}{\alpha}$-Lipschitz continuous and monotone mapping;*
(ii) $\|(I - \lambda Q)x - (I - \lambda Q)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Qx - Qy\|^2$, *for* $\lambda > 0$;
(iii) *if $\lambda \in (0, 2\alpha]$, then $I - \lambda Q$ is a nonexpansive mapping where I is an identity mapping on H.*

Lemma 2.3. *Let $x \in H$ and $z \in C$ be any points. Then the following statements are hold:*

(i) $z = P_C(x) \iff \langle x - z, \ y - z\rangle \geq 0, \ \forall y \in C.$
(ii) $z = P_C(x) \Rightarrow \ \|x - z\|^2 \geq \|x - y\|^2 - \|y - z\|^2, \ \forall y \in C.$
(iii) $\langle P_C(x) - P_C(y), \ x - y\rangle \geq \|P_C(x) - P_C(y)\|^2, \ \forall x, y \in H.$
(iv) $u \in \Upsilon(Q, C) \Leftrightarrow u \in F(P_C(I - \lambda Q)), \ \forall \lambda > 0.$

Lemma 2.4. [15] *For $x, y \in H$ and $\omega \in (0, 1)$ the following statements are hold:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(ii) $\|(1 - \omega)x + \omega y\|^2 = (1 - \omega)\|x\|^2 + \omega\|y\|^2 - \omega(1 - \omega)\|x - y\|^2.$

Lemma 2.5. [10] *Let $\{a_n\}$ be a sequence of real numbers and there exists a subsequence $\{a_{m_j}\}$ of $\{a_n\}$ such that $a_{m_j} < a_{m_j+1}$ for all $j \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers. Then there exists a non decreasing sequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\longrightarrow\infty} n_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$*

$$a_{n_k} \leq a_{n_k+1}, \ a_k \leq a_{n_k+1}.$$

In fact, n_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that $a_n < a_{n+1}$ hold.

Lemma 2.6. [7] *Let $\{a_n\} \subset [0, \infty), \ \{\alpha_n\} \subset [0, 1), \{b_n\} \subset (-\infty, +\infty)$ and $\hat{\alpha} \in [0, 1]$ be such that*

(i) *$\{a_n\}$ is a bounded sequence;*
(ii) $a_{n+1} \leq (1 - \alpha_n)^2 a_n \hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n, \ \forall n \geq 1;$
(iii) *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying*

$$\lim_{k\longrightarrow\infty} \inf(a_{n_{k+1}} - a_{n_k}) \geq 0$$

it follows that

$$\lim_{k\longrightarrow\infty} \sup b_{n_k} \leq 0;$$

(iv) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty.$

Then $\lim_{n\longrightarrow\infty} a_n = 0.$

3. Main Results

First, we prove the following lemma.

Lemma 3.1. *Let $Q : H \longrightarrow H$ be an α-inverse strongly monotone mapping. Let $\Upsilon(Q, C) \neq \emptyset$ be the solution set of (2.1). Then the following are hold:*

1. *the mapping $\Omega : H \longrightarrow C$ is defined by*
$$\Omega = P_C(I - \lambda Q), \text{ for } \lambda \in (0, 2\alpha],$$
is quasi nonexpansive, where I is an identity mapping;
2. *the mapping $I - \Omega : H \longrightarrow H$ is demiclosed at zero, that is, for any sequence $\{x_n\} \subset H$ if $x_n \rightharpoonup x$ and $(I - \Omega)x_n \longrightarrow 0$, then $x = \Omega x$;*
3. *the mapping Ω_β defined by*
$$\Omega_\beta = (I - \beta)I + \beta\Omega, \text{ for } \beta \in (0, 1) \tag{3.1}$$
is strongly quasi nonexpansive mapping and $F(\Omega_\beta) = F(\Omega)$.
4. *$I - \Omega_\beta,\ \beta(0, 1)$ is demiclosed at zero.*

Proof. (i) From Lemma 2.2(iii) and Lemma 2.3(iv), the mapping Ω is nonexpansive and $\Upsilon(Q, C) = F(\Omega) \neq \emptyset$. then this show that Ω is quasi nonexpansive.

(ii) Since Ω is a nonexpansive mapping on C, $I - \Omega$ is demiclosed at zero.

(iii) It is obvious that $F(\Omega_\beta) = F(\Omega)$.
Next, we prove that $\Omega_\beta, \beta \in (0, 1)$ is a strongly quasi nonexpansive mapping.
Let $\{x_n\}$ be any bounded sequence in H and $p \in \Omega_\beta$ be a given point such that
$$\|x_n - p\| - \|\Omega_\beta x_n - p\| \longrightarrow 0. \tag{3.2}$$
First, we prove that $\Omega_\beta, \beta \in (0, 1)$ is a quasi nonexpansive mapping.
From (3.1) and the fact that Ω is quasi nonexpansive, we have
$$\begin{aligned}\|\Omega_\beta x - p\| &= \|(1 - \beta)[x - p] + \beta(\Omega x - p)\| \\ &\leq (1 - \beta)\|x - p\| + \beta\|\Omega x - p\| \\ &\leq \|x - p\|, \ \forall x \in C.\end{aligned}$$
Therefore, Ω_β is a quasi nonexpansive mapping.
Next, we prove that
$$\|\Omega_\beta x_n - x_n\| \longrightarrow 0.$$
In fact, it follows from (3.1) that
$$\begin{aligned}\|\Omega_\beta x_n - p\|^2 &= \|x_n - p - \beta(x_n - \Omega x_n)\|^2 \\ &= \|x_n - p\|^2 - 2\beta\langle x_n - p, x_n - \Omega x_n\rangle + \beta^2\|x_n - \Omega x_n\|^2 \\ &\leq \|x_n - p\|^2 - \beta(1 - \beta)\|x_n - \Omega x_n\|^2.\end{aligned}$$
From (3.2), we have
$$\beta(1 - \beta)\|x_n - \Omega x_n\|^2 \leq \|x_n - p\|^2 - \|\Omega_\beta x_n - p\|^2 \longrightarrow 0.$$
Since $\beta(1 - \beta) > 0$, then
$$\|x_n - \Omega x_n\| \longrightarrow 0.$$
Hence
$$\|x_n - \Omega_\beta x_n\| = \beta\|x_n - \Omega x_n\| \longrightarrow 0.$$

(iv) Since $I-\Omega_\beta=\beta(I-\Omega)$ and $I-\Omega$ is demiclosed at zero, hence $I-\Omega_\beta$ is demiclosed at zero. This completes the proof.

□

Throughout this section, we always assume that the following conditions are satisfied:

(C1) $Q_i : H \longrightarrow H$ is an α_i-inverse strongly monotone mapping and $\Upsilon(Q_i, C) \neq \emptyset$ is the solution set of (2.1) with $Q = Q_i$ $(i = 1, 2, \cdots, N)$.

(C2) Ω_i and $\Omega_{i,\beta}, \beta \in (0, 1)$ are the mappings defined by

$$\begin{aligned} \Omega_i &= P_{C_i}(I-\lambda Q_i),\ \lambda\in(0,2\alpha_i];\\ \Omega_{i,\beta} &= (1-\beta)I+\beta\Omega_i,\ \beta\in(0,1),(i=1,2,\cdots,N)\ \textit{respectively.} \end{aligned} \tag{3.3}$$

Theorem 3.1. *Let Q_i and $\Upsilon(Q_i, C)$ satisfying the conditions $(C1)$ and $f_i : H \longrightarrow H$ be contraction with a contractive constant $\vartheta_i \in (0, 1)$, $(i = 1, 2, \cdots, N)$. Then there exists a unique elements $x_i^* \in \Upsilon(Q_i, C)$ such that the following are hold:*

$$\begin{aligned} \langle x_1^*-f_1(x_2^*),\ x_1-x_1^*\rangle &\geq 0,\ \forall x_1\in\Upsilon(Q_1,C),\\ \langle x_2^*-f_2(x_3^*),\ x_2-x_2^*\rangle &\geq 0,\ \forall x_2\in\Upsilon(Q_2,C),\\ &\vdots\\ \langle x_{N-1}^*-f_{N-1}(x_N^*),\ x_{N-1}-x_{N-1}^*\rangle &\geq 0,\ \forall x_{N-1}\in\Upsilon(Q_{N-1},C),\\ \langle x_N^*-f_N(x_1^*),\ x_N-x_N^*\rangle &\geq 0,\ \forall x_N\in\Upsilon(Q_N,C)(i=1,2,\cdots,N). \end{aligned} \tag{3.4}$$

Proof. The proof is a consequence of Banach's contraction principle but it is given here for the sake of completeness. From Lemma 2.2(iii) and Lemma 2.3(iv), $\Upsilon(Q_i, C)$ $(i = 1, 2, \cdots, N)$ are nonempty closed convex. Therefore the metric projection $P_{\Upsilon(Q_i,C)}$ is well defined for each $i = 1, 2, \cdots, N$. Since $f_i(i = 1, 2, \cdots, N)$ is a contraction mapping. Then

$$P_{\Upsilon(Q_i,C)}f_i$$

and

$$P_{\Upsilon(Q_1,C)}f_1\circ P_{\Upsilon(Q_2,C)}f_2\circ\cdots\circ P_{\Upsilon(Q_N,C)}f_N \tag{3.5}$$

are contraction mappings. Hence there exists a unique element $x^* \in H$ such that

$$x^*=(P_{\Upsilon(Q_1,C)}f_1\circ P_{\Upsilon(Q_2,C)}f_2\circ\cdots\circ P_{\Upsilon(Q_N,C)}f_N)x^*. \tag{3.6}$$

Putting $x_N^* = P_{\Upsilon(Q_N,C)}f_N(x_1^*), \cdots, x_2^* = P_{\Upsilon(Q_2,C)}f_2(x_3^*), x_1^* = P_{\Upsilon(Q_1,C)}f_1(x_2^*)$ and $x_N^* \in \Upsilon(Q_N, C), \cdots, x_1^* \in \Upsilon(Q_1, C)$.

Suppose that $(\bar{x}_1, \cdots, \bar{x}_N) \in \Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that the following are satisfied:

$$\begin{aligned} \langle \bar{x}_1-f_1(\bar{x}_2),\ x_1-\bar{x}_1\rangle &\geq 0,\ \forall x_1\in\Upsilon(Q_1,C),\\ \langle \bar{x}_2-f_2(\bar{x}_3),\ x_2-\bar{x}_2\rangle &\geq 0,\ \forall x_2\in\Upsilon(Q_2,C),\\ &\vdots\\ \langle \bar{x}_{N-1}-f_{N-1}(\bar{x}_N),\ x_{N-1}-\bar{x}_{N-1}\rangle &\geq 0,\ \forall x_{N-1}\in\Upsilon(Q_{N-1},C),\\ \langle \bar{x}_N-f_N(\bar{x}_1),\ x_N-\bar{x}_N\rangle &\geq 0,\ \forall x_N\in\Upsilon(Q_N,C). \end{aligned} \tag{3.7}$$

Then

$$\begin{aligned} \bar{x}_1 &= P_{\Upsilon(Q_1,C)}f_1(\bar{x}_2),\\ \bar{x}_2 &= P_{\Upsilon(Q_2,C)}f_2(\bar{x}_3), \end{aligned}$$

$$\vdots$$

$$\bar{x}_N = \quad P_{\Upsilon(Q_N,C)} f_N(\bar{x}_1). \tag{3.8}$$

Therefore

$$\bar{x}_1 = (P_{\Upsilon(Q_1,C)} f_1 \circ P_{\Upsilon(Q_2,C)} f_2 \circ \cdots \circ P_{\Upsilon(Q_N,C)} f_N)\bar{x}_1. \tag{3.9}$$

This implies that $\bar{x}_1 = x_1^*, \bar{x}_2 = x_2^*, \cdots, \bar{x}_N = x_N^*$, the proof is completed. □

Theorem 3.2. *Let $Q_i, \Upsilon(Q_i, C), \Omega_i$ and $\Omega_{i,\beta}$ satisfying the conditions $(C1 - C2)$, and $f_i : H \longrightarrow H$ be the contraction with a contractive constant $\vartheta_i \in (0,1)(i = 1, 2, \cdots, N)$. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned} x_i^0 &\in H, \qquad i = 1, 2, \cdots, N \\ x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta} x_1^n + \alpha_n f_1(\Omega_{2,\beta} x_2^n), \\ x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta} x_2^n + \alpha_n f_2(\Omega_{3,\beta} x_3^n), \\ &\vdots \\ x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta} x_N^n + \alpha_n f_N(\Omega_{1,\beta} x_1^n), \end{aligned} \tag{3.10}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_i^n\}$ $(i = 1, 2, \cdots, N)$ defined by (3.10) converge to x_i^*, where $(x_1^*, \cdots, x_N^*)$ is the unique elements in $\Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$, verifying (3.4).

Proof. **(i)** We first prove that the sequence $\{x_1^n\}, \cdots, \{x_N^n\}$ are bounded. From Lemma 3.1, it follows that $\Omega_{i,\beta}$ is strongly quasi nonexpansive and $F(\Omega_{i,\beta}) = F(\Omega_i) = \Upsilon(Q_i, C)$ $(i = 1, \cdots, N)$. Since f_i is contraction with constant ϑ_i $(i = 1, \cdots, N)$ and $x_1^* \in F(\Omega_{1,\beta}), x_2^* \in F(\Omega_{2,\beta}), \cdots, x_N^* \in F(\Omega_{N,\beta})$, we have

$$\begin{aligned} \|x_1^{n+1} - x_1^*\| &\leq (1-\alpha_n)\|\Omega_{1,\beta} x_1^n - x_1^*\| + \alpha_n \|f_1(\Omega_{2,\beta} x_2^n) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n \|f_1(\Omega_{2,\beta} x_2^n) - f_1(x_2^*)\| + \alpha_n \|f_1(x_2^*) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n \vartheta_1 \|\Omega_{2,\beta} x_2^n - x_2^*\| + \alpha_n \|f_1(x_2^*) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n \vartheta_1 \|x_2^n - x_2^*\| + \alpha_n \|f_1(x_2^*) - x_1^*\| \\ &\leq (1-\alpha_n)\|x_1^n - x_1^*\| + \alpha_n \vartheta_1 \|x_2^n - x_2^*\| + \alpha_n \|f_1(x_2^*) - x_1^*\|. \end{aligned} \tag{3.11}$$

Similarly, we can also compute that

$$\begin{aligned} \|x_2^{n+1} - x_2^*\| &\leq (1-\alpha_n)\|x_2^n - x_2^*\| + \alpha_n \vartheta_2 \|x_3^n - x_3^*\| + \alpha_n \|f_2(x_3^*) - x_2^*\|, \\ &\vdots \\ \|x_N^{n+1} - x_N^*\| &\leq (1-\alpha_n)\|x_N^n - x_N^*\| + \alpha_n \vartheta_N \|x_1^n - x_1^*\| + \alpha_n \|f_N(x_1^*) - x_N^*\|. \end{aligned} \tag{3.12}$$

This implies that

$$\begin{aligned} &\|x_1^{n+1} - x_1^*\| + \|x_2^{n+1} - x_2^*\| + \cdots + \|x_N^{n+1} - x_N^*\| \leq (1-\alpha_n)[\|x_1^n - x_1^*\| \\ &+ \cdots + \|x_N^n - x_N^*\|] + \alpha_n[\vartheta_N \|x_1^n - x_1^*\| + \vartheta_1 \|x_2^n - x_2^*\| + \cdots + \vartheta_{N-1}\|x_N^n - x_N^*\|] \\ &+\alpha_n[\|f_1(x_2^*) - x_1^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|] \\ &\leq (1-\alpha_n)[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] + \alpha_n \vartheta[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] \\ &+\alpha_n[\|f_1(x_2^*) - x_1^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|] \\ &\leq (1-\alpha_n(1-\vartheta))[\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|] \\ &+\alpha_n(1-\vartheta)\frac{\|f_1(x_2^*) - x_1^*\| + \|f_2(x_3^*) - x_2^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|}{1-\vartheta} \end{aligned}$$

$$\leq \max\{\|x_1^n - x_1^*\| + \cdots + \|x_N^n - x_N^*\|, \frac{\|f_1(x_2^*) - x_1^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|}{1-\vartheta}\}, \quad (3.13)$$

where $\vartheta = \max\{\vartheta_1, \vartheta_2, \cdots, \vartheta_N\}$.
By induction, we have

$$\|x_1^{n+1} - x_1^*\| + \|x_2^{n+1} - x_2^*\| + \cdots + \|x_N^{n+1} - x_N^*\|$$
$$\leq \max\{\|x_1^0 - x_1^*\| + \cdots + \|x_N^0 - x_N^*\|, \frac{\|f_1(x_2^*) - x_1^*\| + \cdots + \|f_N(x_1^*) - x_N^*\|}{1-\vartheta}\}, \forall n \geq 1. \quad (3.14)$$

Hence $\{x_1^n\}, \cdots, \{x_N^n\}$ are bounded, consequently $\{\Omega_{1,\beta} x_1^*\}, \cdots, \{\Omega_{N,\beta} x_N^*\}$ are bounded.

(ii) Next, we prove that for each $n \geq 1$ the following inequalities are hold:

$$\begin{aligned} \|x_1^{n+1} - x_1^*\|^2 + \|x_2^{n+1} - x_2^*\|^2 + \cdots + \|x_N^{n+1} - x_N^*\|^2 \leq (1-\alpha_n)^2(\|x_1^n - x_1^*\|^2 \\ + \|x_2^n - x_2^*\|^2 + \cdots + \|x_N^n - x_N^*\|^2) + 2\alpha_n\vartheta(\|x_1^{n+1} - x_1^*\|\|x_2^n - x_2^*\| \\ + \|x_2^{n+1} - x_2^*\|\|x_3^n - x_3^*\| + \cdots + \|x_N^{n+1} - x_N^*\|\|x_1^n - x_1^*\|) \\ + 2\alpha_n(\langle f_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle + \langle f_2(x_3^*) - x_2^*,\ x_2^{n+1} - x_2^*\rangle \\ + \cdots + \langle f_N(x_1^*) - x_N^*,\ x_N^{n+1} - x_N^*\rangle). \end{aligned} \quad (3.15)$$

From (3.10) and Lemma 2.4, we have

$$\begin{aligned} \|x_1^{n+1} - x_1^*\|^2 &= \|(1-\alpha_n)(\Omega_{1,\beta}(x_1^n) - x_1^*) + \alpha_n(f_1(\Omega_{2,\beta}(x_2^n)) - x_1^*)\|^2 \\ &\leq \|(1-\alpha_n)(\Omega_{1,\beta}(x_1^n) - x_1^*)\|^2 + 2\alpha_n\langle f_1(\Omega_{2,\beta}(x_2^n)) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|\Omega_{1,\beta}(x_1^n) - x_1^*\|^2 + 2\alpha_n\langle f_1(\Omega_{2,\beta}(x_2^n)) - f_1(x_2^*),\ x_1^{n+1} - x_1^*\rangle \\ &\quad + 2\alpha_n\langle f_1(x_2^n) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|x_1^n - x_1^*\|^2 + 2\alpha_n\|f_1(\Omega_{2,\beta}(x_2^n)) - f_1(x_2^*)\|\|x_1^{n+1} - x_1^*\| \\ &\quad + 2\alpha_n\langle f_1(x_2^n) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|x_1^n - x_1^*\|^2 + 2\alpha_n\vartheta_1\|\Omega_{2,\beta}(x_2^n) - x_2^*\|\|x_1^{n+1} - x_1^*\| \\ &\quad + 2\alpha_n\langle f_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle \\ &\leq (1-\alpha_n)^2\|x_1^n - x_1^*\|^2 + 2\alpha_n\vartheta_1\|x_2^n - x_2^*\|\|x_1^{n+1} - x_1^*\| \\ &\quad + 2\alpha_n\langle f_1(x_2^*) - x_1^*,\ x_1^{n+1} - x_1^*\rangle. \end{aligned} \quad (3.16)$$

Similarly, we can also prove that

$$\begin{aligned} \|x_2^{n+1} - x_2^*\|^2 &\leq (1-\alpha_n)^2\|x_2^n - x_2^*\|^2 + 2\alpha_n\vartheta_2\|x_3^n - x_3^*\|\|x_2^{n+1} - x_2^*\| \\ &\quad + 2\alpha_n\langle f_2(x_3^*) - x_2^*,\ x_2^{n+1} - x_2^*\rangle, \\ &\vdots \\ \|x_N^{n+1} - x_N^*\|^2 &\leq (1-\alpha_n)^2\|x_N^n - x_N^*\|^2 + 2\alpha_n\vartheta_N\|x_1^n - x_1^*\|\|x_N^{n+1} - x_N^*\| \\ &\quad + 2\alpha_n\langle f_N(x_1^*) - x_N^*,\ x_N^{n+1} - x_N^*\rangle. \end{aligned} \quad (3.17)$$

Adding (3.16) and (3.17), and assume that $\vartheta = \max\{\vartheta_1, \cdots, \vartheta_N\}$, inequalities (3.15) is proved.

(iii) Next, we prove that if there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$\lim_{k\longrightarrow\infty} \inf\{(\|x_1^{n_k+1} - x_1^*\|^2 + \cdots + \|x_N^{n_k+1} - x_N^*\|^2) - (\|x_1^{n_k} - x_1^*\|^2 + \cdots + \|x_N^{n_k} - x_N^*\|^2)\} \geq 0. \quad (3.18)$$

Then

$$\lim_{k\longrightarrow\infty}\sup\{\langle f_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle+\langle f_2(x_3^*)-x_2^*,\ x_2^{n_k+1}-x_2^*\rangle \\ +\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n_k+1}-x_N^*\rangle\}\leq 0. \quad (3.19)$$

Since the norm $\|\cdot\|^2$ is convex and $\lim_{n\longrightarrow\infty}\alpha_n=0$, by (3.10) we have

$$\begin{aligned} 0 &\leq \lim_{k\longrightarrow\infty}\inf\{(\|x_1^{n_k+1}-x_1^*\|^2+\cdots+\|x_N^{n_k+1}-x_N^*\|^2)-(\|x_1^{n_k}-x_1^*\|^2 \\ &\quad +\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \\ &\leq \lim_{k\longrightarrow\infty}\inf\{(1-\alpha_{n_k})\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2+\alpha_{n_k}\|f_1(\Omega_{2,\beta}(x_2^{n_k}))-x_1^*\|^2 \\ &\quad +(1-\alpha_{n_k})\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|^2+\alpha_{n_k}\|f_2(\Omega_{3,\beta}(x_3^{n_k}))-x_2^*\|^2 \\ &\quad +\cdots+(1-\alpha_{n_k})\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2+\alpha_{n_k}\|f_N(\Omega_{1,\beta}(x_1^{n_k}))-x_N^*\|^2 \\ &\quad -(\|x_1^{n_k}-x_1^*\|^2+\cdots+\|x_N^{n_k}-x_N^*\|^2)\} \\ &\leq \lim_{k\longrightarrow\infty}\inf\{(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2)+(\|\Omega_{2,\beta}(x_2^{n_k})-x_2^*\|^2 \\ &\quad -\|x_2^{n_k}-x_2^*\|^2)+\cdots+(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)\} \\ &\leq \lim_{k\longrightarrow\infty}\sup\{(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2)+(\|\Omega_{2,\beta}(x_2^{n_k})-x_2^*\|^2 \\ &\quad -\|x_2^{n_k}-x_2^*\|^2)+\cdots+(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)\}\leq 0. \quad (3.20) \end{aligned}$$

This implies that

$$\begin{aligned} &\lim_{k\longrightarrow\infty}(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|^2-\|x_1^{n_k}-x_1^*\|^2) \\ &=\lim_{k\longrightarrow\infty}(\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|^2-\|x_2^{n_k}-x_2^*\|^2) \\ &=\cdots=\lim_{k\longrightarrow\infty}(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|^2-\|x_N^{n_k}-x_N^*\|^2)=0. \quad (3.21) \end{aligned}$$

Since the sequences $\{\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|+\|x_1^{n_k}-x_1^*\|\},\{\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|+\|x_2^{n_k}-x_2^*\|\},\cdots,\{\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|+\|x_N^{n_k}-x_N^*\|\}$ are bounded. Therefore, we have

$$\begin{aligned} &\lim_{k\longrightarrow\infty}(\|\Omega_{1,\beta}x_1^{n_k}-x_1^*\|-\|x_1^{n_k}-x_1^*\|) \\ &=\lim_{k\longrightarrow\infty}(\|\Omega_{2,\beta}x_2^{n_k}-x_2^*\|-\|x_2^{n_k}-x_2^*\|) \\ &=\cdots=\lim_{k\longrightarrow\infty}(\|\Omega_{N,\beta}x_N^{n_k}-x_N^*\|-\|x_N^{n_k}-x_N^*\|)=0. \quad (3.22) \end{aligned}$$

By Lemma 3.1, $\Omega_{1,\beta},\cdots,\Omega_{N,\beta}$ are strongly quasi nonexpansive, then

$$\Omega_{1,\beta}x_1^{n_k}-x_1^{n_k}\longrightarrow 0,\ \Omega_{2,\beta}x_2^{n_k}-x_2^{n_k}\longrightarrow 0,\ \cdots,\ \Omega_{N,\beta}x_N^{n_k}-x_N^{n_k}\longrightarrow 0. \quad (3.23)$$

Consequently, we obtain that

$$x_1^{n_k}-x_1^{n_k+1}\longrightarrow 0,\ x_2^{n_k}-x_2^{n_k+1}\longrightarrow 0,\ \cdots,\ x_N^{n_k}-x_N^{n_k+1}\longrightarrow 0. \quad (3.24)$$

It follows from the boundedness of $\{x_1^{n_k}\}$ that there exists a subsequence $\{x_1^{n_{k_\ell}}\}$ of $\{x_1^{n_k}\}$ such that $x_1^{n_{k_\ell}}\rightharpoonup p$ and

$$\begin{aligned} \lim_{\ell\longrightarrow\infty}\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle &=\lim_{k\longrightarrow\infty}\sup\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle \\ &=\lim_{k\longrightarrow\infty}\sup\langle f_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle. \quad (3.25) \end{aligned}$$

By Lemma 3.1, $I-\Omega_{1,\beta}$ is demiclosed at zero and $p\in Fix(\Omega_{1,\beta})=\Upsilon(Q_1,C)$. Hence from (3.4) we have

$$\lim_{\ell\longrightarrow\infty}\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle=\langle f_1(x_2^*)-x_1^*,\ p-x_1^*\rangle\leq 0. \quad (3.26)$$

Therefore

$$\lim_{k\longrightarrow\infty}\sup\langle f_1(x_2^*)-x_1^*,\ x_1^{n_k+1}-x_1^*\rangle=\lim_{\ell\longrightarrow\infty}\langle f_1(x_2^*)-x_1^*,\ x_1^{n_{k_\ell}}-x_1^*\rangle\leq 0. \quad (3.27)$$

Similarly, we can also prove that

$$\begin{aligned}\lim_{k\longrightarrow\infty}\sup\langle f_2(x_3^*)-x_2^*,\ x_2^{n_k+1}-x_2^*\rangle&\leq 0,\\ &\vdots\\ \lim_{k\longrightarrow\infty}\sup\langle f_N(x_1^*)-x_N^*,\ x_N^{n_k+1}-x_N^*\rangle&\leq 0. \end{aligned} \quad (3.28)$$

Hence, we have the desired inequalities.

(iv) Finally, we prove that the sequences $\{x_1^n\},\cdots,\{x_N^n\}$ generated by (3.10) converge to $x_1^*,\cdots,x_N^*$, *respectively.* It is clear that

$$\begin{aligned}&\|x_1^{n+1}-x_1^*\|\|x_2^n-x_2^*\|+\|x_2^{n+1}-x_2^*\|\|x_3^n-x_3^*\|+\cdots\\ &+\|x_N^{n+1}-x_N^*\|\|x_1^n-x_1^*\|\leq\sqrt{\|x_1^n-x_1^*\|^2+\cdots+\|x_N^n-x_N^*\|^2}\times\\ &\sqrt{\|x_1^{n+1}-x_1^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2}. \end{aligned} \quad (3.29)$$

Substituting (3.29) into (3.15) we have

$$\begin{aligned}&\|x_1^{n+1}-x_1^*\|^2+\|x_2^{n+1}-x_2^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2\leq(1-\alpha_n)^2(\|x_1^n-x_1^*\|^2\\ &+\cdots+\|x_N^n-x_N^*\|^2)+2\alpha_n\vartheta\{\sqrt{\|x_1^n-x_1^*\|^2+\cdots+\|x_N^n-x_N^*\|^2}\times\\ &\sqrt{\|x_1^{n+1}-x_1^*\|^2+\cdots+\|x_N^{n+1}-x_N^*\|^2}\}+2\alpha_n(\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle\\ &+\langle f_2(x_3^*)-x_2^*,\ x_2^{n+1}-x_2^*\rangle+\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n+1}-x_N^*\rangle). \end{aligned} \quad (3.30)$$

Set

$$\begin{aligned}a_n&=\|x_1^n-x_1^*\|^2+\|x_2^n-x_2^*\|^2+\cdots+\|x_N^n-x_N^*\|^2,\\ b_n&=2(\langle f_1(x_2^*)-x_1^*,\ x_1^{n+1}-x_1^*\rangle+\cdots+\langle f_N(x_1^*)-x_N^*,\ x_N^{n+1}-x_N^*\rangle). \end{aligned} \quad (3.31)$$

Then, we have the following statements:

(i) From (i), $\{a_n\}$ is bounded sequence.
(ii) From (3.30) $a_{n+1}\leq(1-\alpha_n)^2a_n\vartheta\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_nb_n,\ \forall n\geq 1.$
(iii) From (iii) whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying

$$\lim_{k\longrightarrow\infty}\inf(a_{n_k+1}-a_{n_k})\geq 0, \quad (3.32)$$

it follows that

$$\lim_{k\longrightarrow\infty}\sup b_{n_k}\leq 0.$$

By Lemma 2.6, we have

$$\lim_{n\longrightarrow\infty}(\|x_1^n-x_1^*\|^2+\cdots+\|x_N^n-x_N^*\|^2)=0. \quad (3.33)$$

Hence, we obtain that

$$\lim_{n\longrightarrow\infty}\|x_1^n-x_1^*\|=\lim_{n\longrightarrow\infty}\|x_2^n-x_2^*\|=\cdots=\lim_{n\longrightarrow\infty}\|x_N^n-x_N^*\|=0. \quad (3.34)$$

The proof is completed. □

Theorem 3.3. *Let $Q_i, \Upsilon(Q_i, C), \Omega_i$ and $\Omega_{i,\beta}$ $(i = 1, 2, \cdots, N)$ satisfying the conditions $(C1) - (C2)$ and $F : H \longrightarrow H$ be μ-Lipschitz continuous and r-strongly monotone mapping. Let $\{x_1^n\}, \cdots, \{x_N^n\}$ be the sequences defined by*

$$\begin{aligned} x_1^0, \cdots, x_N^0 &\in H \\ x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta} x_1^n + \alpha_n f_1(\Omega_{2,\beta}(x_2^n)), \\ x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta} x_2^n + \alpha_n f_2(\Omega_{3,\beta}(x_3^n)), \\ &\vdots \\ x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta} x_N^n + \alpha_n f_N(\Omega_{1,\beta}(x_1^n)), \text{ for } n = 0, 1, 2, \cdots \end{aligned} \tag{3.35}$$

where $f_i = I - \eta_i F$ with $\eta_i \in (0, \frac{2r}{\mu})(i = 1, 2, \cdots, N)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_1^n\}, \{x_2^n\} \cdots, \{x_N^n\}$ converges to $x_1^*, x_2^*, \cdots, x_N^*$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $\Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that (2.3) is satisfied.

Proof. It is easy to see that $f_i (i = 1, 2, \cdots, N)$ are contraction mappings and all the conditions in Theorem 3.2 are satisfied. By Theorem 3.2, we have the sequences $\{x_1^n\}, \cdots, \{x_N^n\}$ which converges to $(x_1^*, \cdots, x_N^*) \in \Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that the following are satisfied.

$$\begin{aligned} \langle x_1^* - f_1(x_2^*),\ x_1 - x_1^* \rangle \geq 0,\ \forall x_1 \in \Upsilon(Q_1, C), \\ \langle x_2^* - f_2(x_3^*),\ x_2 - x_2^* \rangle \geq 0,\ \forall x_2 \in \Upsilon(Q_2, C), \\ \vdots \\ \langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^* \rangle \geq 0,\ \forall x_{N-1} \in \Upsilon(Q_{N-1}, C), \\ \langle x_N^* - f_N(x_1^*),\ x_N - x_N^* \rangle \geq 0,\ \forall x_N \in \Upsilon(Q_N, C). \end{aligned} \tag{3.36}$$

Substituting $f_1 = I - \eta_1 F,\ f_2 = I - \eta_2 F, \cdots, f_N = I - \eta_N F$ in (3.36), we obtain that the sequences $\{x_1^n\}, \cdots, \{x_N^n\}$ converges to $(x_1^*, \cdots, x_N^*) \in \Upsilon(Q_1, C) \times \Upsilon(Q_2, C) \times \cdots \times \Upsilon(Q_N, C)$ such that (2.3) are hold and proof is completed. □

If setting $Q_i = I - T_i$, where $T_i : H \longrightarrow H$ is a nonexpansive mapping in Theorem 3.2 and Theorem 3.3, Then, Q_i is $\frac{1}{2}$-inverse strongly monotone and $\Upsilon(Q_i, C) = F(T_i)$ $(i = 1, 2, \cdots, N)$. Hence, we obtain the following corollary.

Corollary 3.2. *Let $T_i : H \longrightarrow H$ be a nonexpansive mapping and $Q_i = I - T_i, \Upsilon(Q_i, C), \Omega_i$ and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $f_i : H \longrightarrow H$ be contraction with a contractive constant $\vartheta_i \in (0, 1)$ for $i = 1, 2, \cdots, N$. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned} x_i^0 \in H, &\qquad i = 1, 2, \cdots, N \\ x_1^{n+1} &= (1-\alpha_n)\Omega_{1,\beta} x_1^n + \alpha_n f_1(\Omega_{2,\beta} x_2^n), \\ x_2^{n+1} &= (1-\alpha_n)\Omega_{2,\beta} x_2^n + \alpha_n f_2(\Omega_{3,\beta} x_3^n), \\ &\vdots \\ x_N^{n+1} &= (1-\alpha_n)\Omega_{N,\beta} x_N^n + \alpha_n f_N(\Omega_{1,\beta} x_1^n), \end{aligned} \tag{3.37}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $F(T_1) \times F(T_2) \times \cdots \times F(T_N)$ such that following are satisfied:

$$\begin{aligned} \langle x_1^* - f_1(x_2^*),\ x_1 - x_1^* \rangle \geq 0,\ \forall x_1 \in F(T_1), \\ \langle x_2^* - f_2(x_3^*),\ x_2 - x_2^* \rangle \geq 0,\ \forall x_2 \in F(T_2), \end{aligned}$$

$$\vdots$$

$$\langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^* \rangle \geq 0,\ \forall x_{N-1} \in F(T_{N-1}),$$
$$\langle x_N^* - f_N(x_1^*),\ x_N - x_N^* \rangle \geq 0,\ \forall x_N \in F(T_N). \tag{3.38}$$

Corollary 3.3. *Let $T_i : H \longrightarrow H$ be a nonexpansive mapping and $Q_i = I - T_i$, $\Upsilon(Q_i, C)$, Ω_i and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $F : H \longrightarrow H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned} & x_i^0 \in H, \qquad i = 1, 2, \cdots, N \\ & x_1^{n+1} = (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}x_2^n), \\ & x_2^{n+1} = (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}x_3^n), \\ & \vdots \\ & x_N^{n+1} = (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}x_1^n), \end{aligned} \tag{3.39}$$

where $f_i = I - \eta_i F$ with $\eta_i \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $F(T_1) \times F(T_2) \times \cdots \times F(T_N)$ such that the following are satisfied:

$$\langle \eta_1 F(x_2^*) + x_1^* - x_2^*,\ x_1 - x_1^* \rangle \geq 0,\ \forall\ x_1 \in F(T_1),$$
$$\langle \eta_2 F(x_3^*) + x_2^* - x_3^*,\ x_2 - x_2^* \rangle \geq 0,\ \forall\ x_2 \in F(T_2),$$
$$\vdots$$
$$\langle \eta_{N-1} F(x_N^*) + x_{N-1}^* - x_N^*,\ x_{N-1} - x_{N-1}^* \rangle \geq 0,\ \forall\ x_{N-1} \in F(T_{N-1}),$$
$$\langle \eta_N F(x_1^*) + x_N^* - x_1^*,\ x_N - x_N^* \rangle \geq 0,\ \forall\ x_N \in F(T_N). \tag{3.40}$$

Corollary 3.4. *Let C_i be a nonempty closed convex subset of H and $Q_i = I - P_{C_i}$, $\Upsilon(Q_i, C)$, Ω_i and $\Omega_{i,\beta}$ satisfying the conditions $(C1) - (C2)$ $(i = 1, 2, \cdots, N)$. Let $f_i : H \longrightarrow H$ be contraction with a contractive constant $\vartheta_i \in (0, 1)$ $(i = 1, 2, \cdots, N)$. Let $\{x_i^n\}$ be the sequences defined by*

$$\begin{aligned} & x_i^0 \in H, \qquad i = 1, 2, \cdots, N \\ & x_1^{n+1} = (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}x_2^n), \\ & x_2^{n+1} = (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}x_3^n), \\ & \vdots \\ & x_N^{n+1} = (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}x_1^n), \end{aligned} \tag{3.41}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $C_1 \times C_2 \times \cdots \times C_N$ such that the following are satisfied:

$$\langle x_1^* - f_1(x_2^*),\ x_1 - x_1^* \rangle \geq 0,\ \forall x_1 \in C_1,$$
$$\langle x_2^* - f_2(x_3^*),\ x_2 - x_2^* \rangle \geq 0,\ \forall x_2 \in C_2,$$
$$\vdots$$
$$\langle x_{N-1}^* - f_{N-1}(x_N^*),\ x_{N-1} - x_{N-1}^* \rangle \geq 0,\ \forall x_{N-1} \in C_{N-1},$$
$$\langle x_N^* - f_N(x_1^*),\ x_N - x_N^* \rangle \geq 0,\ \forall x_N \in C_N. \tag{3.42}$$

Corollary 3.5. *Let C_i be a nonempty closed convex subset of H and $Q_i = I - P_{C_i}$, $\Upsilon(Q_i, C)$, Ω_i and $\Omega_{i,\beta}$ satisfying the conditions $(C1)-(C2)$ $(i = 1, 2, \cdots, N)$. Let $F_i : H \longrightarrow H$ $(i = 1, 2, \cdots, N)$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_i^n\}$ be the sequences defined by*

$$
\begin{aligned}
&x_i^0 \in H, \qquad i = 1, 2, \cdots, N\\
&x_1^{n+1} = (1-\alpha_n)\Omega_{1,\beta}x_1^n + \alpha_n f_1(\Omega_{2,\beta}x_2^n),\\
&x_2^{n+1} = (1-\alpha_n)\Omega_{2,\beta}x_2^n + \alpha_n f_2(\Omega_{3,\beta}x_3^n),\\
&\vdots\\
&x_N^{n+1} = (1-\alpha_n)\Omega_{N,\beta}x_N^n + \alpha_n f_N(\Omega_{1,\beta}x_1^n),
\end{aligned}
\tag{3.43}
$$

where $f_i = I - \eta_i F$ with $\eta_i \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then, the sequences $\{x_i^n\}$ converge to x_i^* $(i = 1, 2, \cdots, N)$, where $(x_1^*, \cdots, x_N^*)$ is an unique elements in $C_1 \times C_2 \times \cdots \times C_N$ such that the following are satisfied:

$$
\begin{aligned}
&\langle \eta_1 F(x_2^*) + x_1^* - x_2^*,\ x_1 - x_1^*\rangle \geq 0,\ \forall\ x_1 \in C_1,\\
&\langle \eta_2 F(x_3^*) + x_2^* - x_3^*,\ x_2 - x_2^*\rangle \geq 0,\ \forall\ x_2 \in C_2,\\
&\vdots\\
&\langle \eta_{N-1} F(x_N^*) + x_{N-1}^* - x_N^*,\ x_{N-1} - x_{N-1}^*\rangle \geq 0,\ \forall\ x_{N-1} \in C_{N-1},\\
&\langle \eta_N F(x_1^*) + x_N^* - x_1^*,\ x_N - x_N^*\rangle \geq 0,\ \forall\ x_N \in C_N.
\end{aligned}
\tag{3.44}
$$

4. Applications

Let $Q_1 = Q_2 = \cdots = Q_N$, $f_1 = \cdots = f_N$ and $x_1^0 = \cdots = x_N^0$ in Theorem 3.2, then we have the following:

Theorem 4.1. *Let Q, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1)-(C2)$ and $f : H \longrightarrow H$ be a contraction with a contractive constant $\vartheta \in (0,1)$. Let $\{x_n\}$ be a sequence suggested by*

$$
\begin{cases}
x_0 \in H,\\
x_{n+1} = (1-\alpha_n)\Omega_\beta x_n + \alpha_n f(\Omega_\beta x_n),\ n = 0, 1, 2, \cdots,
\end{cases}
\tag{4.1}
$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then a sequence $\{x_n\}$ converges to $x^* \in \Upsilon(Q, C)$ such that the following inequality is satisfied:

$$\langle x^* - f(x^*), x - x^*\rangle \geq 0,\ \forall x \in \Upsilon(Q, C).$$

Theorem 4.2. *Let Q, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1)-(C2)$ and $F : H \longrightarrow H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$ be a sequence suggested by*

$$
\begin{cases}
x_0 \in H,\\
x_{n+1} = (1-\alpha_n)\Omega_\beta x_n + \alpha_n (I - \eta F)(\Omega_\beta x_n),\ n = 0, 1, 2, \cdots,
\end{cases}
\tag{4.2}
$$

where $\eta \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then a sequence $\{x_n\}$ converges to $x^* \in \Upsilon(Q, C)$ such that the following are satisfied:

$$\langle F(x^*), x - x^*\rangle \geq 0,\ \forall x \in \Upsilon(Q, C).$$

Corollary 4.1. *Let $T : H \longrightarrow H$ be a nonexpansive mapping and $Q = I - T$, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1) - (C2)$ and $f : H \longrightarrow H$ be contraction with a contractive constant $\vartheta \in (0, 1)$. Let $\{x_n\}$ be a sequence suggested by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n)\Omega_\beta x_n + \alpha_n f(\Omega_\beta x_n), \ n = 0, 1, 2, \cdots, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges to $x^* \in F(T)$ such that the following inequality is satisfied:

$$\langle x^* - f(x^*), x - x^* \rangle \geq 0, \ \forall x \in F(T).$$

Corollary 4.2. *Let $T : H \longrightarrow H$ be a nonexpansive mapping and $Q = I - T$, $\Upsilon(Q, C)$, Ω and Ω_β satisfy the conditions $(C1) - (C2)$ and $F : H \longrightarrow H$ be a μ-Lipschitzian and r-strongly monotone mapping. Let $\{x_n\}$ be a sequence suggested by*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n)\Omega_\beta x_n + \alpha_n (I - \eta F)(\Omega_\beta x_n), \ n = 0, 1, 2, \cdots, \end{cases} \tag{4.4}$$

where $\eta \in (0, \frac{2r}{\mu^2})$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \longrightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then a sequence $\{x_n\}$ converges to $x^* \in F(T)$ such that the following inequality is satisfied:

$$\langle F(x^*), x - x^* \rangle \geq 0, \ \forall x \in F(T).$$

5. Conclusion

We propose a new class of system of generalized hierarchical variational inequality problems in Hilbert spaces, that seems to be a useful extension of the class of hierarchical variational inequality. Further, we established some fundamental properties belonging to this class. Based on these properties and well-known result due to concepts of Mainge′s, we obtained some existence of the solutions of system of generalized hierarchical variational inequality problems. Also, we established a result, that may be viewed as an applications for system of generalized hierarchical variational inequality problems.

6. Acknowledgements

The author wishes to express his gratitude to the Editor Professor Narin Petrot, the two anonymous referees for their detailed comments and useful suggestions that allowed to improve significantly the presentation of this paper.

References

1. M. K. Ahmad and Salahuddin, A stable perturbed algorithms for a new class of generalized nonlinear implicit quasi variational inclusions in Banach spaces, Advances in Pure Mathematics, **2(2)**, 2012, 139–148.
2. J. Bracken and J.M. McGill, Mathematical programs with optimization problems in the constraints, Operation Research, **21** 1973, 37-–44.
3. J. Bracken and J.M. McGill, A method for solving mathematical programs with nonlinear programs in the constraints, Operation Research, **22**, 1974, 1097-–1101.
4. S. S. Chang, J. K. Kim, H. W. Lee and C. K. Chun, On the hierarchical variational inclusion problems in Hilbert spaces, Fixed Point Theory and Applications, **2013**, Article 179 (2013).
5. F. Cianciaruso, V. Calao, L. Muglia and H. K. Xu, On an implicit hierarchical fixed point approach to variational inequalities, Bulletin of Australian Mathematical Society, **80 (1)**, 2009, 117–124.

6. X. P. Ding and Salahuddin, On a system of general nonlinear variational inclusions in Banach spaces, Applied Mathematics and Mechanics, **36(12)**, 2015, 1663–1672.
7. R. Kraikaew and S. Saejung, On Mainge approach for hierarchical optimization problems, Journal of Optimization theory and Applications, **154(1)**, (2012), 71–87.
8. J. L. lions and G. Stampacchia, Variational Inequalities, Communication in Pure Applid Mathematics, **20**, 1967, 493–517.
9. P. E. Mainge, New approach to solving a system of variational inequalities and hierarchical problems, Journal of Optimization theory and Applications, **138(3)**, 2008, 459–477.
10. P. E. Mainge, The viscosity approximation process for quasi nonexpansive mappings in Hilbert spaces, Computational Mathematics and Applications, **59(1)**, 2010, 74–79.
11. P. E. Mainge and A. Moudafi, Strong convergence of an iterative method for hierarchical fixed point problems, Pacific Journal of Optimizations, **3(3)**, 2007, 529–538.
12. G. Marino, V. Colao, L. Muglia and Y. Yao, Krasnoselski-Mann iteration for hierarchical fixed points and equilibrium problems, Bulletin Australian Mathematical Society, **79(2)**, 2009, 187–200.
13. A. Moudafi and P. Mainge, Towards viscosity approximations of hierarchical fixed point problems, Fixed Point Theory and Applications, **2006**, Article 95453 (2006), 10 pages.
14. Salahuddin, Convergence analysis for Hierarchical optimizations, Nonlinear Analysis Forum, **20(8)**, 2015, 229–239.
15. W. Takahashi, Introductionto Nonlinear and Convex Analysis, Yakohama Publishers, Yakohama, Japan, 2009.
16. Y. K. Tang, S. S. Chang and Salahuddin, A System of Nonlinear Set Valued Variational Inclusions, SpringerPlus, **3**(2014) ,318, Doi:10.1186/2193-180-3-318.
17. N. Wairojjana and P. Kumam, Existence and algorithm for the systems of Hierarchical variational inclusion problems, Abstract Applied Analysis, **2014**, ID 589679 (2014), 10 pages.
18. H. K. Xu, Viscosity method for hierarchical fixed point approach to variational inequalities, Taiwanese Journal of Mathematics, **14(2)**, 2010, 463–478.
19. Y. Yao, J. C. Cho and Y. C. Liou, Iterative algorithms for hierarchical fixed points problems and variational inequalities, Mathematics Computer Modelling, **52(9-10)**, 2010, 1697–1705.
20. S. Zhang, J. H. W. Lee and C. K. Chan, Algorithms of common solutions to quasi variational inclusions and fixed point problems, Applied mathematics Mechanics, **29(5)**, 2008, 571–581.