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COMMON FIXED POINTS OF GENERALIZED CO-CYCLIC WEAKLY CONTRACTIVE MAPS

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ABSTRACT. In this paper, we introduce generalized co-cyclic weakly contractive maps and prove the existence of common fixed points in complete metric spaces. We deduce some corollaries from our main results and provide examples in support of our results.

KEYWORDS: cyclic representation, co-cyclic representation, co-cyclic weakly contractive maps, generalized co-cyclic weakly contractive maps.

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1. Introduction

In 1997, Alber and Guerre-Delabriere [2] introduced weakly contractive mappings as a generalization of contraction maps and proved some fixed point results in Hilbert space setting. In 2001, Rhoades [7] extended this concept to Banach spaces. In 2003, Kirk, Srinivasan and Veeramani [6] introduced cyclic contractions and proved fixed point results for not necessarily continuous mappings. In 2013, Harjani, Lopez and Sadarangani [4] proved existence of fixed points of continuous cyclic weakly contractive selfmaps in compact metric spaces. Recently, Alemayehu [1] introduced co-cyclic weakly contractive maps and proved common fixed points results in compact metric spaces.

In this paper, we denote
$\tau = \{\varphi : [0,\infty) \to [0,\infty)/\varphi$ is non-decreasing, $\varphi(0) = 0, \varphi(t) > 0$ for $t > 0\}$, and
$\Phi = \{\varphi : [0,\infty) \to [0,\infty)/\varphi$ is continuous on $[0,\infty)$and $\varphi(t) = 0 \Leftrightarrow t = 0\}$.

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Definition 1.1. [8] Let X be a non-empty set, m a positive integer and $f : X \to X$ a selfmap and $X = \cup_{i=1}^{m} A_i$ is said to be a *cyclic representation of X with respect to the map f* if

(i) $A_i,\ i = 1, 2, ..., m$ are non-empty subsets of X
(ii) $f(A_1) \subset A_2, ..., f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Definition 1.2. [1] Let X be a non-empty set, m a positive integer and $T,\ f : X \to X$ be two selfmaps. $X = \cup_{i=1}^{m} A_i$ is said to be a *co-cyclic representation of X w.r.t. T* and f if

(i) $A_i,\ i = 1, 2, ..., m$ are non-empty subsets of X
(ii) $T(A_1) \subset f(A_2), ..., T(A_{m-1}) \subset f(A_m)$ and $T(A_m) \subset f(A_1)$.

Here we note that, by taking f as the identity map, we get a cyclic representation of X with respect to the selfmap T introduced by Rus [8].

Definition 1.3. [1] Let (X, d) be a metric space, m a positive integer, $A_1, A_2, ..., A_m$ closed non-empty subsets of X and $X = \cup_{i=1}^{m} A_i$. Let $f, T : X \to X$ be two selfmaps. If

(i) $X = \cup_{i=1}^{m} A_i$ is a co-cyclic representation of X w. r. t. T and f, and
(ii) there exists $\varphi \in \tau$ such that

$$d(Tx, Ty) \le d(fx, fy) - \varphi(d(fx, fy)) \tag{1.1}$$

for any $x \in A_i$ and $y \in A_{i+1}$, where $A_{m+1} = A_1$

then we say that T is a *co-cyclic weakly contractive map w.r.t. f with $\varphi \in \tau$.*

Definition 1.4. [5] Two self mappings f and T of a metric space (X, d) are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $fu = Tu$ for $u \in X$ then $fTu = Tfu$.

Remark 1.5. In [1], maps f, T satisfying (i) and (ii) of Definition 2.3 are mentioned as 'co-cyclic weak contractions'. But the terminology 'T is a co-cyclic weakly contractive map w. r. t. f' is more appropriate as the inequality (1.1) is indicating 'weakly contractive' property. For more details on weakly contractive maps, we refer [2] and [7].

Alemayehu [1] proved the following theorem in compact metric spaces.

Theorem 1.1. [1] *Let (X, d) be a compact metric space and let $T,\ f : X \to X$ be two selfmaps. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty subsets of X, $X = \cup_{i=1}^{m} A_i$ and T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \tau$.*
If the pair of operators (f, T) is weakly compatible on X, then f and T have a unique common fixed point in X.

Unfortunately, the proof of Theorem 2.1 contains many argumental errors. For more details, we refer [3]. A rectified version of this theorem is the following.

Theorem 1.2. [3] *Let (X, d) be a compact metric space and let $T,\ f : X \to X$ be two selfmaps. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X, $X = \cup_{i=1}^{m} A_i$ and T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \tau$. If f is one-one and T and f are continuous, then f and T have a coincidence point in X. Further, if the maps f and T are weakly compatible then f and T have a unique common fixed point in X.*

In Definition 1.3, if (i) holds and (ii) holds with $\varphi \in \Phi$ then we say that T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \Phi$.

In Section 2, we prove the existence of common fixed points of a pair of co-cyclic weakly contractive maps with $\varphi \in \Phi$ in complete metric spaces. In Section 3, we define generalized co-cyclic weakly contractive maps w. r. t. f and T by using $\varphi \in \Phi$ and prove the existence of common fixed points in complete metric spaces. In Section 4, we deduce some corollaries from our main results and provide examples in support of our results.

In the following, we prove Theorem 2.2 for the case of complete metric spaces in which the selfmaps f and T are such that T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \Phi$.

2. Common fixed points of co-cyclic weakly contractive maps

Theorem 2.1. *Let (X, d) be a complete metric space. Suppose that m is a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X, $X = \cup_{i=1}^m A_i$. Let $T,\ f : X \to X$ be two selfmaps. Suppose that T is a co-cyclic weakly contractive map w. r. t. f with $\phi \in \Phi$. If f is one-one and $f(A_i)$ is closed, then there exists $z \in \cap_{i=1}^m A_i$ such that z is a coincidence point of f and T.*

Proof. Let $x_0 \in X = \cup_{i=1}^m A_i$. Then $x_0 \in A_i$ for some $i \in \{1, 2, 3, .., m\}$. Then $Tx_0 \in T(A_i) \subset f(A_{i+1})$ and hence $Tx_0 = fx_1 \in f(A_{i+1})$ for some $x_1 \in A_{i+1}$. Now, since $Tx_1 \in T(A_{i+1}) \subset f(A_{i+2})$, we have $Tx_1 = fx_2$ for some $x_2 \in A_{i+2}$. On continuing this process, we get a sequence $\{x_n\} \subset X$ such that

$$Tx_n = fx_{n+1} \text{ for all } n = 1, 2, ... \tag{2.1}$$

Hence, for each n, there exists a positive integer $i_n \in \{1, 2, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ satisfying

$$Tx_n = fx_{n+1}. \tag{2.2}$$

If there exists $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, then we have $Tx_{n_0+1} = Tx_{n_0} = fx_{n_0+1}$ so that f and T have a coincidence point x_{n_0+1}.
Hence, w. l. g., we assume that $x_n \neq x_{n+1}$ for all $n = 1, 2, ...$. Then $fx_n \neq fx_{n+1}$ for all n. Further, from the construction of $\{x_n\}$, we have $Tx_n \neq Tx_{n+1}$ for all $n = 1, 2, ...$.

Now, by (2.2) and since T is a co-cyclic weakly contractive map w. r. t. f with $\varphi \in \Phi$, we have

$$d(fx_n, fx_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq d(fx_{n-1}, fx_n) - \varphi(d(fx_{n-1}, fx_n)) \tag{2.3}$$

for each $n = 1, 2, ...$. Therefore
$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$ for all $n \geq 1$.
Hence $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of non-negative reals and hence converges to a limit r (say), $r \geq 0$.
Now, on letting $n \longrightarrow \infty$ in (2.3) and using the continuity of ϕ we have
$r \leq r - \lim\limits_{n \longrightarrow \infty} \varphi(d(fx_{n-1}, fx_n)) = r - \phi(r)$ and hence $\varphi(r) = 0$ so that $r = 0$.
We now prove that $\{fx_n\}$ is a Cauchy sequence in X.
For this purpose, first we show that for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ with $p - q \equiv 1 \pmod{m}$, then $d(fx_p, fx_q) < \epsilon$.

If it is false, then there exists an $\epsilon > 0$ such that for each $n \in \mathbb{N}$ we can find sequences $\{p_n\}$ and $\{q_n\}$ such that $p_n > q_n \geq n$ with $p_n - q_n \equiv 1 \pmod m$ and $d(fx_{p_n}, fx_{q_n}) \geq \epsilon$.

Now, let n be such that $n > 2m$. Then for $q_n \geq n$ we choose p_n such that p_n is the smallest positive integer greater than q_n satisfying $p_n - q_n \equiv 1 \pmod m$ and $d(fx_{q_n}, fx_{p_n}) \geq \epsilon$, which implies that $d(fx_{q_n}, fx_{p_{n-m}}) < \epsilon$.

By using the triangular inequality, we have
$\epsilon \leq d(fx_{q_n}, fx_{p_n})$
$\leq d(fx_{q_n}, fx_{p_{n-m}}) + \Sigma_{i=1}^{m} d(fx_{p_n-i}, fx_{p_n-i+1}) < \epsilon + \Sigma_{i=1}^{m} d(fx_{p_n-i}, fx_{p_n-i+1})$.
On letting $n \longrightarrow \infty$, by using $\lim\limits_{n\longrightarrow\infty} d(fx_n, fx_{n+1}) = 0$ we have

$$\lim_{n\longrightarrow\infty} d(fx_{q_n}, fx_{p_n}) = \epsilon. \tag{2.4}$$

Again, by the triangular inequality, we have
$\epsilon \leq d(fx_{q_n}, fx_{p_n})$
$\leq d(fx_{q_n}, fx_{q_n+1}) + d(fx_{q_n+1}, fx_{p_n+1}) + d(fx_{p_n+1}, fx_{p_n})$
$\leq d(fx_{q_n}, fx_{q_n+1}) + d(fx_{q_n+1}, fx_{q_n}) + d(fx_{q_n}, fx_{p_n}) + d(fx_{p_n}, fx_{p_n+1}) + d(fx_{p_n+1}, fx_{p_n})$
$\leq 2d(fx_{q_n}, fx_{q_n+1}) + d(fx_{q_n}, fx_{p_n}) + 2d(fx_{p_n+1}, fx_{p_n})$
On letting $n \longrightarrow \infty$ and by using (2.4), we have

$$\lim_{n\longrightarrow\infty} d(fx_{q_n+1}, fx_{p_n+1}) = \epsilon. \tag{2.5}$$

In fact, x_{q_n} and x_{p_n} lie in different adjacently labelled sets A_i and A_{i+1}, for $1 \leq i \leq m$. Now by using the inequality (1.1) with $\varphi \in \Phi$ we have
$d(fx_{q_n+1}, fx_{p_n+1}) = d(Tx_{q_n}, Tx_{p_n})$

$$\leq d(fx_{q_n}, fx_{p_n}) - \varphi(d(fx_{q_n}, fx_{p_n})). \tag{2.6}$$

On letting $n \longrightarrow \infty$, by using the continuity property of φ in (2.6) and using (2.4) we have
$\epsilon \leq \epsilon - \phi(\epsilon)$ so that $\epsilon = 0$,
a contradiction. So we conclude that our assumption is wrong. Therefore given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with $p - q \equiv 1 \pmod m$ then

$$d(fx_p, fx_q) \leq \frac{\epsilon}{2}. \tag{2.7}$$

Since $\lim\limits_{n\longrightarrow\infty} d(fx_n, fx_{n+1}) = 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(fx_n, fx_{n+1}) \leq \frac{\epsilon}{2m} \tag{2.8}$$

for each $n \geq n_1$.

Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, ..., m\}$ such that $s - r \equiv k \pmod m$. We choose $j = m - k + 1$. Then, since $m + 1 \equiv 1 \pmod m$, we have $s+j-r = s+(m-k+1)-r = (s-r)+(m+1)-k \equiv 1 \pmod m$.
$d(fx_r, fx_s) \leq d(fx_r, fx_{s+j}) + d(fx_{s+j}, fx_{s+j-1}) + ... + d(fx_{s+1}, fx_s)$
$d(fx_r, fx_s) \leq \frac{\epsilon}{2} + (j+1).\frac{\epsilon}{2} \leq \frac{\epsilon}{2} + m.\frac{\epsilon}{2m} = \epsilon$.

Therefore, given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(fx_r, fx_s) \leq \epsilon$ for all $r, s \geq n$. Hence $\{fx_n\}$ is a Cauchy sequence. Since (X, d) is complete, we have $\lim\limits_{n\longrightarrow\infty} fx_n = x$ for some $x \in X$. Since $x_0 \in X = \cup_{i=1}^{m} A_i$ implies $x_0 \in A_i$ for some i and $x_l \in A_{i+l}$ for all $l \in \{1, 2, ..., m\}$. In particular, $x_m \in A_{i+m} = A_i$ and $x_{2m} \in A_i, ..., x_{km} \in A_i$ for all $k = 0, 1, 2, ...$. Since $\{x_{km}\} \subset A_i$, we have $\{f(x_{km})\} \subset f(A_i)$. Since $f(A_i)$ is closed and $\{f(x_{km})\}$ is a subsequence of $\{f(x_n)\}$ we have $fx_{km} \longrightarrow x$ as $k \longrightarrow \infty$ and $x \in f(A_i)$.

We now show that $x \in \cap_{i=1}^{m} f(A_i)$. We have $x_{l+km} \in A_{i+l+km} = A_{i+l}$ for all $l = 1, 2, ..., m$ which implies that $f(x_{l+km}) \in f(A_{i+l})$ for all l. So $i+l \equiv i_0 \pmod{m}$ for some $i_0 \in \{1, 2, ..., m\}$. Therefore $f(x_{l+km}) \in f(A_{i_0})$. Now $l \in \{1, 2, ..., m\}$ implies $f(x_{l+km}) \longrightarrow x$ as $k \longrightarrow \infty$. Since $f(A_{i_0})$ is closed, we have $x \in f(A_{i_0})$. Note that, for any $i \in \{1, 2, ..., m\}$ we have $\{i + l/l = 1, 2, ..., m\} = \{1, 2, ..., m\}$ under congruent modulo m. Since this is true for any $l \in \{1, 2, ..., m\}$ it follows that $x \in \cap_{i=1}^{m} f(A_i)$. Hence $x \in f(A_i)$ for each $i = 1, 2, ..., m$ so that there exists $z_i \in A_i$ such that $x = fz_i$ for each $i = 1, 2, ..., m$. *i.e.*, $x = fz_1 = fz_2 = ... = fz_m$ for some $z_1 \in A_1, z_2 \in A_2, ..., z_m \in A_m$. Since f is one-one, we have $z_1 = z_2 = ... = z_m = z$ (say). Hence $x = fz,\ z \in \cap_{i=1}^{m} A_i$.

Now we prove that z is a coincidence point of f and T.
By using the inequality (1.1) with $\varphi \in \Phi$, we have

$$\begin{aligned} d(fx_{l+km}, Tz) &= d(Tx_{l+km-1}, Tz) \\ &\leq d(fx_{l+km-1}, fz) - \varphi(d(fx_{l+km-1}, fz)), \end{aligned}$$

since $x_{l+km-1} \in A_{l+km-1}$ and $z \in A_{l+km}$.
On letting $k \longrightarrow \infty$, we have

$$d(x, Tz) \leq d(x, Tz) - \varphi(d(x, Tz)).$$

Hence $d(fz, Tz) \leq d(fz, Tz) - \varphi(d(fz, Tz))$ which implies that $\varphi(d(fz, Tz)) = 0$. Since $\varphi \in \Phi$ we have $fz = Tz$ and z is a coincidence point of f and T in X. □

Theorem 2.2. *In addition to the hypotheses of Theorem* 3.1*, if the maps T and f are weakly compatible then T and f have a unique common fixed point.*

Proof. By Theorem 3.1, we have
$Tz = fz = u$ (say). Since T and f are weakly compatible, we have
$Tu = Tfz = fTz = fu$ implies $Tu = fu$.
Now, we prove that $Tu = u$.
Since $Tz \in X = \cup_{i=1}^{m} A_i$ implies $Tz \in A_i$ for some i and $z \in \cap_{i=1}^{m} A_i$, we have $z \in A_i$ for all $i \in \{1, 2, ..., m\}$.
Now, by the inequality (1.1) with $\varphi \in \Phi$ we have

$$\begin{aligned} d(Tz, TTz) &\leq d(fz, fTz) - \varphi(d(fz, fTz)) \\ &\leq d(Tz, TTz) - \varphi(d(Tz, TTz)) \end{aligned}$$

so that $Tz = TTz$ and hence $u = Tu = fu$.
Therefore u is a common fixed point of f and T.
We now show that $u \in \cap_{i=1}^{m} A_i$ since $Tu = fu = u$, we have $u \in A_i$ for some i.

$$\begin{aligned} \text{Now, } u \in A_i &\Rightarrow Tu \in T(A_i) \\ &\Rightarrow Tu \in T(A_i) \subset f(A_{i+1}) \\ &\Rightarrow Tu = fv \in f(A_{i+1}) \text{ for some } v \in A_{i+1}. \end{aligned}$$

Therefore $fu = fv$ for some $v \in A_{i+1}$, since f is one-one we have $u = v \in A_{i+1}$ so that $u \in A_{i+1}$. By repeating the same argument, we get $u \in \cap_{i=1}^{m} A_i$.

In the following, we prove the uniqueness of common fixed point of T and f. Let y and z be two common fixed points of T and f. Then we have $Ty = fy = y$ and $Tz = fz = z$ and $y, z \in \cap_{i=1}^{m} A_i$.
From the inequality (1.1) with $\varphi \in \Phi$ we have

$$\begin{aligned} d(y, z) &= d(Ty, Tz) \\ &\leq d(fy, fz) - \varphi(d(fy, fz)) \\ &\leq d(y, z) - \varphi(d(y, z)) \text{ so that } \varphi(d(y, z)) = 0. \end{aligned}$$

Since $\varphi \in \Phi$ it follows that $y = z$. Therefore f and T have a unique common fixed point in X. □

3. Common fixed points of generalized co-cyclic weakly contractive maps

In the following, we introduce generalized co-cyclic weakly contractive maps by using an element $\varphi \in \Phi$.

Definition 3.1. Let (X, d) be a metric space, m a positive integer, $A_1, A_2, ..., A_m$ closed non-empty subsets of X and $X = \cup_{i=1}^m A_i$. Let $f, T : X \to X$ be two selfmaps. If

(i) $\cup_{i=1}^m A_i$ is a co-cyclic representation of X w. r. t. f and T

(ii) there exists $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)) \tag{3.1}$$

for any $x \in A_i$ and $y \in A_{i+1}$, $A_{m+1} = A_1$, where
$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}(d(fx, Ty) + d(Tx, fy))\}$

then we say that T is a *generalized co−cyclic weakly contractive map w. r. t f with* $\phi \in \Phi$.

Theorem 3.1. *Let (X, d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X, $X = \cup_{i=1}^m A_i$. Let T, $f : X \to X$ be two selfmaps. Suppose that T is a generalized co-cyclic weakly contractive map w. r. t. f. If f is one-one and $f(A_i)$ is closed, then there exist $z \in \cap_{i=1}^m A_i$ such that z is a coincidence point of f and T.*

Proof. Let $x_0 \in X = \cup_{i=1}^m A_i$. Then proceeding as in the proof of Theorem 2.1, we obtain a sequence $\{x_n\} \subset X$ satisfying (2.1) and (2.2). Without loss of generality we assume that $x_n \neq x_{n+1}$ for all $n = 1, 2, ...$. Then $fx_n \neq fx_{n+1}$ for all n. Further, from the construction of $\{x_n\}$, we have $Tx_n \neq Tx_{n+1}$ for all $n = 1, 2, ...$.
Now, by applying the inequality (3.1) to the sequence $\{fx_n\}$ we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq M(x_{n-1}, x_n) - \varphi(M(x_{n-1}, x_n)) \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \\ &\qquad \tfrac{1}{2}(d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1}))\} \\ &= \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad \tfrac{1}{2}(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n))\} \\ &\leq \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad \tfrac{1}{2}(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}))\} \\ &= \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\} \leq M(x_{n-1}, x_n) \end{aligned}$$

so that
$M(x_{n-1}, x_n) = max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}$.
If $\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\} = d(fx_n, fx_{n+1})$
then, from (3.2) we have
$d(fx_n, fx_{n+1}) \leq d(fx_n, fx_{n+1}) - \varphi(d(fx_n, fx_{n+1})) < d(fx_n, fx_{n+1})$,
a contradiction.
Hence $M(x_{n-1}, x_n) = d(fx_{n-1}, fx_n)$.
Now, from (3.2) we have

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq d(fx_{n-1}, fx_n) - \varphi(d(fx_{n-1}, fx_n)). \\ &\leq d(fx_{n-1}, fx_n). \end{aligned} \tag{3.3}$$

Therefore $\{d(fx_n, fx_{n+1})\}$ is a decreasing sequence of non-negative reals and hence converges to a limit r (say), $r \geq 0$.

Now on letting $n \longrightarrow \infty$ in (3.3), we have
$r \leq r - \phi(r)$, and hence $\phi(r) = 0$ so that $r = 0$.
We now prove that $\{fx_n\}$ is a Cauchy sequence in X.
Here onwards proceeding as in the proof of Theorem 2.1 we get (2.4) and (2.5).
In fact, x_{q_n} and x_{p_n} lie in different adjacently labelled sets A_i and A_{i+1}, for $1 \leq i \leq m$. Now by using the inequality (3.1), we have

$$d(fx_{q_n+1}, fx_{p_n+1}) = d(Tx_{q_n}, Tx_{p_n})$$

$$\leq M(x_{q_n}, x_{p_n}) - \varphi(M(x_{q_n}, x_{p_n})) \tag{3.4}$$

where

$$\begin{aligned}\epsilon \leq d(fx_{p_n}, fx_{q_n}) \leq M(x_{q_n}, x_{p_n}) &= \max\{d(fx_{q_n}, fx_{p_n}), d(fx_{q_n}, Tx_{q_n}), d(fx_{p_n}, Tx_{p_n}), \\ &\quad \tfrac{1}{2}(d(fx_{q_n}, Tx_{p_n}) + d(fx_{p_n}, Tx_{q_n}))\} \\ &= \max\{d(fx_{q_n}, fx_{p_n}), d(fx_{q_n}, fx_{q_n+1}), d(fx_{p_n}, fx_{p_n+1}), \\ &\quad \tfrac{1}{2}(d(fx_{q_n}, fx_{p_n+1}) + d(fx_{p_n}, fx_{q_n+1}))\} \\ &\leq \max\{d(fx_{q_n}, fx_{p_n}), d(fx_{q_n}, fx_{q_n+1}), d(fx_{p_n}, fx_{p_n+1}), \tfrac{1}{2}(d(fx_{q_n}, fx_{p_n}) \\ &\quad + d(fx_{p_n}, fx_{p_n+1}) + d(fx_{p_n}, fx_{q_n}) + d(fx_{q_n}, fx_{q_n+1}))\} \longrightarrow \epsilon \text{ as } n \longrightarrow \infty\end{aligned}$$

so that $\lim_{n\longrightarrow\infty} M(x_{q_n}, x_{p_n}) = \epsilon$.
Hence, on letting $n \longrightarrow \infty$, using the continuity property of φ in (3.4) and using (2.4) and (2.5) we get that

$$\begin{aligned}\epsilon = \lim_{n\longrightarrow\infty} d(fx_{q_n+1}, fx_{p_n+1}) &= \lim_{n\longrightarrow\infty} M(x_{q_n}, x_{p_n}) - \lim_{n\longrightarrow\infty} \varphi(M(x_{q_n}, x_{p_n})) \\ &= \epsilon - \varphi(\epsilon),\end{aligned}$$

a contradiction.
Therefore, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $p, q \geq n_0$ with
$p - q \equiv 1 \pmod m$ then

$$d(fx_p, fx_q) \leq \frac{\epsilon}{2}. \tag{3.5}$$

Since $\lim_{n\longrightarrow\infty} d(fx_n, fx_{n+1}) = 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(fx_n, fx_{n+1}) \leq \frac{\epsilon}{2m} \text{ for each } n \geq n_1. \tag{3.6}$$

Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, ..., m\}$ such that $s - r \equiv k \pmod m$. We choose $j = m - k + 1$. Then, since $m + 1 \equiv 1 \pmod m$, we have $s+j-r = s+(m-k+1)-r = (s-r)+(m+1)-k \equiv 1 \pmod m$. Now

$$\begin{aligned}d(fx_r, fx_s) &\leq d(fx_r, fx_{s+j}) + d(fx_{s+j}, fx_{s+j-1}) + ... + d(fx_{s+1}, fx_s) \\ &\leq \tfrac{\epsilon}{2} + (j+1).\tfrac{\epsilon}{2} \leq \tfrac{\epsilon}{2} + m.\tfrac{\epsilon}{2m} = \epsilon.\end{aligned}$$

Therefore, given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(fx_r, fx_s) \leq \epsilon$ for all $r, s \geq n$. Hence $\{fx_n\}$ is a Cauchy sequence. Since (X, d) is complete $\lim_{n\longrightarrow\infty} fx_n = x$ for some $x \in X$. From here onwards, again proceeding as in the proof of Theorem 3.1 we have $x = fz,\ z \in \cap_{i=1}^m A_i$.

Now we prove that z is a coincidence point of f and T.
By using the inequality (3.1), we have

$$d(fx_{l+km}, Tz) = d(Tx_{l+km-1}, Tz)$$

$$\leq M(x_{l+km-1}, z) - \varphi(M(x_{l+km-1}, z)) \tag{3.7}$$

where

$$\begin{aligned}M(x_{n_k}, z) &= \max\{d(fx_{l+km-1}, fz), d(fx_{l+km-1}, Tx_{l+km-1}), d(fz, Tz), \\ &\quad \tfrac{1}{2}(d(fx_{l+km-1}, Tz) + d(fz, Tx_{l+km-1}))\} \\ &= \max\{d(fx_{l+km-1}, fz), d(fx_{l+km-1}, fx_{l+km}), d(fz, Tz), \\ &\quad \tfrac{1}{2}(d(fx_{l+km-1}, Tz) + d(fz, Tx_{l+km}))\},\end{aligned}$$

since $x_{l+km-1} \in A_{l+km-1}$ and $z \in A_{l+km}$.
On letting $k \longrightarrow \infty$, we have
$\lim_{k\longrightarrow\infty} M(x_{n_k}, z) = d(fz, Tz)$.
On letting $k \longrightarrow \infty$ in (3.7), we have
$d(x, Tz) \leq d(x, Tz) - \varphi(d(x, Tz))$.
Hence $d(fz, Tz) \leq d(fz, Tz) - \varphi(d(fz, Tz))$ which implies that $\varphi(d(fz, Tz)) = 0$ so that $fz = Tz$.

□

Theorem 3.2. *In addition to the hypotheses of Theorem* 3.1*, if the maps T and f are weakly compatible then T and f have a unique common fixed point.*

Proof. From the proof of Theorem 3.1 we have $Tz = fz = u$ (say). Since T and f are weakly compatible, we have
$Tu = Tfz = fTz = fu$.
Now we prove that $Tu = u$.
Since $Tz \in X = \cup_{i=1}^{m} A_i$ implies $Tz \in A_i$ for some i and $z \in \cap_{i=1}^{m} A_i$, we have $z \in A_i$ for all $i \in \{1, 2, ..., m\}$.
Now, by the inequality (3.1) we have

$$d(Tz, TTz) \leq M(z, Tz) - \varphi(M(z, Tz)) \tag{3.8}$$

where

$$\begin{aligned} M(z, Tz) &= \max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), \tfrac{1}{2}(d(fz, TTz) + d(fTz, Tz))\} \\ &= \max\{d(Tz, TTz), d(Tz, Tz), d(TTz, TTz), d(Tz, TTz)\} \\ &= d(Tz, TTz). \end{aligned}$$

From (3.8), we have
$d(Tz, TTz) \leq d(Tz, TTz) - \varphi(d(Tz, TTz))$ so that $Tz = TTz$ and hence $u = Tu = fu$.
Therefore u is a common fixed point of f and T.
We now show that $u \in \cap_{i=1}^{m} A_i$. Since $Tu = fu = u$, we have $u \in A_i$ for some i.

$$\begin{aligned} \text{Now, } u \in A_i &\Rightarrow Tu \in T(A_i) \\ &\Rightarrow Tu \in T(A_i) \subset f(A_{i+1}) \\ &\Rightarrow Tu = fv \in f(A_{i+1}) \text{ for some } v \in A_{i+1}. \end{aligned}$$

Therefore $fu = fv$ for some $v \in A_{i+1}$, since f is one-one we have $u = v \in A_{i+1}$ so that $u \in A_{i+1}$. By repeating the same argument, we get $u \in \cap_{i=1}^{m} A_i$.

Uniqueness of common fixed point of T and f follows from the inequality (3.1) trivially. □

4. Corollaries and Examples

By choosing $f = I_X$ in Theorem 2.1, we have the following corollary.

Corollary 4.1. *Let (X, d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X and $X = \cup_{i=1}^{m} A_i$. If $T : X \to X$ is a mapping such that*

(i) *$\cup_{i=1}^{m} A_i$ is a cyclic representation of X w. r. t. T*
(ii) *there exists $\varphi \in \Phi$ such that $d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}, A_{m+1} = A_1$*

then there exists $z \in \cap_{i=1}^{m} A_i$ such that $Tz = z$.

By choosing $f = I_X$ in Theorem 3.1, we have the following corollary.

Corollary 4.2. *Let (X, d) be a complete metric space. Suppose that m a positive integer, $A_1, A_2, ..., A_m$ are non-empty closed subsets of X and $X = \cup_{i=1}^m A_i$. If $T : X \to X$ is a mapping such that*

(i) $\cup_{i=1}^m A_i$ *is a cyclic representation of X w. r. t. T*

(ii) *there exists $\varphi \in \Phi$ such that $d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y))$ for any $x \in A_i$ and $y \in A_{i+1}, A_{m+1} = A_1$ where*
$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$

then there exists $z \in \cap_{i=1}^m A_i$ such that $Tz = z$.

Example 4.3. Let $X = \mathbb{R}$ with the usual metric. Let $A_1 = (-\infty, 2]$ and $A_2 = [2, \infty)$. We define $T,\ f : X \longrightarrow X$ by $Tx = \frac{2+x}{2}$ and $fx = 6 - 2x$. We define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ by $\varphi(t) = \frac{t}{2+t}, t \geq 0$. Then $\varphi \in \Phi$. Clearly, $X = A_1 \cup A_2$ is co-cyclic representation of X w.r.t. T and f. Now we verify the inequality (1.1) in the following:
For $x \in A_1$ and $y \in A_2$, then $d(Tx, Ty) = |\frac{x}{2} - \frac{y}{2}|$ and $d(fx, fy) = |2x - 2y|$
$d(Tx, Ty) = |\frac{x}{2} - \frac{y}{2}| \leq |2x - 2y| - \varphi(|2x - 2y|) = d(fx, fy) - \varphi(d(fx, fy))$.
Clearly, T and f are weakly compatible and satisfy all the hypotheses of Theorem 2.2 and 2 is the unique common fixed point of T and f and $2 \in A_1 \cap A_2$.

In the following, we provide examples in support of the results obtained in Section 4.

Example 4.4. Let $X = \{0, 2, 3, 5\}$ with the usual metric. Let $A_1 = \{0, 2\}$ and $A_2 = \{2, 3, 5\}$. We define $T,\ f : X \longrightarrow X$ by $T0 = T2 = 2, T3 = 0, T5 = 2;\ f0 = 0,\ f2 = 2,\ f3 = 5$ and $f5 = 3$. We define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{1+t} & \text{if } 0 \leq t \leq 5 \\ \frac{5}{6}e^{-(t-5)} & \text{if } 5 \leq t < \infty. \end{cases}$$

Here we observe that $\varphi \in \Phi$. Now we verify the inequality (3.1) in the following:
<u>Case</u> (i): $x = 0$ and $y = 3$
then $d(T0, T3) = 2$ and $M(0, 3) = 5$
$d(Tx, Ty) = d(T0, T3) = 2$
$\leq 5 - \varphi(5) = M(0, 3) - \varphi(M(0, 3)) = M(x, y) - \varphi(M(x, y))$.
<u>Case</u> (ii): $x = 2$ and $y = 3$
then $d(T2, T3) = 2$ and $M(2, 3) = 5$
$d(Tx, Ty) = d(T2, T3) = 2$
$\leq 5 - \varphi(5) = M(2, 3) - \varphi(M(2, 3)) = M(x, y) - \varphi(M(x, y))$.
In the other cases the inequality (3.1) trivially holds.
Clearly, T and f are weakly compatible and satisfy all the hypotheses of Theorem 3.2 and 2 is the unique common fixed point of T and f and $2 \in A_1 \cap A_2$.

If we relax the weakly compatibility property of f and T of Theorem 3.2 then T and f may not have a common fixed point.

Example 4.5. Let $X = \{1, 2, 3, 4\}$ with the usual metric. Let $A_1 = \{1, 2, 3\}$ and $A_2 = \{2, 3, 4\}$. We define $T,\ f : X \longrightarrow X$ by $T1 = 2, T2 = T3 = 3, T4 = 4$;
$f1 = 4, f2 = 3,\ f3 = 2$ and $f4 = 1$. We define $\varphi : [0, \infty) \longrightarrow [0, \infty)$ by

$$\varphi(t) = \begin{cases} \frac{t}{2+t} & \text{if } 0 \leq t \leq 4 \\ \frac{2}{3}e^{-(t-4)} & \text{if } 4 \leq t < \infty. \end{cases}$$

Then $\varphi \in \Phi$. Now we verify the inequality (3.1) in the following:
<u>Case</u> (i): $x = 1$ and $y = 2$

then $d(T1, T2) = 1$ and $M(1, 2) = 2$
$$d(Tx, Ty) = d(T1, T2) = 1$$
$$\leq 2 - \varphi(2) = M(1, 2) - \varphi(M(1, 2)) = M(x, y) - \varphi(M(x, y)).$$
<u>*Case*</u> *(ii)*: $x = 1$ and $y = 3$
then $d(T1, T3) = 1$ and $M(1, 3) = 2$
$$d(Tx, Ty) = d(T1, T3) = 1$$
$$\leq 2 - \varphi(2) = M(1, 3) - \varphi(M(1, 3)) = M(x, y) - \varphi(M(x, y)).$$
<u>*Case*</u> *(iii)*: $x = 1$ and $y = 4$
then $d(T1, T4) = 2$ and $M(1, 4) = 3$
$$d(Tx, Ty) = d(T1, T4) = 2$$
$$\leq 3 - \varphi(3) = M(1, 4) - \varphi(M(1, 4)) = M(x, y) - \varphi(M(x, y)).$$
<u>*Case*</u> *(iv)*: $x = 2$ and $y = 3$
In this case, the inequality (3.1) trivially holds.
<u>*Case*</u> *(v)*: $x = 2$ and $y = 4$
then $d(T2, T4) = 1$ and $M(2, 4) = 3$
$$d(Tx, Ty) = d(T2, T4) = 1$$
$$\leq 3 - \varphi(3) = M(2, 4) - \varphi(M(2, 4)) = M(x, y) - \varphi(M(x, y)).$$
<u>*Case*</u> *(vi)*: $x = 3$ and $y = 4$
then $d(T3, T4) = 1$ and $M(3, 4) = 3$
$$d(Tx, Ty) = d(T3, T4) = 1$$
$$\leq 3 - \varphi(3) = M(3, 4) - \varphi(M(3, 4)) = M(x, y) - \varphi(M(x, y))$$
Hence 2 is the coincidence point of T and f and $2 \in A_1 \cap A_2$. Here, we note that f and T are not weakly compatible, since $T2 = 3$ and $f2 = 3$ then $T(f(2)) = T(3) = 3$ and $f(T(2)) = f(3) = 2$ so that $T(f(2)) \neq f(T(2))$. Hence f and T satisfy all the hypotheses of Theorem 3.2 except the weakly compatible property of f and T, and we observe that f and T have no common fixed points in X.

Further, we observe that at $x = 1$ and $y = 2$
$d(T1, T2) = 1 \nleq 1 - \varphi(1) = d(f1, f2) - \varphi d(f1, f2)$ for any $\varphi \in \Phi$ and any $\phi \in \tau$. Therefore T is not a co-cyclic weakly contractive map w. r. t. f with any $\varphi \in \tau$. Hence Theorem 1.2 is not applicable.

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