S. Sharma & M. Chaudhary, J. Nonlinear Anal. Optim. Vol. 10(1) (2019), 35-47

Journal of Nonlinear Analysis and Optimization
Volume 10(1) (2019)
http://www.math.sci.nu.ac.th
ISSN : 1906-9685

J. Nonlinear Anal. Optim.

OPTIMALITY AND DUALITY FOR SET-VALUED FRACTIONAL PROGRAMMING INVOLVING GENERALIZED CONE INVEXITY

SUNILA SHARMA[1] AND MAMTA CHAUDHARY*[2]

[1] Department of Mathematics, Miranda House, University of Delhi, Delhi 110007, India
[2] Department of Mathematics, Satyawati College, University of Delhi, Delhi 110052, India

ABSTRACT. In this paper, a new class of generalized preinvex set-valued maps is introduced and its characterization in terms of their contingent epi-derivatives is obtained. Then we derive necessary and sufficient optimality conditions for a set-valued fractional programming problem using generalized cone invexity. Wolfe and Mond Weir type duals are formulated and various duality results are established.

KEYWORDS: Convex cone; set-valued map; optimality conditions; contingent epiderivative, duality.
AMS Subject Classification: Primary 90C46; Secondary 45N15.

1. INTRODUCTION

In the last decade, there has been increasing interest in the extension of vector optimization to set-valued optimization. The theory of set-valued optimization problems has wide applications in differential inclusion, variational inequality, optimal control, game theory, economic equilibrium problem, viability theory etc. Realizing the importance of the application of the set-valued maps, it becomes essential to study the notion of derivative for a set-valued map as it is most important for the formulation of optimality conditions. Aubin [1] introduced the notion of contingent derivative of a set-valued map. Later it was observed by Corley [4] that in case of contingent derivative, necessary and sufficient optimality conditions do not coincide under standard assumptions. Therefore, while characterizing optimality conditions, derivatives involving epigraph of set-valued maps were considered rather than their graph [7, 9]. These derivatives were termed as epiderivatives of different types. These epiderivatives differed either on the basis of their tangent cones or on the nature of the minimizers. Working in this direction, Jahn and Rauh [7] introduced contingent epiderivative in set-valued analysis.

* *Corresponding author.*
Email address : sunilaomhari@yahoo.co.in (Sunila Sharma), mam.gupta18@gmail.com (Mamta Chaudhary).
Article history : Received 1 July 2018 Accepted 3 November 2018.

Luc and Malivert [8] extended the study of invexity to set-valued maps and vector optimization problems with set-valued data. Sach and Craven [9] proved Wolfe type (WD) and Mond Weir type (MWD) duality theorems for set-valued optimization problems under invexity assumptions. Later on Bhatia and Mehra [3] introduced preinvex set-valued map as an extension of the notion of convex set-valued map. Bao-huai and San-Yang [2] investigated the KKT optimality conditions for preinvex set-valued optimization problems with the help of the generalized contingent epiderivative. Recently Das and Nahak [5, 6] and Yu and Kong [10] studied various types of generalized convexity notions for studying set-valued optimization problem via contingent epiderivatives.

In this paper, we study set-valued fractional programming problem (FP) and its associated parametric problem $(\mathrm{FP})_{\lambda^*}$. It is structured as follows: In Section 2 we recall some well known definitions and results. We also introduce a new class of generalized setvalued cone preinvex maps. Then we give characterization of these maps in terms of contingent epiderivatives. In section 3 necessary and sufficient optimality conditions are obtained for a weak minimizer of the problems (FP) and $(\mathrm{FP})_{\lambda^*}$. In section 4, we formulate Parametric type dual, Wolfe type dual and Mond Weir type dual of (FP) and establish weak duality, strong duality and converse duality results for the same.

2. Definitions and Preliminaries

Throughout this paper, let X and Y be real normed spaces. Let $K \subseteq Y$ be a closed, pointed convex cone with non-empty interior. Then its positive dual cone K^+ is defined as follows:

$$K^+ = \{y^* \in Y : \langle y^*, y\rangle \geq 0 \text{ for all } y \in K\}$$

Several kinds of tangent cones have been studied in literature. We now give the definition of tangent cone namely, the contingent (or Bouligand tangent) cone.

Definition 2.1. Let B be a non-empty subset of Y. Then the contingent (or Bouligand tangent) cone to B at $y^* \in B$ is denoted by $T(B, y^*)$ and is defined as

$$T(B, y^*) = \{y \in Y : \exists\, y_n \longrightarrow y^*, y_n \in B, t_n > 0, n \longrightarrow \infty \text{ such that } t_n(y_n - y^*) \longrightarrow y\}$$

or

$$T(B, y^*) = \{y \in Y : \exists\ y_n \longrightarrow y^*, y_n \in B, t_n \downarrow 0 \text{ such that } y^* + t_n y_n \in B\}$$

Let $F : X \longrightarrow 2^Y$ be a set-valued map where X and Y are real normed spaces. Let the space Y be partially ordered by a closed convex pointed cone $K \subseteq Y$ with nonempty interior. The domain, graph and epigraph of F are defined as

$$\begin{aligned}
&\operatorname{dom} F = \{x \in X : F(x) \neq \phi\};\\
&\operatorname{gr} F = \{(x, y) : x \in X, y \in F(x)\};\\
&\operatorname{epi} F = \{(x, y) : x \in X, y \in F(x) + K\}.
\end{aligned}$$

Jahn and Rauh [7] gave the following notion of contingent epiderivative relating epigraph of the derivative with the contingent cone.

Definition 2.2. A single valued map $DF(x^*, y^*) : X \longrightarrow Y$ whose epigraph is the contingent cone to $\operatorname{epi} F$ at $(x^*, y^*) \in \operatorname{gr} F$, that is,

$$\operatorname{epi} DF(x^*, y^*) = T(\operatorname{epi} F, (x^*, y^*))$$

is called the contingent epiderivative of F at (x^*, y^*).

In the present paper we assume condition C on η defined as follows:

Condition C ([2])**.** Let $\eta : X \times X \longrightarrow X$ be a map. Then η is said to be satisfy Condition C if for any $x, y \in X$.

(C1) $\eta(x,x) = 0$;

(C2) $\bigcup\limits_{x \in X} \eta(x,y) = X,\ \forall\ y \in X$;

(C3) $\eta(\lambda x, \lambda y) = \lambda \eta(x,y)$, $\eta(x - x_0, y - x_0) = \eta(x,y)$, for all $x, x_0, y \in X$

Consider the following set-valued fractional programming problem

$$\text{(FP)} \quad K\text{-minimize } \frac{F(x)}{G(x)} = \left(\frac{F_1(x)}{G(x)}, \frac{F_2(x)}{G(x)}, \dots, \frac{F_m(x)}{G(x)} \right)$$
$$\text{subject to } H(x) \cap (-Q) \neq \phi,$$

where X is a real normed space and S is a non-empty subset of X, $F : S \longrightarrow 2^{R^m}$, $G : S \longrightarrow 2^{R^+}$ and $H : S \longrightarrow 2^{R^k}$ are set-valued maps.

K and Q are closed convex pointed cones in R^m and R^k respectively with non-empty interiors. The feasible set of the problem (FP) is

$$X^0 = \{x \in S : H(x) \cap (-Q) \neq \phi\}$$

Throughout the paper, we denote

$$0_{R^m} = (0, 0, \dots, 0) \in R^m$$

Definition 2.3. A point $\left(x^*, \frac{y^*}{z^*}\right) \in X \times R^m$, with $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a minimizer of the problem (FP) if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} - \frac{y^*}{z^*} \in -K \setminus \{0_{R^m}\}.$$

Definition 2.4. A point $\left(x^*, \frac{y^*}{z^*}\right) \in X \times R^m$, with $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a weak minimizer of the problem (FP) if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$\frac{y}{z} - \frac{y^*}{z^*} \in -\operatorname{int} K.$$

Consider the parametric problem $(\text{FP})_{\lambda^*}$ associated with the set-valued fractional programming problem (FP):

$$(\text{FP})_{\lambda^*} \quad K\text{- }\underset{x \in S}{\text{minimize}}\ F(x) - \lambda^* G(x)$$
$$\text{subject to } H(x) \cap (-Q) \neq \phi.$$

Definition 2.5. A point $(x^*, y^* - \lambda^* z^*) \in X \times R^m$, with $\lambda^* = \frac{y^*}{z^*}$, $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a minimizer of the problem $(\text{FP})_{\lambda^*}$ if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda^* z) - (y^* - \lambda^* z^*) \in -K \setminus \{0_{R^m}\}.$$

Definition 2.6. A point $(x^*, y^* - \lambda^* z^*) \in X \times R^m$, with $\lambda^* = \frac{y^*}{z^*}$, $x^* \in X^0$, $y^* \in F(x^*)$ and $z^* \in G(x^*)$ is called a weak minimizer of the problem $(\text{FP})_{\lambda^*}$ if there exist no $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda^* z) - (y^* - \lambda^* z^*) \in -\operatorname{int} K.$$

Lemma 2.1 (3)**.** *A point* $\left(x^*, \frac{y^*}{z^*}\right) \in X \times R^m$ *is a weak minimizer of the problem (FP) if and only if* $(x^*, 0_{R^m})$ *is a weak minimizer of the problem* $(FP)_{\lambda^*}$*, where* $\lambda^* = \frac{y^*}{z^*}$.

Let X, Y be real normed spaces. Let $\eta : S \times S \longrightarrow X$ be a vector valued function. Let $S \subseteq X$ be a non-empty set.

Definition 2.7. A subset $S \subseteq X$ is said to be an η-invex set if for every $x, x^* \in S$ there exists a map $\eta : S \times S \longrightarrow X$ such that

$$x^* + \lambda\eta(x, x^*) \in S, \text{ for all } \lambda \in [0, 1].$$

Now, we introduce the notion of ρ-cone preinvexity of set-valued maps.

Definition 2.8. Let $S \subseteq X$ be an η-invex set. Let $e \in \operatorname{int} K$ and $F : S \longrightarrow 2^Y$ be a set-valued map. Then F is called $\rho - K - \eta$-preinvex at $x^* \in S$ with respect to e on S if there exists $\rho \in R$ such that

$$(1-\lambda)F(x^*) + \lambda F(x) \subseteq F(x^* + \lambda\eta(x, x^*)) + \lambda(1-\lambda)\rho\|\eta(x, x^*)\|^2 e + K,$$
$$\text{for all } x \in S \text{ and } \lambda \in [0, 1].$$

F is $\rho - K - \eta$-preinvex with respect to e on S if F is $\rho - K - \eta$-preinvex with respect to e for all $x^* \in S$.

Remark 2.1. (i) If $\rho = 0$, then the definition of $\rho - K - \eta$ preinvex reduces to the usual notion of cone $K - \eta$ preinvexity of set-valued maps defined by Bhatia and Mehra [3].
(ii) If $\eta(x, x^*) = x - x^*$, then $\rho - K - \eta$ preinvex functions reduce to $\rho - K$-convex functions defined by Das and Nahak in [6].

Now we give a characterization of ρ-cone preinvexity of set-valued maps in terms of their contingent epiderivatives.

Theorem 2.1. *Let $S \subseteq X$ be an η-invex set, $e \in \operatorname{int} K$ and $F : S \longrightarrow 2^Y$ be $\rho - K - \eta$-preinvex with respect to e on S. Let $x^* \in S$ and $y^* \in F(x^*)$. Suppose that F is contingent epiderivable at (x^*, y^*). Then*

$$F(x) - y^* \subseteq DF(x^*, y^*)\eta(x, x^*) + \rho\|\eta(x, x^*)\|^2 e + K, \textit{ for all } x \in S.$$

Proof. Let $x \in S$ and $y \in F(x)$. As F is $\rho - K - \eta$ preinvex with respect to e on S, therefore

$$(1-\lambda)F(x^*) + \lambda F(x) \subseteq F(x^* + \lambda\eta(x, x^*)) + \rho\lambda(1-\lambda)\|\eta(x, x^*)\|^2 e + K,$$
$$\text{for all } x \in S \text{ and } \lambda \in [0, 1].$$

Define a sequence $\{(x_n, y_n)\}_{n \in N}$ with

$$x_n = x^* + \frac{1}{n}\eta(x, x^*)$$

and

$$y_n = \frac{1}{n}y + \left(1 - \frac{1}{n}\right)y^* - \rho\frac{1}{n}\left(1 - \frac{1}{n}\right)\|\eta(x, x^*)\|^2 e, \text{ for all } n \in N.$$

Therefore

$$y_n \in F\left(x^* + \frac{1}{n}\eta(x, x^*)\right) + K, \quad \text{for all } x \in S.$$

Thus

$$y_n \in F(x_n) + K, \quad \text{for all } x \in S.$$

It is clear that

$$x_n \longrightarrow x^*, \ y_n \longrightarrow y^*, \ n(x_n - x^*) \longrightarrow \eta(x, x^*), \text{when } n \longrightarrow \infty \text{ and}$$
$$n(y_n - y^*) = y - y^* - \rho\left(1 - \frac{1}{n}\right)\|\eta(x, x^*)\|^2 e$$

$$\longrightarrow y - y^* - \rho\|\eta(x, x^*)\|^2 e, \text{ when } \eta \longrightarrow \infty.$$

Therefore

$$(\eta(x, x^*), y - y^* - \rho\|\eta(x, x^*)\|^2 e) \in T(\text{epi}(F), (x, x^*)) = \text{epi}(DF(x^*, y^*)).$$

Consequently

$$y - y^* - \rho\|\eta(x, x^*)\|^2 e \in DF(x^*, y^*)\eta(x, x^*) + K,$$

which is true for all $y \in F(x)$.

Therefore

$$F(x) - y^* \subseteq DF(x^*, y^*)\eta(x, x^*) + \rho\|\eta(x, x^*)\|^2 e + K, \text{ for all } x \in S.$$

□

Remark 2.2. If F satisfies the above condition then it is said to be $\rho - K - \eta$ invex function.

3. Optimality Conditions

We shall use the following Slater type constraint qualification to prove the necessary optimality Kuhn Tucker conditions for $(\text{FP})_{\lambda^*}$.

Definition 3.1. A set-valued map $H : S \longrightarrow 2^{R^k}$ is said to satisfy the generalized Slater's constraint qualification if there exists an element $\hat{x} \in S$ such that $H(\hat{x}) \cap -\text{int}\, Q \neq \phi$.

Bao-Huai and San Yang [2] investigated KKT necessary optimality conditions for a set-valued vector optimization problem in terms of alpha order contingent epiderivatives by assuming the objective and the constraint function to be alpha order preinvex.

If we take $\alpha = 1$, then we can get the following necessary optimality conditions for $(\text{FP})_{\lambda^*}$.

Theorem 3.1 (Karush-Kuhn-Tucker Necessary Optimality Conditions)**.** *Let $S \subseteq X$ be a η-invex set satisfying Condition C. Let $(x^*, y^* - \lambda^* z^*)$ be a weak minimizer of (FP)$_{\lambda^*}$. Let $F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map, $-\lambda^* G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. If H satisfies generalized Slater's constraint qualification and F is contingent epiderivable at (x^*, y^*), $-\lambda^* G$ is contingent epiderivable at $(x^*, -\lambda^* z^*)$ and H is contingent epiderivable at (x^*, w^*), where $w^* \in H(x^*) \cap (-Q)$, then there exists $(\tau^*, \mu^*) \in K^+ \times Q^+$, with $\tau^* \neq 0_{R^m}$ such that*

$$\langle \tau^*, DF(x^*, y^*)\eta(x, x^*) + D(-\lambda^* G)(x^*, -\lambda^* z^*)\eta(x, x^*)\rangle$$
$$\langle \mu^*, DH(x^*, w^*)\eta(x, x^*)\rangle \geq 0, \text{ for all } x \in S. \tag{3.1}$$
$$\langle \mu^*, w^*\rangle = 0 \tag{3.2}$$

In the light of Lemma 2.1, we have the following necessary optimality theorem for (FP).

Theorem 3.2 (Karush-Kuhn-Tucker Necessary Optimality Conditions)**.** *Let $S \subseteq X$ be an η-invex set satisfying Condition C. Let $\left(x^*, \dfrac{y^*}{z^*}\right)$ be a weak minimizer of (FP). Let $z^* F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map, $-y^* G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. If H satisfies generalized Slater's constraint qualification and $z^* F$ is contingent* *epiderivable at* (x^*, y^*z^*), $-y^*G$ *is contingent epiderivable at* $(x^*, -y^*z^*)$ *and* H *is contingent epiderivable at* (x^*, w^*), *where* $w^* \in H(x^*) \cap (-Q)$, *then there exists* $(\tau^*, \mu^*) \in K^+ \times Q^+$, *with* $\tau^* \neq 0_{R^m}$ *such that*

$$\langle \tau^*, D(z^*F)(x^*, y^*) + D(-y^*G)(x^*, -y^*z^*)\eta(x, x^*)\rangle + \langle \mu^*, DH(x^*, w^*)\eta(x, x^*)\rangle \geq 0, \text{ for all } x \in S \tag{3.3}$$

and condition (3.2) *hold.*

Now we establish sufficient optimality conditions for the problems (FP) and $(\text{FP})_{\lambda^*}$ by assuming that the objective and constraint set-valued maps are ρ-cone invex as well as contingent epiderivable.

Theorem 3.3 (Sufficiency)**.** *Let* $S \subseteq X$ *be an* η*-invex set,* $x^* \in X^0$, $y^* \in F(x^*)$, $z^* \in G(x^*)$, $\lambda^* = \frac{y^*}{z^*}$ *and* $w^* \in H(x^*) \cap (-Q)$. *Assume that* F *is* $\rho_1 - K - \eta$ *invex with respect to* e, $-\lambda^*G$ *is* $\rho_2 - K - \eta$ *invex with respect to* e *and* H *is* $\rho_3 - Q - \eta$ *invex with respect to* e *on* S. *Let* F *be contingent epiderivable at* (x^*, y^*), $-\lambda^*G$ *be contingent epiderivable at* $(x^*, -\lambda^*z^*)$ *and* H *be contingent epiderivable at* (x^*, w^*).

Suppose there exist $0 \neq \tau^* \in K^+$ *and* $\mu^* \in Q^+$ *satisfying the conditions* (3.1) *and* (3.2), *then* $(x^*, y^* - \lambda^*z^*)$ *is a weak minimizer of the problem* $(FP)_{\lambda^*}$ *provided*

$$(\rho_1 + \rho_2)\langle \tau^*, e\rangle + \rho_3\langle \mu^*, e\rangle \geq 0 \tag{3.4}$$

Proof. Let if possible $(x^*, y^* - \lambda^*z^*)$ be not a weak minimizer of the problem $(\text{FP})_{\lambda^*}$. Then there exist $x \in X^0$, $y \in F(x)$ and $z \in G(x)$ such that

$$(y - \lambda^*z) - (y^* - \lambda^*z^*) \in -\operatorname{int} K.$$

As $y^* - \lambda^*z^* = 0$, so we have

$$y - \lambda^*z \in -\operatorname{int} K\,.$$

Hence $\langle \tau^*, y - \lambda^*z\rangle < 0$.

Therefore we have $\langle \tau^*, y - \lambda^*z - (y^* - \lambda^*z^*)\rangle < 0$.

Since $x_0 \in X$, there exists an element $w \in H(x) \cap (-Q)$.

Therefore $\langle \mu^*, w\rangle \leq 0$.

So, $\langle \mu^*, w - w^*\rangle \leq 0$.

Hence

$$\langle \tau^*, y - \lambda^*z - (y^* - \lambda^*z^*)\rangle + \langle \mu^*, w - w^*\rangle < 0. \tag{3.5}$$

As F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. We have

$$F(x) - y^* \subseteq DF(x^*, y^*)\eta(x, x^*) + \rho_1\|\eta(x, x^*)\|^2 e + K,$$
$$-\lambda^*G(x) + \lambda^*z^* \subseteq D(-\lambda^*G)(x^*, -\lambda^*z^*)\eta(x, x^*) + \rho_2\|\eta(x, x^*)\|^2 e + K$$

and

$$H(x) - w^* \subseteq DH(x^*, w^*)\eta(x, x^*) + \rho_3\|\eta(x, x^*)\|^2 e + Q.$$

Hence

$$y - y^* \in DF(x^*, y^*)\eta(x, x^*) + \rho_1\|\eta(x, x^*)\|^2 e + K,$$
$$-\lambda^*z + \lambda^*z^* \in D(-\lambda^*G)(x^*, -\lambda^*z^*)\eta(x, x^*) + \rho_2\|\eta(x, x^*)\|^2 e + K$$

and

$$w - w^* \in DH(x^*, w^*)\eta(x, x^*) + \rho_3\|\eta(x, x^*)\|^2 e + Q.$$

This gives

$$y - y^* - DF(x^*, y^*)\eta(x, x^*) - \rho_1\|\eta(x, x^*)\|^2 e \in K,$$

$$-\lambda^* z + \lambda^* z^* - D(-\lambda^* G)(x^*, -\lambda^* z^*)\eta(x, x^*) - \rho_2\|\eta(x, x^*)\|^2 e \in K$$

and

$$w - w^* - DH(x^*, w^*)\eta(x, x^*) - \rho_3\|\eta(x, x^*)\|^2 e \in Q.$$

This further gives

$$\begin{aligned}&\langle \tau^*, y - y^* - \lambda^* z + \lambda^* z^*\rangle - \langle \tau^*, DF(x^*, y^*)\eta(x, x^*) + D(-\lambda^* G)(x^*, -\lambda^* z^*)\eta(x, x^*)\rangle \\ &- (\rho_1 + \rho_2)\langle \tau^*, e\rangle\|\eta(x, x^*)\|^2 + \langle \mu^*, w - w^*\rangle - \langle \mu^*, DH(x^*, w^*)\eta(x, x^*)\rangle \\ &- \rho_3\langle \mu^*, e\rangle\|\eta(x, x^*)\|^2 \geq 0\end{aligned}$$

By condition (3.4), this implies

$$\begin{aligned}&\langle \tau^*, y - y^* - \lambda^* z + \lambda^* z^*\rangle - \langle \tau^*, DF(x^*, y^*)\eta(x, x^*) + D(-\lambda^* G)(x^*, -\lambda^* z^*)\eta(x, x^*)\rangle \\ &+ \langle \mu^*, w - w^*\rangle - \langle \mu^* - DH(x^*, w^*)\eta(x, x^*)\rangle \geq 0\end{aligned}$$

By condition (3.1), this implies

$$\langle \tau^*, y - y^* - \lambda^* z + \lambda^* z^*\rangle + \langle \mu^*, w - w^*\rangle \geq 0$$

which contradicts (3.5).

Therefore $(x^*, y^* - \lambda^* z^*)$ is a weak minimizers of $(\mathrm{FP})_{\lambda^*}$. □

Theorem 3.4 (Sufficiency)**.** *Let $S \subseteq X$ be an η-invex set, $x^* \in X^0$, $y^* \in F(x^*)$, $z^* \in G(x^*)$ and $w^* \in H(x^*) \cap (-Q)$. Assume that z^*F is $\rho_1 - K - \eta$ invex with respect to e, $-y^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Assume that z^*F is a contingent epiderivable at (x^*, y^*z^*), $-y^*G$ is contingent epiderivable at $(x^*, -y^*z^*)$ and H is contingent epiderivable at (x^*, w^*). Further suppose that there exist $(\tau^*, \mu^*) \times K^+ \times Q^+$ with $\tau^* \neq 0_{R^m}$ such that condition*

$$\begin{aligned}&\langle \tau^*, D(z^*F)(x^*, y^*z^*)\eta(x, x^*) + D(-y^*G)(x^*, -y^*z^*)\eta(x, x^*)\rangle \\ &+ \langle \mu^*, DH(x^*, w^*)\eta(x, x^*)\rangle \geq 0, \text{ for all } x \in S\end{aligned}$$

and condition (3.2) are satisfied. Then $\left(x^*, \frac{y^*}{z^*}\right)$ is a weak minimizer of the problem (FP) provided condition (3.4) holds.

4. Duality

We now formulate parametric, Mond-Weir and Wolfe type duals for the problem (FP) and study duality theorems for the same.

Parametric type dual. We associate the following parametric type dual (PD) with the primal problem (FP).

$$\begin{aligned}\text{(PD)}\quad &\text{maximize}\,\lambda \\ &\text{subject to} \\ &\langle \tau, DF(u, v)\eta(x, u) + D(-\lambda G)(u, -\lambda l)\eta(x, u)\rangle \\ &\quad + \langle \mu, DH(u, q)\eta(x, u)\rangle \geq 0, \text{ for all } x_0 \in X, \\ &\langle \mu, q\rangle \geq 0, \\ &u \in S,\ v \in F(u),\ l \in G(u),\ \lambda = \frac{v}{l},\ q \in H(u). \\ &0 \neq \tau \in K^+,\ \mu \in Q^+ \text{ and } \langle \tau, e\rangle = 1.\end{aligned}$$

A point $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ satisfying all the constraints of the problem (PD) is called a feasible point of (PD).

Definition 4.1. A feasible point $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ of the problem (PD) is called a weak maximizer of (PD) if there exists no feasible point $(u, v, l, \lambda, q, \tau, \mu)$ of (PD) such that

$$\lambda - \lambda^* \in \operatorname{int} K.$$

Theorem 4.1 (Weak Duality)**.** *Let $S \subseteq X$ be an η-invex set, $\bar{x} \in X$ and $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (PD). Suppose that F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^* G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Further assume that F is contingent epiderivable at (u^*, v^*), $-\lambda^* G$ is contingent epiderivable at $(u^*, -\lambda^* l)$ and H is contingent epiderivable at (u^*, q^*). Then*

$$\frac{F(\bar{x})}{G(\bar{x})} - \lambda^* \subseteq R^m \setminus -\operatorname{int} K,$$

provided condition (3.4) holds.

Proof. Let if possible for some $\dot{v} \in F(\bar{x})$ and $\dot{l} \in G(\bar{x})$,

$$\frac{\dot{v}}{\dot{l}} - \lambda^* \in \operatorname{int} K.$$

Thus $\dot{v} - \lambda^* \dot{l} \in -\operatorname{int} K$.

Hence $\langle \tau^*, \dot{v} - \lambda^* \dot{l} \rangle < 0$.

Therefore, $\langle \tau^*, \dot{v} - \lambda^* \dot{l} - (v^* - \lambda^* l^*) \rangle < 0$.

Now as $\bar{x} \in X^0$, we have

$$H(\bar{x}) \cap (-Q) \neq \phi.$$

Let $\bar{q} \in H(\bar{x}) \cap (-Q)$. Then

$$\langle \mu^*, \bar{q} \rangle \leq 0.$$

Again, from the constraints of (PD), we have

$$\langle \mu^*, q^* \rangle \geq 0.$$

Hence $\langle \mu^*, \bar{q} - q^* \rangle \leq 0$.

Therefore

$$\langle \tau^*, \dot{v} - \lambda^* \dot{l} - (v^* - \lambda^* l^*) + \langle \mu^*, \bar{q} - q^* \rangle < 0\,. \tag{4.1}$$

As F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^* G$ is $\rho_2 - K - \eta$ with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S, we have

$$F(\bar{x}) - v^* \subseteq DF(u^*, v^*)\eta(\bar{x}, u^*) + \rho_1 \|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$(-\lambda^* G)(\bar{x}) + \lambda^* l^* \subseteq D(-\lambda^* G)(u^*, -\lambda^* l^*) + \eta(\bar{x}, u^*) + \rho_2 \|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$H(\bar{x}) - q^* \subseteq DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3 \|\eta(\bar{x}, u^*)\|^2 e \in Q.$$

Hence

$$\dot{v} - v^* \in DF(u^*, v^*)\eta(\bar{x}, u^*) + \rho_1 \|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$-\lambda^* \dot{l} + \lambda^* l^* \in D(-\lambda^* G)(u^*, -\lambda^* l^*)\eta(\bar{x}, u^*) + \rho_2 \|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$\bar{q} - q^* \in DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3 \|\eta(\bar{x}, u^*)\|^2 e + Q.$$

Hence, from the constraints of (PD) and condition (3.4), we have

$$\langle \tau^*, \dot{v} - \lambda^* \dot{l} - (v^* - \lambda^* l^*) \rangle + \langle u^*, \bar{q} - q^* \rangle \geq 0,$$

which contradicts equation (4.1).

Thus

$$\frac{\dot{v}}{\dot{l}} - \lambda^* \notin \operatorname{int} K.$$

Since $\dot{v} \in F(\bar{x})$ and $\dot{l} \in G(\bar{x})$ are arbitrary, therefore

$$\frac{F(\bar{x})}{G(\bar{x})} - \lambda^* \subseteq R^m \setminus -\operatorname{int} K.$$

By the Theorems 3.1 and 4.1, we get the following result. □

Theorem 4.2 (Strong Duality). *Let $S \subseteq X$ be an η-invex set satisfying Condition C. Let $(x^*, y^* - \lambda^* z^*)$ be a weak minimizer of (FP)$_{\lambda^*}$. Let $F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map, $-\lambda^* G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. Further assume that H satisfies generalized Slater's constraint qualification and F is contingent epiderivable at (x^*, y^*), $-\lambda^* G$ is contingent epiderivable at $(x^*, -\lambda^* z^*)$ and H is contingent epiderivable at (x^*, w^*), $w^* \in H(x^*) \cap -(Q)$. Then there exist $0 \neq \tau^* \in K^+$, $\mu^* \in Q^+$ such $(x^*, y^*, z^*, \lambda^*, w^*, \tau^*, \mu^*)$ is feasible for (PD). Moreover, if for each feasible point of (PD), hypothesis of Weak Duality Theorem 4.1 holds, then $(x^*, y^*, z^*, \lambda^*, w^*, \tau^*, \mu^*)$ is a weak maximizer of (PD).*

Theorem 4.3 (Converse Duality). *Let $S \subseteq X$ be an η-invex set and $(u^*, v^*, l^*, \lambda^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (PD), where $\lambda^* = \frac{v^*}{l^*}$. Suppose that F is $\rho_1 - K - \eta$ invex with respect to e, $-\lambda^* G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Also let F be contingent epiderivable at (u^*, v^*), $-\lambda^* G$ be contingent epiderivable at $(u^*, -\lambda^* l^*)$ and H be contingent epiderivable at (u^*, q^*). If u^* is a feasible point of the problem (FP)$_{\lambda^*}$, then $(u^*, v^* - \lambda^* l^*)$ is a weak minimizer of the problem (FP)$_{\lambda^*}$ provided condition (3.4) holds.*

Proof. Let if possible $(u^*, v^* - \lambda^* l^*)$ be not a weak minimizer of the problem (FP)$_{\lambda^*}$.

Then there exist $x \in X^0$, $v \in F(x)$ and $l \in G(x)$ such that

$$(v - \lambda^* l) - (v^* - \lambda^* l^*) \in -\operatorname{int} K.$$

This gives

$$v - \lambda^* l \in -\operatorname{int} K$$

Since $0_{R^m} \neq \tau^* \in K^+$, so this further gives

$$\langle \tau^*, v - \lambda^* l \rangle < 0$$

Therefore

$$\langle \tau^*, v - \lambda^* l - (v^* - \lambda^* l^*) \rangle < 0.$$

Proceeding on the same lines as in the proof of Theorem 4.1, we will get the result. □

Mond-Weir type dual. We now associate the following Mond-Weir type dual with the primal problem (FP).

$$\text{(MWD)} \quad \text{maximize } \frac{v}{l}$$

subject to

$$\langle \tau, D(lF)(u, vl)\eta(x, u) + D(-vG)(u, -vl)\eta(x, u) \rangle$$

$$+ \langle \mu, DH(u, q)\eta(x, u) \rangle \geq 0, \text{ for all } x \in X^0,$$

$$\langle \mu, q \rangle \geq 0,$$

$$u \in S,\ v \in F(u), l \in G(u),\ q \in H(u),\ 0 \neq \tau \in K^+,\ \mu \in Q^+$$

and $\langle \tau, e\rangle = 1$.

A point $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ which satisfies all the constraints of the dual problem (MWD) is a feasible point of (MWD).

Definition 4.2. A feasible point $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ of the problem (MWD) is called a weak maximizer of (MWD) if there exists no feasible point (u, v, l, q, τ, μ) of (MWD) such that

$$\frac{v}{l} - \frac{v^*}{l^*} \in \operatorname{int} K$$

Theorem 4.4 (Weak Duality)**.** *Let $S \subseteq X$ be an η-invex set, $\bar{x} \in X^0$ and $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (MWD). Suppose that l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Also let l^*F be contingent epiderivable at (u^*, v^*l^*), $-v^*G$ be contingent epiderivable at $(u^*, -v^*l^*)$ and H be contingent epiderivable at (u^*, q^*), then*

$$\frac{F(\bar{x})}{G(\bar{x})} - \frac{v^*}{l^*} \subseteq R^m \setminus -\operatorname{int} K,$$

provided condition (3.4) holds.

Proof. Let if possible for some $\dot{v} \in F(\bar{x})$ and $\dot{l} \in G(\bar{x})$,

$$\frac{\dot{v}}{\dot{l}} - \frac{v^*}{l^*} \in -\operatorname{int} K.$$

As $\dot{l}l^* \in R^+$, so this implies

$$\dot{v}l^* - v^*\dot{l} \in -\operatorname{int} K\,.$$

Thus $\langle \tau^*, \dot{v}l^* - v^*\dot{l}\rangle < 0$.

Now $\bar{x} \in X^0$, so there exists an element $\bar{q} \in H(\bar{x}) \cap (-Q)$.

Therefore

$$\langle \mu^*, \bar{q}\rangle \leq 0$$

Again, from the constraints of (MWD), we have

$$\langle \mu^*, q^*\rangle \geq 0.$$

So

$$\langle \mu^*, \bar{q} - q^*\rangle \leq 0.$$

Hence,

$$\langle \tau^*, \dot{v}l^* - v^*\dot{l}\rangle + \langle \mu^*, \bar{q} - q^*\rangle < 0. \tag{4.2}$$

Since l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S, we have

$$(l^*F)(\bar{x}) - v^*l^* \subseteq D(l^*F)(u^*, v^*l^*)\eta(\bar{x}, u^*) + \rho_1\|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$(-v^*G)(\bar{x}) - v^*l^* \subseteq D(v^*G)(u^*, v^*l^*)\eta(\bar{x}, u^*) + \rho_2\|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$H(\bar{x}) - q^* \subseteq DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3\|\eta(\bar{x}, u^*)\|^2 e + Q\,.$$

This gives

$$\dot{v}l^* - v^*l^* \in D(l^*F)(u^*, v^*l^*)\eta(\bar{x}, u^*) + \rho_1\|\eta(\bar{x}, u^*)\|^2 e + K,$$
$$-v^*\dot{l} + v^*l^* \in D(-v^*G)(u^*, -v^*l^*)\eta(\bar{x}, u^*) + \rho_2\|\eta(\bar{x}, u^*)\|^2 e + K$$

and

$$\bar{q} - q^* \in DH(u^*, q^*)\eta(\bar{x}, u^*) + \rho_3\|\eta(\bar{x}, u^*)\|^2 e + Q.$$

Thus, from the constraints of (MWD) and condition (3.4), we have

$$\langle \tau^*, \dot{v}l^* - v^*\dot{l}\rangle + \langle \mu^*, \bar{q} - q^*\rangle \geq 0$$

which contradicts equation (4.2).

Hence

$$\frac{\dot{v}}{\dot{l}} - \frac{v^*}{l^*} \notin -\operatorname{int} K.$$

Since $\frac{\dot{v}}{\dot{l}} \in \frac{F(\bar{x})}{G(\bar{x})}$ is arbitrary, so

$$\frac{F(\bar{x})}{G(\bar{x})} - \frac{v^*}{l^*} \subseteq R^m \setminus -\operatorname{int} K. \qquad \square$$

By the Theorems 3.2 and 4.4, we will get the following result.

Theorem 4.5 (Strong Duality)**.** *Let $S \subseteq X$ be an η-invex set satisfying Condition C. Let $\left(x^*, \frac{y^*}{z^*}\right)$ be an weak minimizer of (FP). Let $z^*F : S \longrightarrow 2^{R^m}$ be $K - \eta$ preinvex set-valued map $-y^*G : S \longrightarrow 2^{R^+}$ be $K - \eta$ preinvex set-valued map and $H : S \longrightarrow 2^{R^k}$ be $Q - \eta$ preinvex set-valued map. If H satisfies generalized Slater's constraint qualification and z^*F is contingent epiderivable at (x^*, y^*z^*), $-y^*G$ is contingent epiderivable at $(x^*, -y^*z^*)$ and H is contingent epiderivable at (x^*, w^*), where $w^* \in H(x^*) \cap (-Q)$. Then there exists $0_{R^m} \neq \tau^* \in K^+$, $\mu^* \in Q^+$, such that $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is feasible for (MWD). Moreover, if for each feasible point of (MWD), hypothesis of Weak Duality Theorem 4.4 holds, then $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is a weak maximizer of (MWD).*

Theorem 4.6 (Converse Duality)**.** *Let $S \subseteq X$ be an η-invex set and $(u^*, v^*, l^*, q^*, \eta^*, \mu^*)$ be a feasible point of the problem (MWD). Suppose that l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e on S. Also let l^*F be contingent epiderivable at (u^*, v^*l^*), $-v^*G$ be contingent epiderivable at $(u^*, -v^*l^*)$ and H be contingent epiderivable at (u^*, q^*). If u^* is a feasible point of the problem (FP), then $\left(v^*, \frac{v^*}{l^*}\right)$ is a weak minimizer of the problem (FP) provided condition (3.4) holds.*

Proof. Let if possible $\left(v^*, \frac{v^*}{l^*}\right)$ be not a weak minimizer of the problem (FP). Then there exist $x \in X^0$, $v \in F(x)$ and $l \in G(x)$ such that

$$\frac{v}{l} - \frac{v^*}{l^*} \in -\operatorname{int} K.$$

This gives

$$vl^* - v^*l \in -\operatorname{int} K.$$

Thus $\langle \tau^*, vl^* - v^*l\rangle < 0$.

Proceeding on the same lines as in the proof of Theorem 4.4, we will get a contradiction which proves that $\left(u^*, \frac{v^*}{l^*}\right)$ must be a weak minimizer of (FP). $\square$

Wolfe Type Dual. We now associate the following Wolfe-type dual with the primal problem (FP).

$$\text{(WD)} \quad \text{maximize } \frac{v + \langle \mu, q \rangle e}{l}$$

subject to

$$\langle \tau, D(lF)(u, vl)\eta(x,u) + D(-vG)(u, -vl)\eta(x,u) \geq 0, \text{ for all } x \in X^0. \tag{4.3}$$

$$\langle \mu, DH(u,q)\eta(x,u) \rangle \geq 0, \text{ for all } x \in X^0, \tag{4.4}$$

$$u \in S,\ v \in F(u),\ l \in G(u),\ 0_{R^m} \neq \tau \in K^+,\ \mu \in Q^+$$

and $\langle \tau, e \rangle = 1$.

Definition 4.3. A feasible point $\langle u^*, v^*, l^*, q^*, \tau^*, \mu^* \rangle$ of the problem (WD) is called a weak maximizer of (WD) if there exists no feasible point (u, v, lq, τ, μ) of (WD) such that

$$\frac{v + \langle \mu, q \rangle}{l} - \frac{v + \langle \mu^*, q^* \rangle}{l^*} \in \operatorname{int} K.$$

The following results can easily be established for the Wolfe type dual.

Theorem 4.7 (Weak Duality)**.** *Let $S \subseteq X$ be an η-invex set, $\bar{x} \in X^0$ and $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (WD). Suppose l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e, on S. Also l^*F is contingent epiderivable at (u^*, v^*l^*), $-v^*G$ is contingent epiderivable at $(u^*, -v^*l^*)$ and H is contingent epiderivable at (u^*, q^*). Then*

$$\frac{F(\bar{x})}{G(\bar{x})} - \frac{v^* + \langle \mu^*, q^* \rangle e}{l^*} \subseteq R^m \setminus -\operatorname{int} K,$$

provided

$$(\rho_1 + \rho_2)\langle \tau^*, e \rangle \geq 0 \text{ and } \rho_3 \langle \mu^*, e \rangle \geq 0 \tag{4.5}$$

Theorem 4.8 (Strong Duality)**.** *Let $\left(x^*, \frac{y^*}{z^*}\right)$ be a weak minimizer of the problem (FP) and $w^* \in H(x^*) \cap (-Q)$. Suppose that there exists $\tau^* \in K^+$, $\mu^* \in Q^+$ with $\langle \tau^*, e \rangle \geq 1$ such that conditions (4.3) and (4.4) are satisfied at $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$. Then $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is a feasible solution of the problem (WD). Furthermore, if for each feasible point of (WD) the conditions of Weak Duality Theorem 4.7 hold, then $(x^*, y^*, z^*, w^*, \tau^*, \mu^*)$ is a weak maximizer of (WD).*

Theorem 4.9 (Converse Duality)**.** *Let $S \subseteq X$ be an η-invex set and $(u^*, v^*, l^*, q^*, \tau^*, \mu^*)$ be a feasible point of the problem (WD) and $\langle \mu^*, q^* \rangle = 0$. Suppose that l^*F is $\rho_1 - K - \eta$ invex with respect to e, $-v^*G$ is $\rho_2 - K - \eta$ invex with respect to e and H is $\rho_3 - Q - \eta$ invex with respect to e, on S. Also let l^*F be contingent epiderivable at (u^*, v^*l^*), $-v^*G$ be contingent epiderivable at $(u^*, -v^*l^*)$ and H be contingent epiderivable at (u^*, q^*). If u^* is a feasible point of the problem (FP), then $\left(u^*, \dfrac{v^*}{l^*}\right)$ is a weak minimizer of the problem (FP) provided condition (4.5) hold.*

Acknowledgements

We express our sincere thanks to Dr. Surjeet K. Suneja for her valuable comments and suggestions that helped in improving the quality of our paper.

References

1. J. P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, In: L. Nachbin (ed.), Mathematical Analysis and Applications, Part A pp. 160 – 229, Academic Press, New York (1981).
2. S. Bao-Huai and I. San-Yang, Kuhn-Tucker Condition and Wolfe-duality of preinvex set-valued optimization, Appl. Math. Mech. Eng. 27 (12) (2006), 1655 – 1644.
3. D. Bhatia and A. Mehra, Langrangian duality for preinvex set-valued function, J. Math. Anal. Appl. 214 (1997), 599 – 612.
4. H. W. Corley, Existence and Lagrangian duality for maximization of set-valued functions, J. Optim. Theory Appl. 54 (1987), 489 – 501.
5. K. Das and C. Nahak, Sufficient Optimality Conditions and duality theorems for set-valued optimization problem under generalized cone convexity, Rend. Circ. Mat. Palermo 63 (2014), 329 – 345.
6. K. Das and C. Nahak, Set-Valued fractional programming problems under generalized cone convexity, Opsearch 53(1) (2016), 157 – 177.
7. J. Jahn and R. Rauh, Contingent epiderivative and set-valued optimization, Math. Methods Oper. Res. 46 (1997), 193 – 211.
8. D. T. Luc and C. Malivert, Invex optimization problems, Bull. Aust. Math. Soc. 49 (1992), 47 – 66.
9. P. H. Sach and B. D. Craven, Invex multifunction and duality, Numer. Funct. Anal. Opt. 12 (1991), 575 – 591.
10. G. Yu and X. Kong, Approximate solutions for nonconvex set-valued optimization and vector variational inequality, Journal of Inequalities and Applications 2015 (2015), 324.