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FIXED POINT FOR MAPPINGS SATISFYING KANNAN TYPE INEQUALITY IN FUZZY METRIC SPACES INVOLVING T-NORMS WITH EQUI-CONTINUOUS ITERATES

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ABSTRACT. In this paper, we define coupled weak compatible condition and use it to derive certain coupled coincidence point theorems for four mappings in fuzzy metric spaces. We use here a t-norm which has equicontinuous iterates at 1. Some coupled fixed point results in metric spaces are obtained by applications of the results. Our results are obtained without any assumption of continuity on the mappings. Our main result is supported by an illustrative example. Some corollaries are also obtained.

KEYWORDS: Fuzzy metric space, Hadzic type t-norm, Ψ-function, Cauchy sequence, Weak compatible mappings, Coupled fixed point, Coupled coincidence point.

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1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Afterwards fuzzy concepts made quick headways in almost all branches of mathematics. In particular, fuzzy metric space was introduced by Kramosil and Michalek [2]. George and Veeramani modified the definition of Kramosil and Michalek in [3] for topological reasons. The topology in the space introduced by George and Veeramani is a Hausdroff topology. There are several fixed point results for mappings defined on fuzzy metric spaces in the sense of George and Veeramani. We have

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noted some of these works in [4, 5, 6, 7, 8, 9, 10] and [11].

Coupled fixed point results attracted renewed interest after the publication of a coupled contraction mapping theorem in partially ordered metric spaces by Gnana-Bhasker and Lashmikanthm [12]. An interesting application of this result was also given in the same paper. The result in [12] was extended to coincidence point results in [13] and [14] under separate sets of sufficient conditions. Several other works on coupled fixed points have appeared in recent times. Some other works in this line of research are noted in [15, 16, 17]. Coupled fixed point problems have also been studied in probabilistic metric spaces [18], in cone metric spaces [19, 20] and in G- metric spaces [21]. We establish here coupled coincidence and fixed point results for the cases of coupled Kannan type mappings. Kannan type of mappings are considered to be important in metric fixed point theory for several reasons. We mention two of these in the following.

Banach contraction is continuous. A natural question is that whether there exists a class of mappings satisfying some contractive inequality which necessary have fixed points in complete metric spaces but need not necessarily be continuous. Kannan type mappings are such mappings [22, 23]. Another reason is its connection with metric completeness. A Banach contraction mapping may have a fixed point in a metric space which is not complete. In fact, Connell in [24] has given an example of a metric space which is not complete but every Banach contraction defined on which has a fixed point. It has been established in [25] that the metric completeness is implied by the necessary existence of fixed points of the class of Kannan type mappings. Some of the recent works on Kannan type mappings are noted in [26, 27, 28]. It may be noted that fuzzy functional analysis is a vast area of study of which some instances are [29, 30, 31, 32, 33, 34, 35].

In this paper we establish a common fixed point and coupled fixed point result for four mappings. The name 'Kannan type' is suggested by the form of the inequality we use here. We apply our result to obtain a new coupled Kannan type common fixed point result in metric spaces. An example illustrates our result in fuzzy metric spaces. In this paper we use Hadzic type t-norm which is a t-norm for which the iterates are equicontinuous at 1.

2. Preliminaries

Definition 2.1[36] A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a t-norm if the following properties are satisfied:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.
Generic examples of $t-$norms are $a *_1 b = \min\{a, b\}$, $a *_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $0 < \lambda < 1$, $a *_3 b = ab$ and $a *_4 b = \max\{a + b - 1, 0\}$.

Kramosil and Michalek defined fuzzy metric space by extending probabilistic metric spaces.

Definition 2.2[2] The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of Kramosil and Michalek if X is a non-empty set, $*$ is a t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:
(i) $M(x, y, 0) = 0$,
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and

(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is left-continuous, where $t, s > 0$ and $x, y, z \in X$.
George and Veeramani in their paper [3] introduced a modification of the above definition. The motivation was to make the corresponding induced topology necessarily into a Hausdroff topology.
Definition 2.3[3] The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani if X is a non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:
(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.
The following details of this space are described in the introductory paper [3].
Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let τ denote the family of all open subsets of X. Then τ is a topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

In the present work we will only consider the space as described in definition 2.3 and will refer this space simply as fuzzy metric space.
There are several examples of the fuzzy metric space for which we refer to [3].
Lemma 2.4[37] Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is nondecreasing for all $x, y \in X$.
Definition 2.5[2] Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
Lemma 2.6[38] M is a continuous function on $X^2 \times (0, \infty)$.
Definition 2.7[12] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \; F(y, x) = y.$$

Further Lakshmikantham and Ćirić have introduced the concept of coupled coincidence point.
Definition 2.8[14] Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $h : X \to X$ if

$$F(x, y) = hx, \; F(y, x) = hy.$$

Definition 2.9[14] Let X be a nonempty set and the mappings $F : X \times X \to X$ and $h : X \to X$ are commuting if for all $x, y \in X$

$$hF(x, y) = F(hx, hy).$$

Definition 2.10[39] A t-norm $*$ is said to be Hadzic type t-norm if the family $\{*^p\}_{p \geq 0}$ of its iterates defined for each $s \in [0, 1]$ by

$*^0(s) = 1$, $*^{p+1}(s) = *(*^p(s), s)$ for all $p \geq 0$, is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0, 1)$ such that

$1 \geq s > \eta(\lambda) \Rightarrow *^{(p)}(s) > 1 - \lambda$ for all $p \geq 0$.

We will require the result of the following recently established lemma to prove our results.

Lemma 2.11 [40] Let $(X, M, *)$ be a fuzzy metric space with a Hadzic type t-norm $*$ such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. If the sequences $\{x_n\}$ and $\{y_n\}$ in X are such that, for all $n \geq 1$, $t > 0$,

$$M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) \geq M(x_{n-1}, x_n, \tfrac{t}{k}) * M(y_{n-1}, y_n, \tfrac{t}{k})$$

where $0 < k < 1$, then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

We will use the following class of real mappings.

Definition 2.12 (Ψ-function) A function $\psi : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a ψ-function if

(i) ψ continuous and monotone increasing in both the variables,

(ii) $\psi(t, t) \geq t$ for all $0 \leq t \leq 1$.

3. Main Results

We next give the following definition.

Definition 3.1. Two maps $F : X \times X \to X$ and $h : X \to X$, where X is a nonempty set, are weakly compatible pair if they commute at their coincidence point, that is, for any $x, y \in X$, $hx = F(x, y)$ and $hy = F(y, x)$ implies that $h(F(x, y)) = F(hx, hy)$ and $h(F(y, x)) = F(hy, hx)$.

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space with a Hadzic type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $F : X \times X \to X$, $G : X \times X \to X$, $h : X \to X$ and $g : X \to X$ be four mappings satisfying the following conditions:*

(i) $F(X \times X) \subseteq g(X), G(X \times X) \subseteq h(X)$ and $h(X), g(X)$ are two closed subsets of X,

(ii) (F, h) and (G, g) are weakly compatible pairs,

(iii) $M(F(x, y), G(u, v), kt) \geq \psi(M(hx, F(x, y), t), M(gu, G(u, v), t))$, (3.1)

where $x, y, u, v \in X$, $t > 0$, $0 < k < 1$ and ψ is Ψ-function. Then there exist $x, y \in X$ such that $x = hx = gx = F(x, y) = G(x, y)$ and $y = hy = gy = F(y, x) = G(y, x)$, that is, there exist $x, y \in X$ such that x and y are common fixed points of h and g, and that (x, y) is a unique coupled fixed point of F and G.

Proof. Let x_0, y_0 be two points in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows, for all $n \geq 0$,

$$p_{2n} = gx_{2n+1} = F(x_{2n}, y_{2n}) \text{ and } q_{2n} = gy_{2n+1} = F(y_{2n}, x_{2n}). \quad (3.2)$$

$$p_{2n+1} = hx_{2n+2} = G(x_{2n+1}, y_{2n+1}) \text{ and } q_{2n+1} = hy_{2n+1} = G(y_{2n+1}, x_{2n+1}). \quad (3.3)$$

This construction is possible by the condition $F(X \times X) \subseteq g(X)$, $G(X \times X) \subseteq h(X)$. Now, for all $t > 0$, $n \geq 1$, we have

$$\begin{aligned} M(p_{2n}, p_{2n+1}, kt) &= M(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}), kt) \text{ (by(3.2) and (3.3))} \\ &\geq \psi(M(hx_{2n}, F(x_{2n}, y_{2n}), t), M(gx_{2n+1}, G(x_{2n+1}, y_{2n+1}), t)) \text{ (by(3.1))} \\ &\geq \psi(M(p_{2n-1}, p_{2n}, t), M(p_{2n}, p_{2n+1}, t)). \end{aligned}$$

If, for some $s>0$, and for some n, $M(p_{2n-1},p_{2n},s)\geq M(p_{2n},p_{2n+1},s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(p_{2n+1},p_{2n},ks) &\geq \psi(M(p_{2n},p_{2n+1},s),M(p_{2n+1},p_{2n},s)) \\ &\geq M(p_{2n},p_{2n+1},s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(p_{2n},p_{2n+1},ks)>M(p_{2n-1},p_{2n},s) \text{ for all } n>0.$$

Thus, for all $n>0$ and $t>0$, we have

$$M(p_{2n},p_{2n+1},kt)\geq \psi(M(p_{2n-1},p_{2n},t),M(p_{2n-1},p_{2n},t)),$$

that is, for all $n>0$, $t>0$, we have

$$M(p_{2n},p_{2n+1},kt)\geq M(p_{2n-1},p_{2n},t), \text{ (using the properties of } \psi\text{- function).} \tag{3.4}$$

Again, for all $t>0$, $n\geq 0$, we have

$$\begin{aligned} M(p_{2n+1},p_{2n+2},kt) &= M(F(x_{2n+2},y_{2n+2}),G(x_{2n+1},y_{2n+1}),kt) \text{ (by(3.2) and (3.3))} \\ &\geq \psi(M(hx_{2n+2},F(x_{2n+2},y_{2n+2}),t),M(gx_{2n+1},G(x_{2n+1},y_{2n+1}),t)) \\ &\qquad \text{(by(3.1))} \\ &\geq \psi(M(p_{2n+1},p_{2n+2},t),M(p_{2n},p_{2n+1},t)). \end{aligned}$$

If, for some $s>0$, and for some n, $M(p_{2n},p_{2n+1},s)\geq M(p_{2n+1},p_{2n+2},s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(p_{2n+1},p_{2n+2},ks) &\geq \psi(M(p_{2n+1},p_{2n+2},s),M(p_{2n+1},p_{2n+2},s)) \\ &\geq M(p_{2n+1},p_{2n+2},s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(p_{2n+1},p_{2n+2},ks)>M(p_{2n},p_{2n+1},s) \text{ for all } n>0.$$

Thus, for all $n>0$ and $t>0$, we have

$$M(p_{2n+1},p_{2n+2},kt)\geq \psi(M(p_{2n},p_{2n+1},t),M(p_{2n},p_{2n+1},t)),$$

that is, for all $n>0$, $t>0$, we have

$$M(p_{2n+1},p_{2n+2},kt)\geq M(p_{2n},p_{2n+1},t), \text{ (using the properties of } \psi\text{- function).} \tag{3.5}$$

From (3.4) and (3.5), for all $t>0$, $n\geq 1$, we have

$$M(p_n,p_{n+1},kt)\geq M(p_{n-1},p_n,t), \text{ (using the properties of } \psi\text{- function).} \tag{3.6}$$

Now, for all $t>0$, $n\geq 1$, we have

$$\begin{aligned} M(q_{2n},q_{2n+1},kt) &= M(F(y_{2n},x_{2n}),G(y_{2n+1},x_{2n+1}),kt) \text{ (by(3.2) and (3.3))} \\ &\geq \psi(M(hy_{2n},F(y_{2n},x_{2n}),t),M(gy_{2n+1},G(y_{2n+1},x_{2n+1}),t)) \\ &\qquad \text{(by(3.1))} \\ &\geq \psi(M(q_{2n-1},q_{2n},t),M(q_{2n},q_{2n+1},t)). \end{aligned}$$

If, for some $s>0$, and for some n, $M(q_{2n-1},q_{2n},s)\geq M(q_{2n},q_{2n+1},s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(q_{2n+1},q_{2n},ks) &\geq \psi(M(q_{2n},q_{2n+1},s),M(q_{2n+1},q_{2n},s)) \\ &\geq M(q_{2n},q_{2n+1},s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(q_{2n},q_{2n+1},ks)>M(q_{2n-1},q_{2n},s) \text{ for all } n>0.$$

Thus, for all $n>0$ and $t>0$, we have

$$M(q_{2n},q_{2n+1},kt)\geq \psi(M(q_{2n-1},q_{2n},t),M(q_{2n-1},q_{2n},t)),$$

that is, for all $n>0$, $t>0$, we have

$$M(q_{2n},q_{2n+1},kt)\geq M(q_{2n-1},q_{2n},t), \text{ (using the properties of } \psi\text{- function).} \tag{3.7}$$

Again, for all $t>0$, $n\geq 1$, we have

$$M(q_{2n+1},q_{2n+2},kt) = M(F(y_{2n+2},x_{2n+2}),G(y_{2n+1},x_{2n+1}),kt) \text{ (by(3.2) and (3.3))}$$

$$\geq \psi(M(hy_{2n+2}, F(y_{2n+2}, x_{2n+2}), t), M(gy_{2n+1}, G(y_{2n+1}, x_{2n+1}), t)) \quad \text{(by(3.1))}$$
$$\geq \psi(M(q_{2n+1}, q_{2n+2}, t), M(q_{2n}, q_{2n+1}, t)).$$

If, for some $s > 0$, and for some n, $M(q_{2n}, q_{2n+1}, s) \geq M(q_{2n+1}, q_{2n+2}, s)$, then, from the above inequality, and using the properties of ψ, we obtain

$$\begin{aligned} M(q_{2n+1}, q_{2n+2}, ks) &\geq \psi(M(q_{2n+1}, q_{2n+2}, s), M(q_{2n+1}, q_{2n+2}, s)) \\ &\geq M(q_{2n+1}, q_{2n+2}, s). \end{aligned}$$

But this contradicts our assumption that M is strictly increasing in the third variable. Hence we have

$$M(q_{2n+1}, q_{2n+2}, ks) > M(q_{2n}, q_{2n+1}, s) \text{ for all } n \geq 0.$$

Thus, for all $n \geq 0$ and $t > 0$, we have

$$M(q_{2n+1}, q_{2n+2}, kt) \geq \psi(M(q_{2n}, q_{2n+1}, t), M(q_{2n}, q_{2n+1}, t)),$$

that is, for all $n \geq 0$, $t > 0$, we have

$$M(q_{2n+1}, q_{2n+2}, kt) \geq M(q_{2n}, q_{2n+1}, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.8)$$

From (3.7) and (3.8), for all $t > 0$, $n \geq 1$, we have

$$M(q_n, q_{n+1}, kt) \geq M(q_{n-1}, q_n, t), \text{ (using the properties of } \psi\text{- function).} \quad (3.9)$$

By (3.6) and (3.9), for all $n > 0$, $t > 0$, we have

$$M(p_n, p_{n+1}, kt) * M(q_n, q_{n+1}, kt) \geq M(p_{n-1}, p_n, t) * M(q_{n-1}, q_n, t). \quad (3.10)$$

In view of (3.10), by Lemma 2.11, we conclude that $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$p_{2n} = gx_{2n+1} = F(x_{2n}, y_{2n}) = p_{2n+1} = hx_{2n+2} = G(x_{2n+1}, y_{2n+1}) \to x \text{ as } n \to \infty$$

and

$$q_{2n} = gy_{2n+1} = F(y_{2n}, x_{2n}) = q_{2n+1} = hy_{2n+1} = G(y_{2n+1}, x_{2n+1}) \to y \text{ as } n \to \infty.$$

Also $x, y \in h(X) \bigcap g(X)$.

Since, $G(X \times X) \subseteq h(X)$, there exists $u \in X$ such that $hu = x$ and also there exists $v \in X$ such that $hv = y$.

Now for all $t > 0$, we have

$$M(F(u, v), G(x_{2n+1}, y_{2n+1}), kt) \geq \psi(M(hu, F(u, v), t), M(gx_{2n+1}, G(x_{2n+1}, y_{2n+1}), t)).$$

Taking $n \to \infty$ on the both sides, for all $t > 0$, we have

$$M(F(u, v), x, kt) \geq \psi(M(x, F(u, v), t), M(x, F(u, v), t))$$
$$M(F(u, v), x, kt) \geq M(x, F(u, v), t),$$

which implies that $F(u, v) = x$.

Therefore, $F(u, v) = hu = x$.

Similarly, we can prove $F(v, u) = hv = y$.

Since, $F(X \times X) \subseteq g(X)$, there exists $r \in X$ such that $gr = x$ and also there exists $z \in X$ such that $gz = y$.

Now for all $t > 0$, we have

$$\begin{aligned} M(x, G(r, z), kt) &= M(F(x_{2n}, y_{2n}), G(r, z), kt) \\ &\geq \psi(M(hx_{2n}, F(x_{2n}, y_{2n}), t), M(gr, G(r, z), t)) \text{ (by (3.1))} \end{aligned}$$

Taking $n \to \infty$ on the both sides, for all $t > 0$, we have

$$M(x, G(r, z), kt) \geq \psi(M(x, G(r, z), t), M(x, G(r, z), t))$$
$$M(x, G(r, z), kt) \geq M(x, G(r, z), t),$$

which implies that $G(r, z) = x$.

Therefore, $G(r, z) = gr = x$.

Similarly, we can prove $G(z, r) = gz = y$.

Therefore, $F(u, v) = hu = G(r, z) = gr = x$ and $F(v, u) = hv = G(z, r) = gz = y$.

Since, (F, h) is weakly compatible,

therefore $hF(u, v) = F(hu, hv)$ and $hF(v, u) = F(hv, hu)$,

which implies $hx = F(x, y)$ and $hy = F(y, x)$.
Since, (G, g) is weakly compatible,
therefore $gG(r, z) = G(gr, gz)$ and $gG(z, r) = G(gz, gr)$,
which implies $gx = G(x, y)$ and $gy = G(y, x)$.
Now we will prove $x = hx = F(x, y)$.
For $t > 0$, we have

$$\begin{aligned} M(x, F(x, y), kt) &= M(F(x, y), G(r, z), kt) \\ &\geq \psi(M(hx, F(x, y), t), M(gr, G(r, z), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore, $F(x, y) = x$. So $F(x, y) = hx = x$.
Similarly, $F(y, x) = hy = y$.
Again we will prove $x = gx = G(x, y)$.
For $t > 0$, we have

$$\begin{aligned} M(x, G(x, y), kt) &= M(F(u, v), G(x, y), kt) \\ &\geq \psi(M(hu, F(u, v), t), M(gx, G(x, y), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore, $G(x, y) = x$. So $G(x, y) = gx = x$.
Similarly, $G(y, x) = gy = y$.
Therefore, $F(x, y) = G(x, y) = hx = gx = x$ and $F(y, x) = G(y, x) = hy = gy = y$.
So, (x, y) is the coupled common fixed point of F and G.
To show uniqueness, let (e_1, e_2) be another coupled common fixed point of F and G.
Therefore, $F(e_1, e_2) = G(e_1, e_2) = he_1 = ge_1 = e_1$
and

$$F(e_2, e_1) = G(e_2, e_1) = he_2 = ge_2 = e_2.$$

$$\begin{aligned} M(x, e_1, kt) &= M(F(x, y), G(e_1, e_2), kt) \\ &\geq \psi(M(hx, F(x, y), t), M(ge_1, G(e_1, e_2), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore $e_1 = x$.

$$\begin{aligned} M(y, e_2, kt) &= M(F(y, x), G(e_2, e_1), kt) \\ &\geq \psi(M(hy, F(y, x), t), M(ge_2, G(e_2, e_1), t)) \text{ (by (3.1))} \\ &\geq \psi(1, 1) \\ &\geq 1. \end{aligned}$$

Therefore $e_2 = y$.
Therefore (x, y) is the unique coupled common fixed point of F and G.
Thus completes the proof. □

Corollary 3.3 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadzic type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $F : X \times X \to X$, $h : X \to X$ be two mappings satisfying the following conditions:
(i) $M(F(x, y), F(u, v), kt) \geq \psi(M(hx, F(x, y), t), M(hu, F(u, v), t))$, (3.11)
(ii) $F(X \times X) \subseteq h(X)$ and $h(X)$ are two closed subsets of X,
(iii) (F, h) is weakly compatible pair,
for every $x, y, u, v \in X$, for $t > 0$, $0 < k < 1$ and ψ is Ψ-function. Then there exist $x, y \in X$ such that $x = hx = F(x, y)$ and $y = hy = F(y, x)$, that is, there exist $x, y \in X$ such that x and y are fixed points of h, and that (x, y) is a unique coupled fixed point of F.

Proof. The proof follows by putting $F = G$, $h = g$ in Theorem 3.2.

Corollary 3.4 Let $(X, M, *)$ be a complete fuzzy metric space with a Hadzic type t-norm where $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $F : X \times X \to X$ be a mapping satisfies the following condition:

$$M(F(x, y), F(u, v), kt) \geq \psi(M(x, F(x, y), t), M(u, F(u, v), t)),$$

for every $x, y, u, v \in X$, $0 < k < 1$ and ψ is Ψ-function. Then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$, that is, F has unique coupled fixed point in X.

Proof. The proof follows by putting $F = G$, $h = g = I$, the identity function, in Theorem 3.2.

Example 3.5 Let $X = [0, 1]$. Let $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ and $a * b = \min\{a, b\}$. Then $(X, M, *)$ is a complete fuzzy metric space with the property that M is strictly increasing in t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let the mappings $F : X \times X \to X$ and $G : X \times X \to X$ be defined as follows:

$$F(x, y) = G(x, y) = \begin{cases} 1, & \text{if } x > 1, \\ 0, & \text{if } \ otherwise, \end{cases}$$

and the mappings $h : X \to X$ and $g : X \to X$ be defined as follows:

$$hx = gx = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 200, & \text{if } x > 1, \end{cases}$$

Let, for all $x, y \in X$, $\psi(x, y) = \min\{x, y\}$. Then all the conditions of Theorem 3.2 are satisfied. Here $(0, 0)$ is unique coupled common fixed point F and G, and 0 is a fixed point of h and g.

4. Applications to result in metric spaces

In this section we present a coupled coincidence point result in metric spaces. This is obtained by an application of the theorem established in the previous section.

Theorem 4.1 Let (X, d) be a complete metric space. Let $F : X \times X \to X$ and $h : X \to X$ be two mappings satisfying the following conditions:
(i) $F(X \times X) \subseteq h(X)$ and $h(X)$ is closed subsets of X,
(ii) (h, F) is weakly compatible pair,
(iii) $d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(hx, F(x, y)) + d(hu, F(u, v))]$, (4.1)
for all $x, y, u, v \in X$ and $0 < k < 1$. Then there exist $x, y \in X$ such that $x = hx = F(x, y)$ and $y = hy = F(y, x)$, that is, F and h have unique coupled common fixed point in X.

Proof. For all $x, y \in X$ and $t > 0$, we define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

and $a * b = \min\{a, b\}$. Then, as noted earlier, $(X, M, *)$ is a fuzzy metric space.
Further, from the above definition of M, $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$.
Next we show that the inequality (4.1) implies (3.11) with $\psi(x, y) = \min\{x, y\}$. If otherwise, from (3.11), for some $t > 0$, $x, y, u, v \in X$ we have

$$\frac{t}{t + \frac{1}{k} d(F(x, y), F(u, v))} < min\{\frac{t}{t + d(hx, F(x, y))}, \frac{t}{t + d(hu, F(u, v))}\},$$

that is, $t + \frac{1}{k} d(F(x, y), F(u, v)) > t + d(hx, F(x, y))$ and

$$t + \frac{1}{k} d(F(x, y), F(u, v)) > t + d(hu, F(u, v)).$$

Combining the above two inequalities, we have that

$$d(F(x,y),F(u,v)) > \frac{k}{2}[d(hx,F(x,y)) + d(hu,F(u,v))],$$

which is a contradiction with (4.1).
The proof is then completed by an application of corollary 3.2. □

5. Conclusion

We use Hadzic type t-norm in our main result. It has an advantage for use due to the fact that the iterates are equicontinuous at $s = 1$. The fuzzy metric space theory depends strongly on the type of t-norm used in its description. Our main theorem depends on a lemma which in turn depends on the aforesaid equicontinuous of the t-norm. Also the coupled fixed point for two maps are obtained under the assumption of weak compatibility between ordinary maps and coupled maps which is a concept defined in this paper. It may be used elsewhere under different conditions to obtain other fixed point results. Since this definition does not depend on the structure associated with the set on which the mappings are defined, it is possible that such definitions are used in some other spaces as well.

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