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BALL CONVERGENCE OF AN EIGHTH ORDER- ITERATIVE SCHEME WITH HIGH EFFICIENCY ORDER IN BANACH SPACE

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ABSTRACT. We present a local convergence analysis of an eighth order- iterative method in order to approximate a locally unique solution of an equation in Banach space setting. Earlier studies such as [13, 18] have used hypotheses up to the fourth derivative although only the first derivative appears in the definition of these methods. In this study, we only use the hypothesis of the first derivative. This way we expand the applicability of these methods. Moreover, we provide a radius of convergence, a uniqueness ball and computable error bounds based on Lipschitz constants. Numerical examples computing the radii of the convergence balls as well as examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

KEYWORDS: Banach space; eighth-order of convergence; local convergence; efficiency index.
AMS Subject Classification: Primary 65D10; Secondary 65D99, 65E99.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) [1–30]. Closed form solutions of these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore solutions of these nonlinear equations (1.1) are approximated by iterative methods.

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In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1–30]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–30].

Newton's method is undoubtedly the most popular method for approximating a locally unique solution x^* provided that the initial point is close enough to the solution. In order to obtain a higher order of convergence Newton-like methods have been studied such as Potra-Ptak, Chebyshev, Cauchy Halley and Ostrowski method [3, 6, 23, 26]. The number of function evaluations per step increases with the order of convergence. In the scalar case the efficiency index [3, 6, 21] $EI = p^{\frac{1}{m}}$ provides a measure of balance where p is the order of the method and m is the number of function evaluations.

It is well known that according to the Kung-Traub conjuncture the convergence of any multi-point method without memory cannot exceed the upper bound 2^{m-1} [21] (called the optimal order). Hence the optimal order for a method with three function evaluations per step is 4. The corresponding efficiency index is $EI = 4^{\frac{1}{3}} = 1.58740...$ which is better than Newtons method which is $EI = 2^{\frac{1}{2}} = 1.414....$ Therefore, the study of new optimal methods of order four is important.

We present the local convergence analysis of the eighth-order method defined for each $n = 0, 1, 2...$ by

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n)\\
w_n &= \frac{1}{2}(y_n + x_n)\\
z_n &= \frac{1}{3}(4w_n - x_n)\\
u_n &= w_n + (F'(x_n) - 3F'(z_n))^{-1}F(x_n)\\
v_n &= u_n + 2(F'(x_n) - 3F'(z_n))^{-1}F(u_n)\\
x_{n+1} &= v_n + (F'(x_n) - 3F'(z_n))^{-1}F(v_n),
\end{aligned} \tag{1.2}$$

where x_0 is an initial point. The local convergence analysis of method (1.2) was given in [13] in the special case when $X = Y = \mathbb{R}^m$. The semi-local convergence analysis of method (1.2) in a Banach space was given in [18]. The computational efficiency of method (1.2) was also given in [18]. However, the convergence hypotheses for method (1.2) in these references require hypotheses up to the fourth derivative of operator F. These hypothesis limit the applicability of method (1.2) and the other comparable methods given in [13, 18]. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned}
F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; F'(1) = 3,\\
F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x\\
F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22.
\end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. Notice that, in-particular there is a plethora of iterative methods for approximating solutions of nonlinear equations defined on $\mathbb{R}$ [1–30]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) in Section 2. The same technique can be used to other methods.

In the present study we extend the applicability of the method (1.2) by using hypotheses up to the first derivative of function F and contractions on a Banach space setting. Moreover we avoid Taylor expansions and use instead Lipschitz parameters. Moreover, we do not have to use higher order derivatives to show the convergence of method (1.2). This way we expand the applicability of method (1.2).

The paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding Section 3.

2. Local Convergence Analysis

We present the local convergence analysis of the method (1.2) in this section. Let $L_0 > 0$, $L > 0$ and $M \geq 1$ be parameters. It is convenient for the local convergence analysis of method (1.2) that follows to introduce some scalar functions and parameters. Define functions g_1, g_2 on the interval $[0, 1/L_0)$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)},$$

$$g_2(t) = \frac{1}{2}(1 + g_1(t))$$

and parameters r_A, r_0 by

$$r_A = \frac{2}{2L_0 + L},\ r_0 = \frac{1}{3L_0}.$$

Moreover, define functions g_3, h_3, g_4, h_4, g_5 and h_5 on the interval $[0, r_0)$ by

$$g_3(t) = \frac{1}{2(1 - L_0 t)}(L + \frac{8ML_0}{1 - 3L_0 t})t,\ h_3(t) = g_3(t) - 1,$$

$$g_4(t) = (1 + \frac{2M}{1 - 3L_0 t})g_3(t),\ h_4(t) = g_4(t) - 1,$$

$$g_5(t) = (1 + \frac{2M}{1 - 3L_0 t})g_4(t)$$

and

$$h_5(t) = g_5(t) - 1.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to r_0^-$. It then follows from the intermediate value theorem that function h_3 has zeros in the interval $(0, r_0)$. Denote by r_3 the smallest such zero. We also have that $h_4(0) = -1 < 0$ and $h_4(r_3) = \frac{2M}{1 - 3L_0 r_3} > 0$, since $g_3(r_3) = 1$ and $1 - 3L_0 r_3 > 0$. Denote by r_4 the smallest zero of function h_4 in the interval $(0, r_3)$. Finally, we have $h_5(0) = -1 < 0$ and $h_5(r_4) = \frac{2M}{1 - 3L_0 r_4} > 0$. Denote by r_5 the smallest zero of function h_5 in the interval $(0, r_4)$. Set

$$r = \min\{r_A, r_5\}. \tag{2.1}$$

Then, we have that

$$0 < r < r_A \tag{2.2}$$

and for each $t \in [0, r)$

$$0 \leq g_1(t) < 1 \tag{2.3}$$
$$0 \leq g_2(t) < 1 \tag{2.4}$$
$$0 \leq g_3(t) < 1 \tag{2.5}$$
$$0 \leq g_4(t) < 1 \tag{2.6}$$

and

$$0 \leq g_5(t) < 1. \tag{2.7}$$

Let $U(\gamma, \rho)$, $\bar{U}(\gamma, \rho)$, respectively the open and closed balls in X with center $r \in X$ and of radius $r \in X$ and of $\rho > 0$. Next, we present the local convergence analysis of the method (1.2), using the preceding notation.

Theorem 2.1. *Let $F : D \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$ and $M \geq 1$ such that for each $x, y \in D$*

$$F(x^*) = 0, F'(x^*)^{-1} \in L(Y, X), \tag{2.8}$$
$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|, \tag{2.9}$$
$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - x^*\|, \tag{2.10}$$
$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{2.11}$$

and

$$\bar{U}(x^*, \frac{5}{3}r) \subseteq D, \tag{2.12}$$

where the radius r is given by (2.1). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined , remains in $U(x^*, r)$ for each $n = 0, 1, 2.....$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.13}$$
$$\|w_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$
$$\|z_n - x^*\| \leq \frac{1}{3}(4\|w_n - x^*\| + \|x_n - x^*\|) < \frac{5}{3}\|x_n - x^*\|, \tag{2.15}$$
$$\|u_n - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.16}$$
$$\|v_n - x^*\| \leq g_4(\|x_n - x^*)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.17}$$

and

$$\|x_{n+1} - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*, \| \tag{2.18}$$

where the "g" functions are defined above Theorem 2.1. Furthermore, if there exist $T \in [r, \frac{2}{L_0})$ and $\bar{U}(x^*, T) \in D$, then the limit point x^* is the only solution of the equation $F(x) = 0$ in $\bar{U}(x^*, T) \cap D$.

Proof: We shall show estimates (2.13)-(2.18) using mathematical induction. By (2.1), (2.9) and hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.19}$$

It follows from (2.19) and Banach Lemma on invertible operators [3,6,19,23,24,28] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r}. \tag{2.20}$$

Hence, y_0, w_0 and z_0 are well defined. Using the first sub-step of method (1.2) for $n = 0$, (2.1), (2.2), (2.8), (2.10) and (2.20), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\
&\le \|F'(x_0)^{-1}F'(x^*)\| \| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) \\
&- F'(x_0))(x_0 - x^*)d\theta\| \\
&\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \qquad (2.21)
\end{aligned}$$

which shows (2.13) for $n = 0$ and $y_0 \in U(x^*, r)$. Then, by the second sub-step of method (1.2) for $n = 0$, (2.1), (2.3) and (2.21), we obtain that

$$\begin{aligned}
\|w_0 - x^*\| &\le \frac{1}{2}(\|y_0 - x^*\| + \|x_0 - x^*\|) \\
&\le \frac{1}{2}(1 + g_1(\|x_0 - x^*\|))\|x_0 - x^*\| \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \qquad (2.22)
\end{aligned}$$

which shows (2.14) and $w_0 \in U(x^*, r)$. In view of third sub-step of method (1.2) for $n = 0$, (2.1) and (2.22), we get that

$$\begin{aligned}
\|z_0 - x^*\| &\| \frac{1}{3}(4(w_0 - x^*) - (x_0 - x^*)\| \\
&\le \frac{1}{3}(4\|w_0 - x^*\| + \|x_0 - x^*\|) \\
&\le \frac{1}{3}(4\|x_0 - x^*\| + \|x_0 - x^*\|) \\
&= \frac{5}{3}\|x_0 - x^*\| < \frac{5}{3}r, \qquad (2.23)
\end{aligned}$$

which shows (2.15) for $n = 0$ and $z_0 \in U(x^*, \frac{5}{3}r) \subset D$ (by (2.12)). Next, we shall show that $(F'(x_0) - 3F'(z_0))^{-1} \in L(Y, X)$. Using (2.1), (2.9) and (2.23), we get that

$$\begin{aligned}
&\|(-2F'(x^*))^{-1}[F(x_0) - 3F'(z_0) - F'(x^*) + 3F'(x^*)]\| \\
&\le \frac{1}{2}[\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\
&+ 3\|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\|] \\
&\le \frac{L_0}{2}[\|x_0 - x^*\| + 3\|z_0 - x^*\|] \\
&< \frac{L_0}{3}(\|x_0 - x^*\| + 3(\frac{5}{3})\|x_0 - x^*\|) \\
&= 3L_0\|x_0 - x^*\| < 3L_0 r < 1. \qquad (2.24)
\end{aligned}$$

Hence, we get that u_0 is well defined by the fourth sub-step of method (1.2) for $n = 0$ and

$$\begin{aligned}
\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\| &\le \frac{1}{2(1 - \frac{L_0}{2}(\|x_0 - x^*\| + 3\|z_0 - x^*\|))} \\
&\le \frac{1}{2(1 - 3L_0\|x_0 - x^*\|)}. \qquad (2.25)
\end{aligned}$$

Hence, u_0, v_0 and x_1 are well defined. We can write by (2.8) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.26}$$

Notice that $\|x^*+\theta(x_0-x^*)-x^*\| = \theta\|x_0-x^*\| < r$. That is $x^*+\theta(x_0-x^*) \in U(x^*, r)$. Using (2.11) and (2.26), we obtain that

$$\begin{aligned}\|F'(x^*)^{-1}F(x_0)\| &= \|\int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta\| \\ &\leq M\|x_0 - x^*\|. \end{aligned}\tag{2.27}$$

We can write in turn by the first, second and fourth sub-step of method (1.2) for $n = 0$ that

$$u_0 - x^* = \frac{1}{2}(y_0 - x^*) + \frac{1}{2}(x_0 - x^*) + (F'(x_0) - 3F'(z_0))^{-1}F(x_0) \tag{2.28}$$

$$\begin{aligned} &= \frac{1}{2}(y_0 - x^*) + \frac{1}{2}(x_0 - x^* - F'(x_0)^{-1}F(x_0)) \\ &\quad +\frac{1}{2}F'(x_0)^{-1}F(x_0) + (F'(x_0) - 3F'(z_0))^{-1}F(x_0) \\ &= y_0 - x^* \\ &\quad +\frac{1}{2}F'(x_0)^{-1}[F'(x_0) - 3F'(z_0) + 2F'(x_0)](F'(x_0) - 3F'(z_0))^{-1}F(x_0) \\ &= y_0 - x^* \\ &\quad +\frac{3}{2}F'(x_0)^{-1}(F'(x_0) - F'(z_0))(F'(x_0) - 3F'(z_0))^{-1}F(x_0). \end{aligned}\tag{2.29}$$

Using (2.1), (2.5), (2.20), (2.21), (2.25), (2.27) and (2.29), we obtain in turn that

$$\begin{aligned}\|u_0 - x^*\| &\leq \|y_0 - x^*\| + \frac{3}{2}\|F'(x_0)^{-1}F(x_0)\| \\ &\quad \times [\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\|] \\ &\quad \times \|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{3ML_0(\|x_0 - x^*\| + \|z_0 - x^*\|)\|x_0 - x^*\|}{2(1 - L_0\|x_0 - x^*\|)(1 - 3L_0\|x_0 - x^*\|)} \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned}\tag{2.30}$$

which shows (2.16) for $n = 0$ and $u_0 \in U(x^*, r)$ (where, we also used the estimate $\|x_0 - x^*\| + \|z_0 - x^*\| \leq \|x_0 - x^*\| + \frac{5}{3}\|x_0 - x^*\| = \frac{8}{3}\|x_0 - x^*\| < \frac{8}{3}r$). Then, as in (2.27) for $x_0 = w_0$, we obtain that

$$\|F'(x^*)^{-1}F(w_0)\| \leq M\|w_0 - x^*\|. \tag{2.31}$$

Using the fifth sub-step of method (1.2) for $n = 0$, (2.1), (2.6), (2.25), (2.30) and (2.31), we have that

$$\begin{aligned}\|v_0 - x^*\| &\leq \|u_0 - x^*\| + 2\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(u_0)\| \\ &\leq \|u_0 - x^*\| + \frac{2M\|u_0 - x^*\|}{1 - 3L_0\|x_0 - x^*\|} \\ &= (1 + \frac{2M}{1 - 3L_0\|x_0 - x^*\|})\|u_0 - x^*\| \\ &\leq g_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned}\tag{2.32}$$

which shows (2.17) for $n = 0$ and $v_0 \in U(x^*, r)$.

Then, again as in (2.27) for $x_0 = v_0$, we get that

$$\|F'(x^*)^{-1}F(v_0)\| \leq M\|v_0 - x^*\|. \tag{2.33}$$

Using the sixth sub-step of method (1.2) for $n = 0$, (2.1), (2.7), (2.25), (2.32) and (2.33), we have that

$$\begin{aligned} \|x_1 - x^*\| &\leq \|v_0 - x^*\| + 2\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(v_0)\| \\ &= (1 + \frac{2M}{1 - 3L_0\|x_0 - x^*\|})\|v_0 - x^*\| \\ &= g_5(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{2.34}$$

which shows (2.18) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing $x_0, y_0, w_0, z_0, v_0, x_1$ by $x_k, y_k, w_k, z_k, v_k, x_{k+1}$ in the preceding estimates we arrive at estimates (2.13) – (2.18). Then, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (2.9) we get that

$$\begin{aligned} |F'(x^*)^{-1}(Q - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ &\leq \int_0^1 L_0(1-\theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}T < 1. \end{aligned} \tag{2.35}$$

It follows from (2.35) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$.

□

Remark 2.1.

1. In view of (2.9) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

   condition (2.11) can be dropped and be replaced by

$$M(t) = 1 + L_0 t,$$

   or

$$M = M(t) = 2,$$

   since $t \in [0, \frac{1}{L_0})$.
2. The results obtained here can be used for operators F satisfying autonomous differential equations [3, 6, 17] of the form

$$F'(x) = G(F(x))$$

   where T is a continuous operator. Then, since $F'(x^*) = G(F(x^*)) = G(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.
3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3–7].

4. The parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3,6]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0,1,2,\cdots \tag{2.36}$$

under the conditions (2.8)–(2.10). It follows from the definitions of radii r that the convergence radius r of these preceding methods cannot be larger than the convergence radius r_A of the second order Newton's method (2.26). As already noted in [3,6] r_A is at least as large as the convergence ball given by Rheinboldt [26]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ as \ \frac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [28].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding Theorems instead of the stronger conditions used in [13, 18]. Moreover, the preceding Theorems we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

3. Numerical Examples

The numerical examples are presented in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.9) conditions, we get $L_0 = e - 1, L = e, M = 2$. The parameters are

$$r_A = 0.3249, r_0 = 0.3880, r_3 = 0.0471, r_4 = 0.0117, r_5 = 0.0026 = r.$$

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15$, $M = 2$. The parameters for method are

$$r_A = 0.0667, r_0 = 0.0889, r_3 = 0.0106, r_4 = 0.0026, r_5 = 0.0006 = r.$$

Example 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073$, $M = 2$. The parameters are

$$r_A = 0.0045 = r_0, r_3 = 0.0006, r_4 = 0.0001 = r, r_5 = 0.0091.$$

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