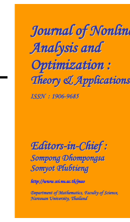
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 7, No. 2, (2016), 115-128
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



OPTIMAL SYNCHRONIZATION AND ANTI-SYNCHRONIZATION FOR A CLASS OF CHAOTIC SYSTEMS

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ABSTRACT. In this study, we apply the optimal adaptive control for synchronization and anti-synchronization of chaotic T-system with complete uncertain parameters on finite and infinite time intervals. Based on the Lyapunov stability theorem and Hamilton Jacobian Bellman(HJB) technique, optimal controls and parameters estimations laws are obtained. For this aim, conditions ensuring asymptotic stability of error system and minimizing cost function are used. The derived control laws make asymptotically synchronization and anti-synchronization of two identical T-systems. Finally, numerical simulations are presented to illustrate the ability and effectiveness of the proposed method.

KEYWORDS: Lyapunov stability; Synchronization; Optimal adaptive control; Chaos; T-system.
AMS Subject Classification: 49J15, 34D06.

1. INTRODUCTION

Chaos, as an interesting phenomenon in nonlinear dynamical systems, has been studied over the last four decades [21, 28, 35, 29, 15, 34, 2]. Chaotic and hyperchaotic systems serving as nonlinear deterministic systems display complex and unpredictable behavior. In addition, these systems are sensitive with respect to initial conditions. The chaotic and hyperchaotic systems have many important applications in nonlinear sciences such as laser physics, secure communications, nonlinear circuits, control, neural networks, and active wave propagation [15, 30, 26, 19, 5, 3, 8, 13, 27].

The synchronization of chaotic systems have been investigated since their introduction in the paper by Pecora and Carrol in 1990 [29] and have been widely investigated in many fields, such as physics, chemistry, ecological sciences, and secure communications [14, 6, 34]. Various techniques and methods have been

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Article history : Received 9 May 2016 Accepted 1 February 2018.

proposed to achieve chaos synchronization and anti-synchronization such as adaptive control [22, 33, 9, 16, 18, 17], active control [10] and nonlinear control [1]. Fortunately, some existing synchronization methods can be generalized to anti-synchronization of chaotic systems. Recently, synchronization of chaotic complex systems was studied in [25]. Furthermore, the anti-synchronization and adaptive anti-synchronization of two different chaotic systems were investigated in [24, 23].

Most methods mentioned in previous paragraph, synchronize two identical chaotic systems using adaptive methods. However, synchronization of systems with these methods is far from optimal synchronization. In general, the adaptive synchronization does not necessarily satisfy the optimality conditions.

The problem of the minimal control synthesis algorithm for controlling and synchronizing chaotic systems was studied in [12] and the optimal control for the chaos synchronization of Rössler system with complete uncertain parameters, was discussed in [11].

In this study, we present the optimal adaptive synchronization and anti-synchronization schemes between two identical chaotic T-system, with three unknown parameters. By this method, we can achieve synchronization and anti-synchronization of drive and response systems, and identify the unknown parameters. Based on the Lyapunov stability theorem and Hamilton-Jacobi- Bellman(HJB) equation in finite and infinite time intervals, optimal adaptive controllers with parameters estimation rules are designed to synchronizing and anti-synchronizing chaotic T-systems asymptotically.

The rest of this paper is organized as follows: In section 2 we introduce the chaotic T-systems briefly. In Section 3, the synchronization and anti-synchronization of two identical T-system with optimal controllers and parameters estimation rules in infinite and finite time intervals are presented. Besides, numerical simulations are computed to check the analytical expressions of optimal controllers and estimation laws. Finally, concluding remarks are given in Section 4.

2. T-system

In 2005, Tigen [32] introduced a new real chaotic nonlinear system which is called T-system, as follows

$$\begin{cases} \dot{x_1} = a(x_2 - x_1) \\ \dot{x_2} = (c-a)x_1 - ax_1x_3 \\ \dot{x_3} = x_1x_2 - bx_3, \end{cases} \tag{2.1}$$

where x_1, x_2 and x_3 are the state variables and a, b and c are real positive parameters. Taking $a = 2.1$, $b = 0.6$ and some values of $0 < c < 40$, the positiveness of one of the Lyapunov exponents in figure 1 shows that the system (2.1) is a chaotic system. Furthermore, the negativity of sum of its Lyapunov exponents implies that the system is dissipative. figure 2 displays an attractor of T-system for some parameters and initial conditions. The attractors are bounded but not fixed points and limit cycles as a property of chaotic systems [4]. Synchronization and anti-synchronization of this system can be used for cryptography and decryption of data in secure communication.

3. Synchronization And Anti-synchronization

The drive and response systems are defined as follows

$$\dot{x} = f(x); \tag{3.1}$$

$$\dot{y} = g(y) + u, \tag{3.2}$$

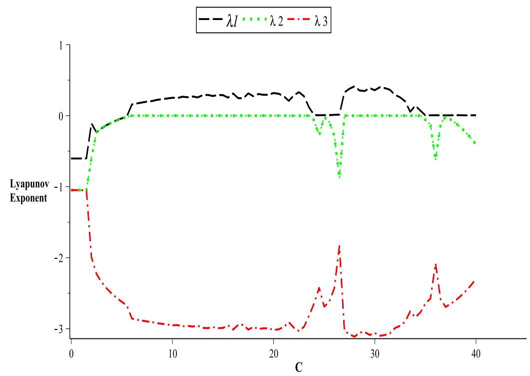


FIGURE 1. Lyapunov exponents of system (2.1), for $a = 2.1$, $b = 0.6$ and $0 < c < 40$.

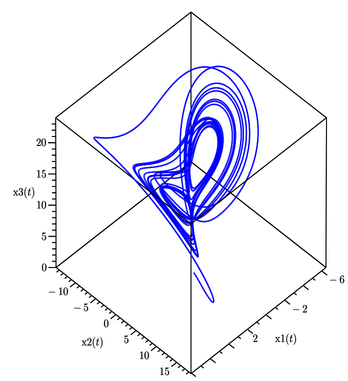


FIGURE 2. Attractor of T-system for $a = 2.1$, $b = 0.6$ and $c = 28$ with initial conditions $(x_1(0), x_2(0), x_3(0)) = (1, 3, 0)$.

where $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in \mathbb{R}^n$ are state vectors of the systems (3.1) and (3.2), $u = (u_1, u_2, \cdots, u_n)^T$ is a vector control and $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Synchronization error is $e = y - x$ and anti-synchronization error is $e = y + x$, therefore we can show the synchronization and the anti-synchronization errors with $e = y + kx$, for $k = -1$ and $k = 1$, respectively. Then, the error system between drive and response systems is

$$\dot{e} = \dot{y} + k\dot{x} = g(y) + kf(x) + u = g(e - kx) + kf(x) + u, \qquad k = \mp 1. \tag{3.3}$$

The goal is to design an appropriate optimal controller u such that for any initial condition y_0 and x_0, we have

$$\lim_{t \longrightarrow \infty} \|e\| = \lim_{t \longrightarrow \infty} \|y(t) + kx(t)\| = 0, \qquad k = \mp 1,$$

where $\|.\|$ is the Euclidean norm [22, 29].

Concerning synchronization and anti-synchronization of two chaotic T-system, assume that (2.1) is the drive system. The response system can be considered in two cases: First, it is identical with the drive system, but in the second instance, it can be different from the drive system. In this paper, we consider the response system to be identical with drive system, with unknown parameters. Then, the response system is defined as follow

$$\begin{cases} \dot{y_1} = \hat{a}(y_2 - y_1) + u_1 \\ \dot{y_2} = (\hat{c} - \hat{a})y_1 - \hat{a}y_1y_3 + u_2 \\ \dot{y_3} = y_1y_2 - \hat{b}y_3 + u_3, \end{cases} \tag{3.4}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are uncertain parameters and u_1, u_2 and u_3 are control functions. Then, the error system between the drive and response systems, is

$$\begin{cases} \dot{e_1} = & \hat{a}(e_2 - e_1) + k(\hat{a} - a)(x_1 - x_2) + u_1 \\ \dot{e_2} = & (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 - ke_1x_3 - ke_3x_1] - (\hat{a} + ka)x_1x_3 \\ & -k(\hat{c} - c - \hat{a} + a)x_1 + u_2 \\ \dot{e_3} = & e_1e_2 - ke_1x_2 - ke_2x_1 + (k+1)x_1x_2 - \hat{b}e_3 + k(\hat{b} - b)x_3 + u_3. \end{cases} \tag{3.5}$$

Now, we discuss the optimal synchronization and anti-synchronization of (2.1), (3.4).

3.1. **Optimal synchronization.**

For synchronization, let $k = -1$ in all the above mentioned relations; then, $e_i = y_i - x_i (i = 1, 2, 3)$, and the error system is

$$\begin{cases} \dot{e_1} = (\hat{a} - a)(x_2 - x_1) + U_1 \\ \dot{e_2} = (\hat{c} - c)x_1 - (\hat{a} - a)(x_1 + x_1x_3) + U_2 \\ \dot{e_3} = -(\hat{b} - b)x_3 + U_3, \end{cases} \tag{3.6}$$

where

$$\begin{cases} U_1 = \hat{a}(e_2 - e_1) + u_1 \\ U_2 = (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 + e_1x_3 + e_3x_1] + u_2 \\ U_3 = e_1e_2 + e_1x_2 + e_2x_1 - \hat{b}e_3 + u_3. \end{cases} \tag{3.7}$$

In the following subsection, we find the controllers and dynamics of uncertain parameters for synchronization on infinite and finite time intervals.

3.1.1. *Optimal synchronization for infinite time.*

Concerning optimal synchronization on infinite time, similar to papers [11, 12], we consider the following extended dynamics for estimating the unknown parameters

$$\begin{cases} \dot{\hat{a}} = -\frac{1}{\beta_1}\left[\alpha_1e_1(x_2 - x_1) - \alpha_2e_2(x_1 + x_1x_3) + \frac{k_1}{2}(\hat{a} - a)\right] \\ \dot{\hat{b}} = \frac{1}{\beta_2}\left[\alpha_3e_3x_3 - \frac{k_2}{2}(\hat{b} - b)\right] \\ \dot{\hat{c}} = -\frac{1}{\beta_3}\left[\alpha_2e_2x_1 + \frac{k_3}{2}(\hat{c} - c)\right], \end{cases} \tag{3.8}$$

where β_i and $\alpha_i (i = 1, 2, 3)$, are real positive and $k_i (i = 1, 2, 3)$ are nonnegative constants. We design controllers $u_i (i = 1, 2, 3)$ such that the dynamic systems (3.6) and (3.8) are applied as follows when $t \to \infty$,

$$e_1 = e_2 = e_3 = 0, \hat{a} = a, \hat{b} = b, \hat{c} = c. \tag{3.9}$$

For this aim, we minimize the following cost function with respect to control vector $\vec{U} = (U_1, U_2, U_3)$

$$\begin{aligned} I & = \int_{t_0}^{\infty} \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U})dt = \\ & = \int_{t_0}^{\infty} \{k_1(\hat{a} - a)^2 + k_2(\hat{b} - b)^2 + k_3(\hat{c} - c)^2 + \textstyle\sum_1^3(c_ie_i^2 + \eta_iU_i^2)\}dt, \end{aligned} \tag{3.10}$$

where t_0 is a fixed time moment, c_i and $\eta_i (i = 1, 2, 3)$ are real positive constants.

Assuming the minimum of (3.10) is $\vec{U} = U^*$, we define

$$V(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t) = \min_{\vec{U}} \int_{t_0}^{\infty} \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U})dt. \tag{3.11}$$

In fact, the function $V(e_1,e_2,e_3,\hat{a},\hat{b},\hat{c},t)$ is the value of the performance index evaluated along the optimal trajectory of the error and updating system parameter which begins at $(e_1,e_2,e_3,\hat{a},\hat{b},\hat{c})$. We shall show that $V(e_1,e_2,e_3,\hat{a},\hat{b},\hat{c})$ can be selected as the Lyapunov function of our system. Now, we find the optimal controller $\vec{U}$ such that it stabilizes the state (3.9) and minimizes I on (3.10). This requirement can be met by applying HJB technique of dynamic programming problem[20, 11]; that is

$$\begin{aligned}&\tfrac{\partial V}{\partial e_1}\dot{e_1}+\tfrac{\partial V}{\partial e_2}\dot{e_2}+\tfrac{\partial V}{\partial e_3}\dot{e_3}+\tfrac{\partial V}{\partial \hat{a}}\dot{\hat{a}}+\tfrac{\partial V}{\partial \hat{b}}\dot{\hat{b}}+\tfrac{\partial V}{\partial \hat{c}}\dot{\hat{c}}+k_1(\hat{a}-a)^2\\&+k_2(\hat{b}-b)^2+k_3(\hat{c}-c)^2+\textstyle\sum_1^3(c_ie_i^2+\eta_iU_i^{*2})=0.\end{aligned}\tag{3.12}$$

Substituting the relations(3.6), into the partial differential equation(3.12), we have

$$\begin{aligned}&\tfrac{\partial V}{\partial e_1}[(\hat{a}-a)(x_2-x_1)+U_1^*]+\tfrac{\partial V}{\partial e_2}[(\hat{c}-c)x_1-(\hat{a}-a)(x_1+x_1x_3)+U_2^*]\\&+\tfrac{\partial V}{\partial e_3}[-(\hat{b}-b)x_3+U_3^*]+\tfrac{\partial V}{\partial \hat{a}}\dot{\hat{a}}+\tfrac{\partial V}{\partial \hat{b}}\dot{\hat{b}}+\tfrac{\partial V}{\partial \hat{c}}\dot{\hat{c}}+k_1(\hat{a}-a)^2+k_2(\hat{b}-b)^2\\&+k_3(\hat{c}-c)^2+\textstyle\sum_1^3(c_ie_i^2+\eta_iU_i^{*2})=0,\end{aligned}\tag{3.13}$$

where $\vec{U^*}$ is the optimal control. Minimizing (3.13) with respect to $\vec{U^*}$ [7], gives

$$U_i^*=-\frac{1}{2\eta_i}\frac{\partial V}{\partial e_i}(i=1,2,3).\tag{3.14}$$

To ensure stability of the error and parameter estimation systems in equilibrium points, we must define the Lyaponuv function such that it be positive definite and its derivative be negative definite. For this purpose, we consider

$$\psi(e_1,e_2,e_3,\hat{a},\hat{b},\hat{c})=\beta_1(\hat{a}-a)^2+\beta_2(\hat{b}-b)^2+\beta_3(\hat{c}-c)^2+\sum_1^3\alpha_ie_i^2,\tag{3.15}$$

Besides, in the sequel we choose its coefficient so that it has mentioned properties and is considered as a solution of (3.13). In other words, $\psi = V$ and ψ is a Lyaponuv function. From (3.14) and (3.15) we have the optimal controllers as

$$U_i^*=\frac{-\alpha_i}{\eta_i}e_i(i=1,2,3).$$

Then

$$\begin{cases}u_1=-\frac{\alpha_1}{\eta_1}e_1-\hat{a}(e_2-e_1)\\u_2=-\frac{\alpha_2}{\eta_2}e_2-(\hat{c}-\hat{a})e_1+\hat{a}[e_1e_3+e_1x_3+e_3x_1]\\u_3=-\frac{\alpha_3}{\eta_3}e_1-e_1e_2-e_1x_2-e_2x_1+\hat{b}e_3,\end{cases}\tag{3.16}$$

where the constants α_i,η_i are positive. A simple calculation implies with $\frac{\alpha_i}{\eta_i}=c_i$, we have

$$\dot{\psi}=-\left[k_1(\hat{a}-a)^2+k_2(\hat{b}-b)^2+k_3(\hat{c}-c)^2+\sum_1^3 2c_ie_i^2\right].\tag{3.17}$$

The relation (3.17) shows that $\dot{\psi}$ is negative definite for all nonnegative c_i and k_i. This result shows that the solution (3.9) is asymptotically stable in the Lyapunov sense via optimal control [20].

3.1.2. *Optimal synchronization in a finite time.* In this subsection, we try to finding optimal controllers $u_i(i=1,2,3)$ such that the equilibrium state of error and updating parameters systems be asymptotically stable. These controllers minimize the value of the following integral performance index through the finite time interval $[0, T]$ [12]

$$\begin{array}{ll} I & = \int_0^T \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U}, t)dt \\ & = \int_0^T \{(\hat{a}-a)^2 + (\hat{b}-b)^2 + (\hat{c}-c)^2 + \sum_1^3 (e_i^2 + lU_i^2)\}dt \end{array}. \tag{3.18}$$

Assume that function $\phi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t)$ is minimum of the integral performance index (3.18) along the optimal trajectory of the system that consists of both the error and update system parameters; consider

$$\phi = \min_{\vec{U}} I = \min_{\vec{U}} \int_0^T \Omega(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, \vec{U}, t)dt. \tag{3.19}$$

Using dynamical programming, there exists optimal controller $U_i^*(i = 1, 2, 3)$, such that ϕ satisfies in HJB equation

$$\begin{array}{l} \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial e_1}\dot{e_1} + \frac{\partial \phi}{\partial e_2}\dot{e_2} + \frac{\partial \phi}{\partial e_3}\dot{e_3} + \frac{\partial \phi}{\partial \hat{a}}\dot{\hat{a}} + \frac{\partial \phi}{\partial \hat{b}}\dot{\hat{b}} + \frac{\partial \phi}{\partial \hat{c}}\dot{\hat{c}} + (\hat{a}-a)^2 \\ +(\hat{b}-b)^2 + (\hat{c}-c)^2 + \sum_1^3 (e_i^2 + lU_i^{*2}) = 0, \end{array} \tag{3.20}$$

where the optimal controllers $U_i^*(i = 1, 2, 3)$ are related to the optimal Lyapunov function

$$U_i^* = -\frac{1}{2l}\frac{\partial \phi}{e_i}, (i = 1, 2, 3), \tag{3.21}$$

with

$$\phi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, T) = 0. \tag{3.22}$$

We can choose Lyapunov function as follows so that it is satisfied in the partial differential equation (3.20)

$$\phi(e_1, e_2, e_3, \hat{a}, \hat{b}, \hat{c}, t) = \sqrt{l}p(t)[\sum_1^3 e_i^2 + (\hat{a}-a)^2 + (\hat{b}-b)^2 + (\hat{c}-c)^2], \tag{3.23}$$

where l is positive constant and $p(t)$ is a positive function and is defined under dynamical programming. The boundary condition (3.22) implies

$$p(T) = 0. \tag{3.24}$$

By applying (3.23) in (3.20), we obtain costate equation as follow

$$\begin{cases} \sqrt{l}\frac{dp}{dt} - p^2 + 1 = 0 \\ p(T) = 0. \end{cases} \tag{3.25}$$

By solving the costate equation (3.25), we have

$$p(t) = \tanh(\frac{-t+T}{\sqrt{l}}).$$

Suppose that the estimation rules of unknown parameters are given by the following differential equations

$$\begin{cases} \dot{\hat{a}} = -e_1(x_1 - x_2) + e_2x_1 + e_1x_1x_3 - \frac{p(t)}{2\sqrt{l}}(\hat{a}-a) \\ \dot{\hat{b}} = e_3x_3 - \frac{p(t)}{2\sqrt{l}}(\hat{b}-b) \\ \dot{\hat{c}} = -e_2x_1 - \frac{p(t)}{2\sqrt{l}}(\hat{c}-c). \end{cases} \tag{3.26}$$

TABLE 1. Initial conditions and parameters for example of synchronization

initial condition	Value	Parameter	Value	Parameter	Value
$x_1(0)$	3	α_1	12	k_1	21
$x_2(0)$	-5	α_2	40	k_2	20
$x_3(0)$	1	α_3	21	k_3	11
$y_1(0)$	-2	β_1	10	l	2
$y_2(0)$	2	β_2	1	T	40
$y_3(0)$	-4	β_3	1.2	a	2.1
$\hat{a}(0)$	0.2	η_1	10	b	0.6
$\hat{b}(0)$	-1	η_2	50	c	30
$\hat{c}(0)$	20	η_3	1	–	–

By using (3.26), (3.20) and (3.21), we get

$$\begin{cases} u_1 = -\hat{a}(e_2 - e_1) - \frac{p(t)e_1}{\sqrt{l}} \\ u_2 = -(\hat{c} - \hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1] - \frac{p(t)e_2}{\sqrt{l}} \\ u_3 = -e_1e_2 - e_1x_2 - e_2x_1 + \hat{b}e_3 - \frac{p(t)e_3}{\sqrt{l}}, \end{cases} \tag{3.27}$$

where the function $p(t)$ is positive in $t \in [0, T]$ and

$$\dot{\phi} = -[\sum_1^3 e_i^2 + (\hat{a} - a)^2 + (\hat{b} - b)^2 + (\hat{c} - c)^2] \leq 0, \tag{3.28}$$

shows that the (3.26) and (3.6) are asymptotically stable in the Lyapunov sense in finite time [12, 31].

In finite time case, the response phase trajectory doesn't track the drive phase trajectory, after $t = T$. Because the lyapunov stability condition after a finite time $t = T$, is not established.

3.1.3. *Numerical simulation of optimal synchronization.* To test the validity of the proposed scheme, we present and discuss numerical results for synchronization of chaotic T-system. For synchronization in infinite time interval, systems (2.1), (3.4) and (3.8) with controllers (3.16) are solved numerically by Maple 16 using CK45 method. In finite time interval, systems (2.1), (3.4) and (3.26) with controllers (3.27) are considered. For numerical simulation of synchronization, the initial conditions and parameters are given in table 1.

The results of chaotic synchronization of two identical chaotic T-systems (2.1) and (3.4), via optimal controllers in infinite and finite time intervals are shown in figures 3 and 4. Clearly, the response system tracks the drive system after a relatively slight time. The synchronization errors are plotted in figure 5. figure 6 shows the estimations of uncertain parameters $\hat{a}(t)$, $\hat{b}(t)$ and $\hat{c}(t)$. As expected from analytical considerations, the synchronization errors e_i and error of estimated parameter are converged to zero in infinite and finite time intervals as $t \longrightarrow \infty$ and $t \longrightarrow T$, respectively. In optimal synchronization in finite time, as after the time $t = T = 40$ the controllers will be disabled, the errors of synchronization and estimated parameters are not zero.

3.2. **Optimal anti-synchronization.** For anti-synchronization, assume $k = 1$ in the (3.3) and (3.5). Then $e_i = y_i + x_i (i = 1, 2, 3)$ and the error system between drive

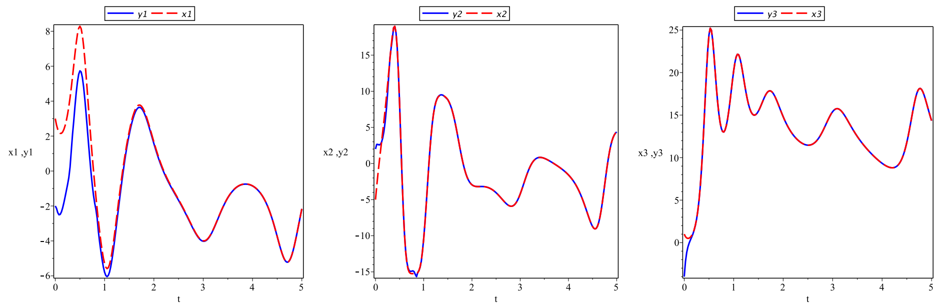


FIGURE 3. Time series trajectories for synchronization via optimal control in infinite time intervals.

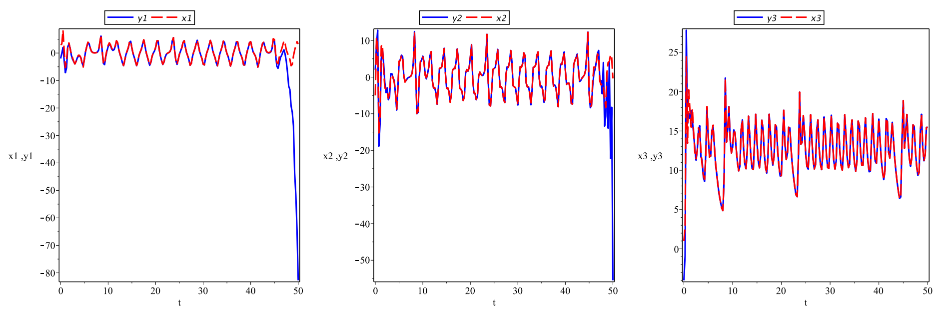
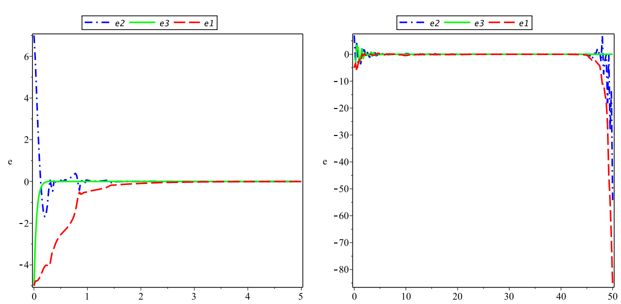


FIGURE 4. Time series trajectories for synchronization via optimal control in finite time intervals.



(a) (b)

FIGURE 5. The error due to synchronization. a: infinite time interval b: finite time interval.

and response systems is written as follows

$$\begin{cases} \dot{e_1} = (\hat{a} - a)(x_1 - x_2) + U_1 \\ \dot{e_2} = -(\hat{c} - c)x_1 + (\hat{a} - a)x_1 + U_2 \\ \dot{e_3} = (\hat{b} - b)x_3 + U_3, \end{cases} \tag{3.29}$$

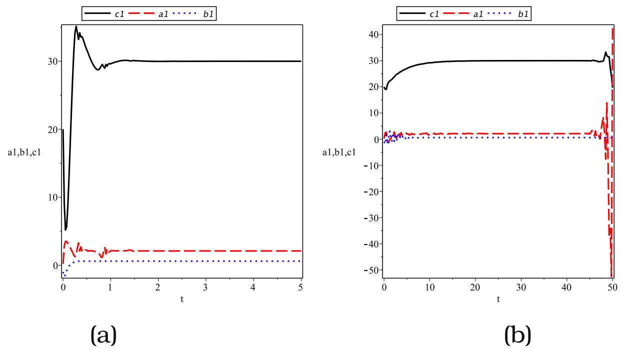


FIGURE 6. Estimation of parameters for synchronization. a: infinite time interval b: finite time interval.

where

$$\begin{cases} U_1 = \hat{a}(e_2 - e_1) + u_1 \\ U_2 = (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 - e_1x_3 - e_3x_1] - (\hat{a} + a)x_1x_3 + u_2 \\ \quad = (\hat{c} - \hat{a})e_1 - \hat{a}[e_1e_3 - e_1x_3 - e_3x_1] - 2\hat{a}x_1x_3 + u_2 \\ U_3 = e_1e_2 - e_1x_2 - e_2x_1 + 2x_1x_2 - \hat{b}e_3 + u_3. \end{cases} \tag{3.30}$$

In the following subsection, we obtain the controllers and dynamic of uncertain parameters that are estimated for anti-synchronization in infinite and finite time intervals.

3.2.1. *Optimal anti-synchronization on infinite time*. For optimal anti-synchronization on infinite time interval, we consider the following extended dynamics to estimate unknown parameters

$$\begin{cases} \dot{\hat{a}} = \frac{1}{\beta_1}\left[\alpha_1 e_1(x_2 - x_1) - \alpha_2 e_2 x_1 - \frac{k_1}{2}(\hat{a} - a)\right] \\ \dot{\hat{b}} = \frac{1}{\beta_2}\left[-\alpha_3 e_3 x_3 - \frac{k_2}{2}(\hat{b} - b)\right] \\ \dot{\hat{c}} = \frac{1}{\beta_3}\left[\alpha_2 e_2 x_1 - \frac{k_3}{2}(\hat{c} - c)\right], \end{cases} \tag{3.31}$$

$k_i (i = 1, 2, 3)$ is non-negative and α_i and $\beta_i (i = 1, 2, 3)$ are positive constants.

Now, we design the controllers $u_i (i = 1, 2, 3)$ such that the dynamical systems (3.29) and (3.31) have the following solution

$$e_1 = e_2 = e_3 = 0, \ \hat{a} = a, \ \hat{b} = b, \ \hat{c} = c. \tag{3.32}$$

This solution presents the equilibrium of systems (3.29) and (3.31). To this end, we consider (3.10) and (3.11) with $e_i = x_i + y_i$. In this case, the HJB for anti-synchronization will be similar to (3.12) and (3.13). Therefore, by applying HJB technique of dynamic programming problem and minimizing it, we have the optimal controller as follows

$$U_i^* = -\frac{1}{2\eta_i}\frac{\partial V}{e_i}, (i = 1, 2, 3). \tag{3.33}$$

Substituting the equation (3.33) into the HJB partial differential equation, it can be solved for the optimal Lyapunov V. The Lyapunov function (3.15) should satisfy in HJB equation of anti-synchronization. After lengthy manipulation, the optimal controllers can be obtained as follow

$$U_i^* = \frac{-\alpha_i}{\eta_i}e_i = \frac{-\alpha_i}{\eta_i}(x_i + y_i),$$

then

$$\begin{cases} u_1 = -\frac{\alpha_1}{\eta_1}e_1 - \hat{a}(e_2 - e_1) \\ u_2 = -\frac{\alpha_2}{\eta_2}e_2 - (\hat{c} - \hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1 + 2x_1x_3] \\ u_3 = -\frac{\alpha_3}{\eta_3}e_1 - e_1e_2 - e_1x_2 - e_2x_1 - 2x_1x_3 + \hat{b}e_3 \end{cases} \quad (3.34)$$

and

$$\dot{V} = -\left[k_1(\hat{a} - a)^2 + k_2(\hat{b} - b)^2 + k_3(\hat{c} - c)^2 + \sum_0^3 2c_ie_i^2\right]. \quad (3.35)$$

Hence, the time derivative of V is negative definite for all nonnegative c_i and k_i. This shows that the solution (3.32) is asymptotically stable in the lyapunov sense via optimal control [20].

3.2.2. *Optimal anti-synchronization on finite time.* In this subsection, we consider problem of finding the optimal controllers $u_i (i = 1, 2, 3)$ such that the steady state of error and updating parameters systems will be asymptotically stable. As such, assume that $e_i = y_i + x_i$ in (3.18)-(3.25). We consider (3.18) and (3.20) as cost functions and HJB equation, respectively. Now similar to calculation as in subsection 3.1.2, we obtain the optimal anti-synchronization controller $U_i (i = 1, 2, 3)$. Since the function U should be satisfied in the HJB equation (3.20), then we have

$$U_i^* = -\frac{1}{2l}\frac{\partial\phi}{\partial e_i} = -\frac{\sqrt{l}}{l}p(t)(x_i + y_i), \quad (i = 1, 2, 3), \quad (3.36)$$

where ϕ is a Lyapunov function similar to (3.23) for anti-synchronization. The ϕ is a solution of the partial differential equation (3.20) and satisfies the boundary condition (3.24). Now, by applying (3.23) in (3.20), costate equation (3.25) with solution $p(t) = \tanh(\frac{-t+T}{\sqrt{l}})$ is obtained. Hence, update rules of estimation of uncertain parameters and optimal controllers are

$$\begin{cases} \dot{\hat{a}} = -e_1(x_1 - x_2) - e_2x_1 - \frac{p(t)}{2\sqrt{l}}(\hat{a} - a) \\ \dot{\hat{b}} = -e_3x_3 - \frac{p(t)}{2\sqrt{l}}(\hat{b} - b) \\ \dot{\hat{c}} = e_2x_1 - \frac{p(t)}{2\sqrt{l}}(\hat{c} - c) \end{cases} \quad (3.37)$$

and

$$\begin{cases} u_1 = -\hat{a}(e_2 - e_1) - \frac{p(t)e_1}{\sqrt{l}} \\ u_2 = -(\hat{c} - \hat{a})e_1 + \hat{a}[e_1e_3 + e_1x_3 + e_3x_1 + x_1x_3] - \frac{p(t)e_2}{\sqrt{l}} \\ u_3 = -e_1e_2 - e_1x_2 - e_2x_1 - 2x_1x_3 + \hat{b}e_3 - \frac{p(t)e_3}{\sqrt{l}}, \end{cases} \quad (3.38)$$

where the function $p(t)$ is positive, in finite time interval $[0, T]$. The time derivative of ϕ is as follow

$$\dot{\phi} = -[\sum_1^3 e_i^2 + (\hat{a} - a)^2 + (\hat{b} - b)^2 + (\hat{c} - c)^2] \leq 0. \quad (3.39)$$

The time derivative of ϕ is negative definite which shows the solution (3.9) is asymptotically stable in the Lyapunov sense via optimal control [12, 31]. This results are similar to those obtained in synchronization case.

TABLE 2. Initial conditions and parameters for example of anti-synchronization

initial condition	Value	Parameter	Value	Parameter	Value
$x_1(0)$	3	α_1	12	k_1	21
$x_2(0)$	-5	α_2	40	k_2	20
$x_3(0)$	1	α_3	21	k_3	11
$y_1(0)$	-2	β_1	10	l	2
$y_2(0)$	2	β_2	1	T	40
$y_3(0)$	-4	β_3	1.2	a	2.1
$\hat{a}(0)$	0.2	η_1	10	b	0.6
$\hat{b}(0)$	-1	η_2	50	c	30
$\hat{c}(0)$	20	η_3	1	–	–

3.2.3. *Numerical simulation of optimal anti-synchronization.* Similar to the synchronization case, to demonstrate and verify the validity of the proposed scheme, we present and discuss numerical results for anti-synchronization of chaotic T-system. For anti-synchronization in infinite time interval, systems(2.1), (3.4) and (3.31) with controllers (3.34) are solved numerically by Maple 16 with CK45. In finite time interval, systems (2.1), (3.4) and (3.37) with controllers (3.38) are considered. For numerical simulation of anti-synchronization, the initial conditions and parameters are given in table 2.

The results of chaotic anti-synchronization of two identical chaotic T-systems (2.1) and (3.4), via optimal controllers in infinite and finite time intervals are shown in FIGURES 7 and 8. As shown in these figures, it is clear that anti synchronization occur after small time.

figure 9 and figure 10 show error of anti-synchronization and estimates of uncertain parameters.

As expected from analytical considerations, the anti-synchronization errors $e_i = y_i + x_i$ and error of estimated parameter are converged to zero in infinite and finite time intervals as $t \longrightarrow \infty$ and $t \longrightarrow T$, respectively. In optimal anti-synchronization in finite time, since after the time $t = T = 15$ the controllers will be disabled, the errors of anti-synchronization and estimated parameters will not be zero.

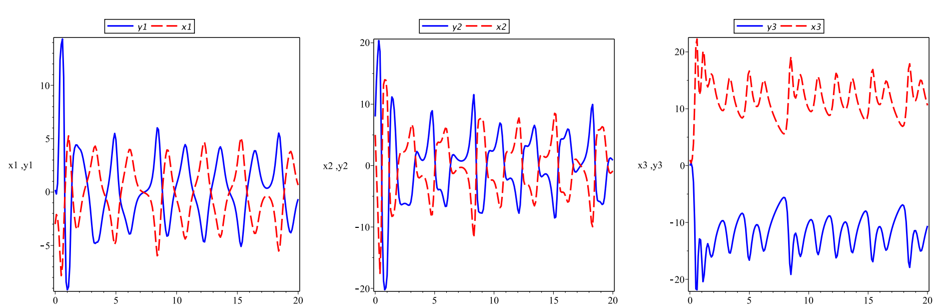


FIGURE 7. Time series trajectories for anti-synchronization via optimal control in infinite time intervals.

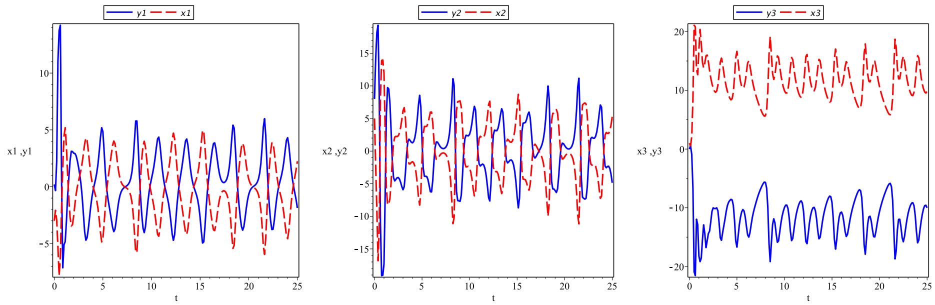
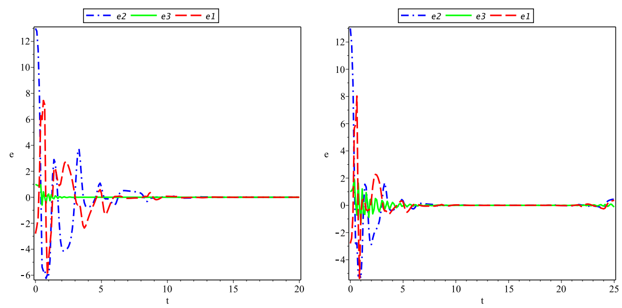
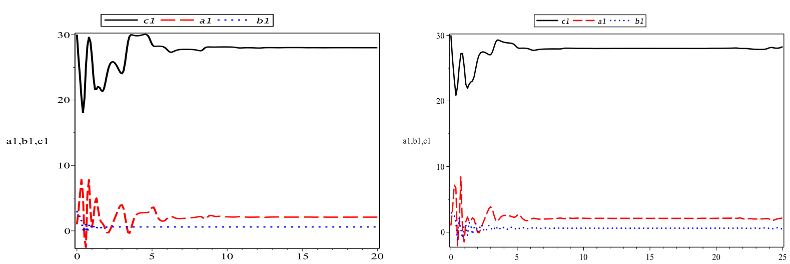


FIGURE 8. Time series trajectories for anti-synchronization via optimal control in finite time intervals.



(a) (b)

FIGURE 9. The error due to anti-synchronization. a: infinite time interval b: finite time interval.



(a) (b)

FIGURE 10. Estimation of parameters for anti-synchronization. a: infinite time interval b: finite time interval.

4. CONCLUSIONS

We designed, new optimal adaptive controller method for synchronization and anti-synchronization of two identical chaotic T-system with uncertain parameters in finite and infinite time intervals. The obtained optimal control laws and parameter estimation rules were satisfied in Lyapunov stability theorem and HJB technique. Estimation of uncertain parameters were designed successfully. The results of synchronization and anti-synchronization showed that errors and the parameter estimation are converged after a short time asymptotically. It was also noted that in finite time case, the synchronization and anti-synchronization fail after $t = T$ time. Finally, numerical examples showed the effectiveness of the proposed method.

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