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IMPROVED CONVERGENCE FOR KING-WERNER-TYPE DERIVATIVE FREE METHODS

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ABSTRACT. We present an improved semilocal and local convergence analysis of some efficient King-Werner-type methods of order $1 + \sqrt{2}$ free of derivatives in a Banach space setting using our new idea of restricted convergence domains. In particular, a more precise convergence domain is determined containing the iterates than in earlier studies leading to: smaller Lipschitz constants, larger radii of convergence and tighter error bounds on the distances involved. Numerical examples are presented to illustrate the theoretical results.

KEYWORDS :King's method; Werner's method; Secant-type method; Banach space; semilocal and local convergence analysis; Fréchet-derivative; efficiency index.
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1. Introduction

In [12], Argyros and Ren studied King-Werner-type methods of order $1+\sqrt{2}$ free of derivatives for approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is Fréchet-differentiable operator defined on a convex subset of a Banach space $\mathcal{B}_1$ with values in a Banach space $\mathcal{B}_2$. In the present study, we extend the applicability of the method considered in [12] using the idea of restricted convergence domains.

Precisely, in [12], Argyros and Ren studied the semilocal convergence analysis of method defined for $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1}F(x_n) \\ y_{n+1} &= x_{n+1} - A_n^{-1}F(x_{n+1}), \end{aligned} \tag{1.2}$$

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where x_0, y_0 are initial points, $A_n = [x_n, y_n; F]$ and $[x, y; F]$ denotes a divided difference of order one for operator F at points $x, y \in \Omega$ [2, 5, 7, 12, 14] satisfying

$$[x, y; F](x - y) = F(x) - F(y) \quad for\ each\ x, y \in \Omega\ with\ x \neq y. \tag{1.3}$$

If F is Fréchet-differentiable on Ω, then $F'(x) = [x, x; F]$ for each $x \in \Omega$.

Method (1.2) is a useful alternative for methods involving $F'(x)$, since the computation of the inverse of $F'(x)$ may be very expensive or impossible rendering such methods useless. The local convergence analysis of method (1.2) was given in [10, 17] in the special case when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$. The convergence order of method (1.2) was shown to be $1 + \sqrt{2}$.

The paper is organized as follows: Section 2 contains the semilocal convergence analysis of method (1.4), and Section 3 contains the local convergence analysis of method (1.4). The numerical examples including favorable comparisons with earlier studies such as [9,10,13,14] are presented in the concluding Section 4.

2. Semilocal convergence of method (1.2)

We present the semilocal convergence of method (1.2) in this section. We first need an auxiliary result on majorizing sequences for method (1.2).

LEMMA 2.1. ([12, Lemma 2.1] Let $L_0 > 0$, $L > 0$, $s_0 \geq 0$, $t_1 \geq 0$ be given parameters. Denote by α the only root in the interval $(0, 1)$ of polynomial p defined by

$$p(t) = L_0 t^3 + L_0 t^2 + 2Lt - 2L. \tag{2.1}$$

Suppose that

$$0 < \frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \leq \alpha \leq 1 - \frac{2L_0 t_1}{1 - L_0 s_0}, \tag{2.2}$$

where

$$s_1 = t_1 + L(t_1 + s_0)t_1. \tag{2.3}$$

Then, scalar sequence $\{t_n\}$ defined for each $n = 1, 2, \ldots$ by

$$\begin{aligned} t_0 = 0, s_{n+1} &= t_{n+1} + \tfrac{L(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{1 - L_0(t_n - t_0 + s_n + s_0)}, \quad for\ each\ n = 1, 2, \ldots, \\ t_{n+2} &= t_{n+1} + \tfrac{L(t_{n+1} - t_n + s_n - t_n)(t_{n+1} - t_n)}{1 - L_0(t_{n+1} - t_0 + s_{n+1} + s_0)}, \quad for\ each\ n = 0, 1, 2, \ldots \end{aligned} \tag{2.4}$$

is well defined, increasing, bounded above by

$$t^{\star\star} = \frac{t_1}{1 - \alpha} \tag{2.5}$$

and converges to its unique least upper bound $t^\star$ which satisfies

$$t_1 \leq t^\star \leq t^{\star\star}. \tag{2.6}$$

Moreover, the following estimates hold

$$s_n - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0), \tag{2.7}$$

$$t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n(t_1 - t_0) \tag{2.8}$$

and

$$t_n \leq s_n \tag{2.9}$$

for each $n = 1, 2, \ldots$.

Denote by $U(w, \xi)$, $\overline{U}(w, \xi)$, the open and closed balls in $\mathcal{B}_1$, respectively, with center $w \in \mathcal{B}_1$ and of radius $\xi > 0$. Next, we present the semilocal convergence of method (1.4) using $\{t_n\}$ as a majorizing sequence.

THEOREM 2.2. Let $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exists a divided differentiable $[.,.,;F]$ of order one for operator F on $\Omega \times \Omega$. Moreover, suppose that there exist $x_0, y_0 \in \Omega$, $L_0 > 0$, $L > 0$, $s_0 \geq 0$, $t_1 \geq 0$ such that

$$A_0^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \tag{2.10}$$

$$\|A_0^{-1}F(x_0)\| \leq t_1, \tag{2.11}$$

$$\|x_0 - y_0\| \leq s_0, \tag{2.12}$$

$$\|A_0^{-1}([x, y; F] - A_0)\| \leq L_0(\|x - x_0\| + \|y - y_0\|), \text{ for each } x, y \in \Omega, \tag{2.13}$$

$$\|A_0^{-1}([x, y; F]-[z, v; F])\| \leq L(\|x-z\|+\|y-v\|), \text{ for each } x, y, z \in \Omega \cap U(x_0, \frac{1 - L_0 s_0}{2L_0}), \tag{2.14}$$

$$\overline{U}(x_0, t^\star) \subseteq \Omega \tag{2.15}$$

and hypotheses of Lemma 2.1 hold, where $A_0 = [x_0; y_0; F]$ and $t^\star$ is given in Lemma 2.1. Then, sequence $\{x_n\}$ generated by method (1.4) is well defined, remains in $\overline{U}(x_0, t^\star)$ and converges to a unique solution $x^\star \in \overline{U}(x_0, t^\star)$ of equation $F(x) = 0$. Moreover, the following estimates hold for each $n = 0, 1, 2, ...$

$$\|x_n - x^\star\| \leq t^\star - t_n. \tag{2.16}$$

Furthermore, if there exists $R > t^\star$ such that

$$U(x_0, R) \subseteq \Omega \tag{2.17}$$

and

$$L_0(t^\star + R + s_0) < 1, \tag{2.18}$$

then, the point $x^\star$ is the only solution of equation $F(x) = 0$ in $U(x_0, R)$.

Proof. It follows from the corresponding proof in [12] by simply noting that the iterates x_n lie in Ω_0 which is a more precise location than Ω used in [12], since $\Omega_0 \subseteq \Omega$.

REMARK 2.3. (a) In [12] condition (2.14) holds in Ω. That is,

$$\|F'(x^\star)^{-1}([x, y; F] - [u, v; F])\| \leq \frac{L_1}{2}(\|x - y\| + \|y - v\|) \tag{2.19}$$

for all $x, y, u, v \in \Omega$. Then, we have $L \leq L_1$, since $\Omega_0 \subset \Omega$ and $L_0 \leq L_1$. If $L < L_1$, we obtain a larger radius of convergence and more precise error bounds on the distances $\|x_n - x^\star\|$ than in [11]. These advantages are obtained under the same commputational cost, since in practice the computation of L_1 requires the computation of L or L_0 as a special case.

(b) The limit point $t^\star$ can be replaced by $t^{\star\star}$ given in closed form by (2.5) in Theorem 2.1.

(c) It follows from the proof of Theorem 2.2 that hypothesis (2.13) is not needed to compute an upper bound for $\|A_0^{-1}F(x_1)\|$. Hence, we can define the more precise (than $\{t_n\}$) majorizing sequence $\{\bar{t}_n\}$ (for $\{x_n\}$) by

$$\begin{aligned}&\bar{t}_0 = 0, \bar{t}_1 = t_1, \bar{s}_0 = s_0, \bar{s}_1 = \bar{t}_1 + L_0(\bar{t}_1 + \bar{s}_0)\bar{t}_1,\\ &\bar{s}_{n+1} = \bar{t}_{n+1} + \tfrac{L(\bar{t}_{n+1}-\bar{t}_n+\bar{s}_n-\bar{t}_n)(\bar{t}_{n+1}-\bar{t}_n)}{1-L_0(\bar{t}_n-\bar{t}_0+\bar{s}_n+\bar{s}_0)} \quad \text{for each } n = 1, 2, ...\end{aligned} \tag{2.20}$$

and

$$\bar{t}_{n+2} = \bar{t}_{n+1} + \tfrac{L(\bar{t}_{n+1}-\bar{t}_n+\bar{s}_n-\bar{t}_n)(\bar{t}_{n+1}-\bar{t}_n)}{1-L_0(\bar{t}_{n+1}-\bar{t}_0+\bar{s}_{n+1}+\bar{s}_0)} \quad \text{for each } n = 0, 1, ...$$

Then, using a simple induction argument we have that

$$\bar{t}_n \leq t_n, \tag{2.21}$$

$$\bar{s}_n \leq s_n, \tag{2.22}$$
$$\bar{t}_{n+1} - \bar{t}_n \leq t_{n+1} - t_n, \tag{2.23}$$
$$\bar{s}_n - \bar{t}_n \leq s_n - t_n \tag{2.24}$$

and

$$\bar{t}^\star = \lim_{n\to\infty} \bar{t}_n \leq t^\star.$$

Furthermore, if $L_0 < L$, then (2.43)-(2.46) are strict for $n \geq 2$, $n \geq 1$, $n \geq 1$, $n \geq 1$, respectively. Clearly, sequence $\{\bar{t}_n\}$ increasing converges to $\bar{t}^\star$ under the hypotheses of Lemma 2.1 and can replace $\{t_n\}$ as a majorizing sequence for $\{x_n\}$ in Theorem 2.2.

3. Local convergence of method (1.2)

We present the local convergence of method (1.2) in this section. We have

THEOREM 3.1. Let $F : \Omega \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exist $x^\star \in \Omega$, $l_0 > 0$ and $l > 0$ such that

$$F(x^\star) = 0, \; F'(x^\star)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{3.1}$$

$$\|F'(x^\star)^{-1}([x, y; F] - F'(x^\star))\| \leq l_0(\|x - x^\star\| + \|y - x^\star\|) \text{ for each } x, y \in \Omega \tag{3.2}$$

$$\|F'(x^\star)^{-1}([x, y; F]-[z, u; F])\| \leq l(\|x-z\|+\|y-u\|), \text{ for each } x, y \in \Omega_1 := \Omega \cap U(x^\star, \frac{1}{2l_0}) \tag{3.3}$$

and

$$\overline{U}(x^\star, \rho) \subseteq \Omega, \tag{3.4}$$

where

$$\rho = \frac{1}{(1 + \sqrt{2})l + 2l_0}. \tag{3.5}$$

Then, sequence $\{x_n\}$ generated by method (1.4) is well defined, remains in $\overline{U}(x^\star, \rho)$ and converges to $x^\star$ with order of $1 + \sqrt{2}$ at least, provided that $x_0, y_0 \in U(x^\star, \rho)$. Moreover, the following estimates

$$\|x_{n+2} - x^\star\| \leq \frac{\sqrt{2} - 1}{\rho^2}\|x_{n+1} - x^\star\|^2\|x_n - x^\star\| \tag{3.6}$$

and

$$\|x_n - x^\star\| \leq (\frac{\sqrt{\sqrt{2} - 1}}{\rho})^{F_{n-1}}\|x_1 - x^\star\|^{F_n} \tag{3.7}$$

hold for each $n = 1, 2, \ldots$, where F_n is a generalized Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = 2F_{n+1} + F_n$.

REMARK 3.2. (a) Let l_1 be the Lipschitz constant for $x, y \in \Omega$ used in [12]. We have that $l_0 \leq l$ and $l \leq l_1$. Hence, the old radius

$$\rho_1 = \frac{1}{(1 + \sqrt{2})l_1 + 2l_0} < \rho, \tag{3.8}$$

if $l < l_1$.

(b) For the special case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, the radius of convergence ball for method (1.2) is given in [10] by

$$\rho_\star = \frac{s^\star}{M}, \tag{3.9}$$

where $s^\star \approx 0.55279$ is a constant and $M > 0$ is the upper bound for $|F(x^\star)^{-1}F''(x)|$ in the given domain Ω. Using (3.2) we have

$$\|F'(x^\star)^{-1}(F'(x) - F'(y))\| \leq 2l\|x - y\| \quad \textit{for any } x, y \in \Omega. \tag{3.10}$$

That is, we can choose $l = \frac{M}{2}$. Simply set $l_0 = l$, we have from (3.5) that

$$\rho = \frac{2}{(3+\sqrt{2})M} = \frac{2(3-\sqrt{2})}{5M} \approx \frac{0.63432}{M} > \frac{s^\star}{M} = \rho_\star. \tag{3.11}$$

Therefore, even in this special case, a bigger radius of convergence ball for method (1.2) has been given in Theorem 3.1.

4. Numerical examples

We present some numerical examples in this section.

EXAMPLE 4.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = (-1, 1)$ and define F on Ω by

$$F(x) = e^x - 1. \tag{4.1}$$

Then, $x^\star = 0$ is a solution of Eq. (1.1), and $F'(x^\star) = 1$. Note that for any $x, y, z, u \in \Omega$, we have

$$\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F] - [z,u;F])| = |\int_0^1 (F'(tx+(1-t)y) - F'(tz+(1-t)u))dt| \\
&= |\int_0^1\int_0^1 (F''\big(\theta(tx+(1-t)y) + (1-\theta)(tz+(1-t)u)\big)\big(tx+(1-t)y-(tz+(1-t)u)\big)d\theta dt| \\
&= |\int_0^1\int_0^1 (e^{\theta(tx+(1-t)y)+(1-\theta)(tz+(1-t)u)}\big(tx+(1-t)y-(tz+(1-t)u)\big)d\theta dt| \\
&\le \int_0^1 e|t(x-z)+(1-t)(y-u)|dt \\
&\le \tfrac{e}{2}(|x-z| + |y-u|)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&|F'(x^\star)^{-1}([x,y;F] - [x^\star,x^\star;F])| = |\int_0^1 F'(tx+(1-t)y)dt - F'(x^\star)| \\
&= |\int_0^1 (e^{tx+(1-t)y} - 1)dt| \\
&= |\int_0^1 (tx+(1-t)y)(1 + \tfrac{tx+(1-t)y}{2!} + \tfrac{(tx+(1-t)y)^2}{3!} + \cdots)dt| \\
&\le |\int_0^1 (tx+(1-t)y)(1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots)dt| \\
&\le \tfrac{e-1}{2}(|x-x^\star| + |y-x^\star|).
\end{aligned} \tag{4.3}$$

That is to say, the Lipschitz condition (3.2) and the center-Lipschitz condition (3.3) are true for $l_1 = \frac{e}{2}, l = \frac{e^{\frac{1}{l_0}}}{2}$ and $l_0 = \frac{e-1}{2}$, respectively. Using (3.5) in Theorem 3.1, we can deduce that the radius of convergence ball for method (1.2) is given by

$$\rho = \frac{1}{(1+\sqrt{2})e^{\frac{1}{l_0}} + 2l_0} \approx 0.17907908501857289, \tag{4.4}$$

which is bigger than the corresponding radius

$$\rho' = \frac{1}{(1+\sqrt{2})l + 2l} \approx 0.14147448123384420 \tag{4.5}$$

obtained by only using the Lipschitz condition (3.2).

Let us choose $x_0 = 0.2, y_0 = 0.199$. Suppose sequences $\{x_n\}$ and $\{y_n\}$ are generated by method (1.2). Table 1 gives a comparison results of error estimates for Example 4.1, which shows that tighter error estimates can be obtained from (3.7) by using both the Lipschitz condition (3.2) and the center-Lipschitz condition (3.3) instead of by using only the Lipschitz condition (3.2).

TABLE 1. The comparison results of error estimates for Example 4.1

n	the right-side of (3.7) by using both l and l_0	the right-side of (3.7) by using only l	the right-side of (3.7) by using both l and l_1
2	0.018640103	0.018640103	0.37609425349
3	6.70547E-05	9.65617E-05	0.000000000001
4	8.67742E-10	2.5913E-09	7.572039969341e-78
5	5.2275E-22	9.66721E-21	8.874190155368e-284

EXAMPLE 4.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$, equipped with the max norm and $\Omega = \overline{U}(0,1)$. Define function F on Ω, given by

$$F(x)(s) = x(s) - 5\int_0^1 stx^3(t)dt, \tag{4.6}$$

and the divided difference of F is defined by

$$[x,y;F] = \int_0^1 F'(tx + (1-t)y)dt. \tag{4.7}$$

Then, we have

$$[F'(x)y](s) = y(s) - 15\int_0^1 stx^2(t)y(t)dt, \quad for \;\; all \;\; y \in \Omega. \tag{4.8}$$

We have $x^\star(s) = 0$ for all $s \in [0,1]$, $l_0 = 3.75$ and $l_1 = l = 7.5$ [2]. Using Theorem 3.1, we can deduce that the radius of convergence ball for method (1.2) is given by

$$\rho = \frac{1}{(1+\sqrt{2})l + 2l_0} \approx 0.039052429, \tag{4.9}$$

which is bigger than the corresponding radius

$$\rho' = \frac{1}{(1+\sqrt{2})l + 2l} \approx 0.030205456 \tag{4.10}$$

obtained by only using the Lipschitz condition (3.2) [13, 16].

EXAMPLE 4.3. Let also $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$ equipped with the max norm and $\Omega = U(0,r)$ for some $r > 1$. Define F on Ω by

$$F(x)(s) = x(s) - y(s) - \mu\int_0^1 G(s,t)x^3(t)dt, \quad x \in C[0,1], \; s \in [0,1].$$

$y \in C[0,1]$ is given, μ is a real parameter and the Kernel G is the Green's function defined by

$$G(s,t) = \begin{cases} (1-s)t & \text{if } t \le s \\ s(1-t) & \text{if } s \le t. \end{cases}$$

Then, the Fréchet derivative of F is defined by

$$(F'(x)(w))(s) = w(s) - 3\mu\int_0^1 G(s,t)x^2(t)w(t)dt, \quad w \in C[0,1], \; s \in [0,1].$$

Let us choose $x_0(s) = y_0(s) = y(s) = 1$ and $|\mu| < \frac{8}{3}$. Then, we have that

$$\begin{aligned} &\|I - A_0\| \le \tfrac{3}{8}\mu, \quad A_0^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \\ &\|A_0^{-1}\| \le \tfrac{8}{8-3|\mu|}, \quad s_0 = 0, \quad t_1 = \tfrac{|\mu|}{8-3|\mu|}, \quad L_0 = \tfrac{3(1+r)|\mu|}{2(8-3|\mu|)}, \end{aligned}$$

and

$$L = \frac{3r|\mu|}{8-3|\mu|}.$$

Let us choose $r = 3$ and $\mu = \frac{1}{2}$. Then, we have that

$$t_1 = 0.076923077, \quad L_0 \approx 0.461538462, \quad L_1 = L \approx 0.692307692$$

and

$$\frac{L(t_1 + s_0)}{1 - L_0(t_1 + s_1 + s_0)} \approx 0.057441746, \quad \alpha \approx 0.711345739, \quad 1 - \frac{2L_0 t_1}{1 - L_0 s_0} \approx 0.928994083.$$

That is, condition (2.2) is satisfied and Theorem 2.2 applies.

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