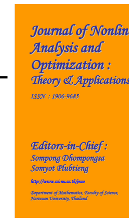
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 7, No. 2, (2016), 91-96
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



BALL CONVERGENCE RESULTS FOR A METHOD WITH MEMORY OF EFFICIENCY INDEX 1.8392 USING ONLY FUNCTIONAL VALUES

I. K. ARGYROS AND S. GEORGE*

Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-575 025

ABSTRACT. We present a local convergence analysis for an one-step iterative method with memory of efficiency index 1.8392 to solve nonlinear equations. If the function is twice differentiable, then it was shown that the $R-$order of convergence is 1.8392. In this paper we use hypotheses up to the first derivative. This way we extend the applicability of this method. Moreover, the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented to illustrate the theoretical results.

KEYWORDS : Halley's method; local convergence; order of convergence; efficiency index.
AMS Subject Classification: 65D10, 65D99

1. Introduction

Let $f : D \subseteq S \to S$ be a nonlinear function, D is a convex subset of S where S is $\mathbb{R}$ or $\mathbb{C}$. Consider the problem of approximating a locally unique solution x^* of equation

$$f(x) = 0. \tag{1.1}$$

Newton-like methods are famous for finding solution of (1.1), these methods are usually studied based on: semi-local and local convergence. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls [3, 4, 19, 20, 23].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1]-[23] require the evaluation of the second derivative f'' at each step, which in general is very expensive. That is why many authors have used higher order multipoint

* *Corresponding author.*
Email address : iargyros@cameron.edu, sgeorge@nitk.edu.in.
Article history : Received 30 August 2016 Accepted 19 January 2018.

methods [1]-[23]. In this paper, we study the local convergence of the one-step method with memory defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} x_{n+1} &= \frac{(x_n + x_{n-1}) f_n f_{n-1} f[x_n, x_{n-1}]}{2 f_n f_{n-1} f[x_n, x_{n-1}] - (f_{n-1}^2 f[x_n, x_{n-1}] + f_n^2 f[x_{n-1}, x_{n-2}])}, \\ &\quad - \frac{(f_{n-1}^2 x_n f[x_n, x_{n-1}] + f_n^2 x_{n-1} f[x_{n-1}, x_{n-2}])}{2 f_n f_{n-1} f[x_n, x_{n-1}] - (f_{n-1}^2 f[x_n, x_{n-1}] + f_n^2 f[x_{n-1}, x_{n-2}])}, \end{aligned} \tag{1.2}$$

where x_{-2}, x_{-1}, x_0 are initial points , $f_n = f(x_n), f[x, y]$ denotes a divided difference of order one for function f at the point x, y [3, 4, 23] defined by

$$f[x, y] = \frac{f(x) - f(y)}{x - y} \quad \text{if } x \neq y \tag{1.3}$$

and $f[x, x] = f'(x)$. Method (1.2) uses only one function evaluation per step, f_n and it was shown using the Herberger's matrix method in [17] that the $R-$order of convergence is $\frac{1}{3}(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}} \approx 1.8392$. Moreover, the efficiency index is

$$I = (1.8392)^{\frac{1}{1}} = 1.8392.$$

These results are obtained provided that the function f is twice differentiable. This hypotheses limits the applicability of method (1.2) although only function evaluations are needed to carry out the computation of each step. As a motivational example, let us define function f on $D = [-1, 2]$ by

$$f(x) = \begin{cases} x^2 \ln x + x^4 - x^3, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then, we have that

$$f'(x) = 2x \ln x + 4x^3 - 3x^2 + x,$$

and

$$f''(x) = 2 \ln x + 12x^2 - 6x + 3.$$

Hence, function f'' is unbounded on D. In the present paper, we study the local convergence of method (1.2) using hypotheses only on the first derivative. Moreover, we provide: the radius of convergence, computable error bounds on the distances $|x_n - x^*|$ and a uniqueness result.

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (1.2). The numerical examples are given in the concluding Section 3.

2. Local convergence for method (1.2)

We present the local convergence analysis of method (1.2) in this section. We first simplify method (1.2) by using formula (1.3) to obtain that

$$x_{n+1} - x^* = \frac{A_n}{B_n}, \tag{2.1}$$

where

$$\begin{aligned} A_n &= f_{n-1}(f_n - f_{n-1})^2 (x_{n-1} - x_{n-2})(x_n - x^*) \\ &\quad + f_n[-(f_n - f_{n-1})^2 (x_{n-1} - x_{n-2}) \\ &\quad + f_n((f_n - f_{n-1})(x_{n-1} - x_{n-2}) \\ &\quad - (f_{n-1} - f_{n-2})(x_n - x_{n-1})](x_{n-1} - x^*) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} B_n &= 2f_n f_{n-1}(f_n - f_{n-1})(x_{n-1} - x_{n-2}) - f_{n-1}^2(f_n - f_{n-1})(x_{n-1} - x_{n-2}) \\ &\quad - f_n^2(f_n - f_{n-1})(x_n - x_{n-1}). \end{aligned} \tag{2.3}$$

Let us also define function g by

$$g(t) = 40M^3t^4 + 4M^3t^2 - 1 \text{ for some } M > 0. \tag{2.4}$$

Notice that

$$r = \frac{1}{2M}\sqrt{\frac{2M}{M^2 + \sqrt{M^4 + 10M}}} \tag{2.5}$$

is the only positive root of equation $g(t) = 0$. We also have that

$$0 \leq \frac{24M^3t^4}{1 - 4M^3t^2(1 + 4t^2)} < 1 \text{ for each } t \in [0, r). \tag{2.6}$$

Let $U(v, \rho), \bar{U}(v, \rho)$ stand for the open and closed balls in S, respectively, with center $v \in S$ and of radius $\rho > 0$.

Using the preceding notation we present the local convergence analysis of method (1.2).

THEOREM 2.1. *Let $f : D \subseteq S \to S$ be a differentiable function. Suppose that there exist $x^* \in D, M > 0$ such that for each $x \in D$*

$$f(x^*) = 0, \tag{2.7}$$

$$|f'(x)| \leq M, \tag{2.8}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.9}$$

where r is defined by (2.5). Then, the sequence $\{x_n\}$ generated by method (1.2) for for $x_{-2}, x_{-1}, x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$,

$$|x_{n+1} - x^*| \leq \frac{a_n}{b_n}|x_{n-1} - x^*||x_n - x^*| < |x_n - x^*| < r, \tag{2.10}$$

where

$$a_n = 2M^3(|x_n - x^*| + |x_{n-1} - x^*|)(|x_{n-1} - x^*| + |x_{n-2} - x^*|)(2|x_n - x^*| + |x_{n-1} - x^*|)$$

and

$$b_n = 1 - M^3(|x_n - x^*| + |x_{n-1} - x^*|)(|x_{n-1} - x^*| + |x_{n-2} - x^*|)(1 + (|x_{n-1} - x^*| + |x_n - x^*|)^2).$$

Proof. By hypothesis $x_{-2}, x_{-1}, x_0 \in U(x^*, r)$. We can write using (2.7) that

$$f_{-2} = f_{-2} - f(x^*) = \int_0^1 f'(x^* + \theta(x_{-2} - x^*))(x_{-2} - x^*)d\theta. \tag{2.11}$$

Then, by (2.11) and (2.8) we get that

$$|f_{-2}| = |\int_0^1 f'(x^* + \theta(x_{-2} - x^*))(x_{-2} - x^*)d\theta| \leq M|x_{-2} - x^*|. \tag{2.12}$$

Similarly, we have that

$$|f_{-1}| \leq M|x_{-1} - x^*|. \tag{2.13}$$

and

$$|f_0| \leq M|x_0 - x^*|. \tag{2.14}$$

We shall show that B_0 given by (2.3) is invertible. We have in turn by (2.3), (2.6), (2.12)- (2.14) and the triangle inequality that

$$\begin{aligned}
|B_0 - 1| &= |2f_0 f_{-1}(f_0 - f_{-1})(x_{-1} - x_{-2}) - f_{-1}^2(f_0 - f_{-1})(x_{-1} - x_{-2}) \\
&\quad - f_0^2(f_{-1} - f_{-2})(x_0 - x_{-1}) - (x_0 - x_{-1})(x_{-1} - x_{-2})| \\
&= |f_{-1}(f_0 - f_{-1})^2(x_{-1} - x_{-2}) + f_0(f_{-1}(f_0 - f_{-1})(x_{-1} - x_{-2}) \\
&\quad - f_0(f_{-1} - f_{-2})(x_0 - x_{-1})) - (x_0 - x_{-1})(x_{-1} - x_{-2})| \\
&\leq M|x_{-1} - x^*|M^2(|x_0 - x^*| + |x_{-1} - x^*|)^2(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M|x_0 - x^*|(M|x_{-1} - x^*|M(|x_0 - x^*| + |x_{-1} - x^*|) \\
&\quad \times(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M|x_0 - x^*|M(|x_{-1} - x^*| + |x_{-2} - x^*|)(|x_0 - x^*| + |x_{-1} - x^*|)) \\
&\quad + (|x_0 - x^*| + |x_{-1} - x^*|)(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&= 1 - b_0 < 4M^3r^2(1 + 4r^2) < 1. \qquad (2.15)
\end{aligned}$$

It follows from (2.15) and the Banach lemma on invertible functions [3, 4] that B_0 is invertible and

$$|B_0^{-1}| \leq \frac{1}{b_0} < \frac{1}{1 - 4M^3r^2(1 + 4r^2)}. \qquad (2.16)$$

Next, we need an estimate on A_0. It follows from (2.2), (2.12)-(2.14) and triangle inequality that

$$\begin{aligned}
|A_0| &\leq M|x_{-1} - x^*|M^2(|x_{-1} - x^*| + |x_0 - x^*|)^2 \\
&\quad \times(|x_{-1} - x^*| + |x_{-2} - x^*|)|x_0 - x^*| \\
&\quad + M|x_0 - x^*||x_{-1} - x^*|[M^2(|x_0 - x^*| + |x_{-1} - x^*|)^2 \\
&\quad \times(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M|x_0 - x^*|(M(|x_0 - x^*| + |x_{-1} - x^*|)(|x_{-1} - x^*| + |x_{-2} - x^*|) \\
&\quad + M(|x_{-1} - x^*| + |x_{-2} - x^*|)(|x_0 - x^*| + |x_{-1} - x^*|)] \\
&\leq a_0|x_{-1} - x^*||x_0 - x^*| \leq 24M^3r^4|x_0 - x^*|. \qquad (2.17)
\end{aligned}$$

Then, using (2.1), (2.6), (2.16) and (2.17), we get that

$$\begin{aligned}
|x_{n+1} - x^*| &\leq \frac{a_0|x_{-1} - x^*||x_0 - x^*|}{b_0} \\
&< \frac{24M^3r^4}{1 - 4M^3r^2(1 + 4r^2)}|x_0 - x^*| = |x_0 - x^*| < r,
\end{aligned}$$

which shows (2.10) for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_{-2}, x_{-1}, x_0 by x_{k-2}, x_{k-1}, x_k in the preceding estimates, we arrive at (2.10). Then from the estimate $|x_{k+1} - x^*| < |x_k - x^*| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Next, we present a uniqueness result for method (1.2).

THEOREM 2.2. *Suppose that the hypotheses of Theorem 2.1 hold and there exists $L_0 > 0$ such that for each $x \in D$, $f'(x^*) \neq 0$,*

$$|f'(x^*)^{-1}(f'(x) - f'(x^*))| \leq L_0|x - x^*| \qquad (2.18)$$

and

$$L_0 r < 2. \qquad (2.19)$$

Then, the limit point x^* is the only solution of equation $f(x) = 0$ in $\bar{U}(x^*, r)$.

Proof. The existence of the solution x^* in $U(x^*, r)$ has been established in Theorem 2.1. To show the uniqueness part, let $y^* \in \bar{U}(x^*, T)$ with $f(y^*) = 0$. Define $T = \int_0^1 f'(y^* + \theta(x^* - y^*)d\theta$. Using (2.18) and (2.19) we get that

$$|f'(x^*)^{-1}(T - f'(x^*))| \leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \leq \int_0^1 (1-\theta)|x^* - y^*|d\theta \leq \frac{L_0}{2} r < 1. \quad (2.20)$$

It follows from (2.20) that T is invertible. Finally, from the identity $0 = f(x^*) - f(y^*) = T(x^* - y^*)$, we deduce that $x^* = y^*$.

□

REMARK 2.3. *The computation of the order of convergence involves estimates of higher order derivative of operator f. So one may use the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

3. Numerical Examples

We present a numerical example in this section.

EXAMPLE 3.1. *Let $D = [-1, 1]$. Define function f of D by*

$$f(x) = e^x - 1. \quad (3.1)$$

Using (3.1) and $x^* = 0$, we get that $M = e, L_0 = e - 1, r = 0.1058$ and $\xi_1 = 1.5586$. Notice that (2.19) is satisfied, so the solution $x^* = 0$ is unique in $\bar{U}(0, 1)$.

References

1. S. Amat, M.A. Hernández, N. Romero, A modified Chebyshev's iterative method with at least sixth order of convergence, Appl. Math. Comput., **206**(1), (2008), 164-174.
2. S. Amat, S. Busquier, S. Plaza, Dynamics of the King's and Jarratt iterations, Aequationes. Math., **69**, (2005), 212-213.
3. I.K. Argyros, "Convergence and Application of Newton-type Iterations," Springer, 2008.
4. I. K. Argyros and Said Hilout, Computational methods in nonlinear Analysis, World Scientific Publ. Co. , New Jersey, USA, 2013.
5. I. K. Argyros D.Chen, Q. Quian, The Jarratt method in Banach space setting, J.Comput.Appl.Math., **51**,(1994), 103-106.
6. V. Candela, A. Marquina, Recurrence relations for rational cubic methods I: The Halley method, Computing, **44**, (1990), 169-184.
7. J.Chen, Some new iterative methods with three-order convergence, Appl. Math. Comput., **181**, (2006), 1519-1522.
8. C.Chun, B. Neta, M. Scott, Basins of attraction for optimal eighth order methods to find simple roots of nonlinear equations, Appl. Math. Comput., **227**, (2014), 567-592.
9. A. Cordero, J. Torregrosa, Variants of Newton's method using fifth order quadrature formulas, Appl.Math.Comput., **190**,(2007), 686-698.
10. A. Cordero, J. Maimo, J. Torregrosa, M.P. Vassileva, P. Vindel, Chaos in King's iterative family, Appl. Math. Lett., **26**, (2013), 842-848.
11. A. Cordero, A. Magrenan, C. Quemada, J.R. Torregrosa, Stability study of eight-order iterative methods for solving nonlinear equations, J. Comput. Appl. Math, **291**, 1, (2016), 348–357.
12. J. A. Ezquerro, M.A. Hernández, A uniparametric Halley-type iteration with free second derivative, Int. J.Pure and Appl. Math., **6**, 1, (2003), 99-110.

13. J. A. Ezquerro, M.A. Hernández, New iterations of R-order four with reduced computational cost. BIT Numer. Math., **49**, (2009), 325- 342.
14. M. Frontini, E. Sormani, Some variants of Newton's method with third order convergence, Appl. Math. Comput., **140**, (2003), 419–426.
15. J.M. Gutiérrez, M.A. Hernández, Recurrence relations for the super-Halley method, Computers Math. Applic., **36** (7), (1998), 1-8.
16. M.A. Hernández, M.A. Salanova, Sufficient conditions for semilocal convergence of a fourth order multipoint iterative method for solving equations in Banach spaces. Southwest J. Pure Appl. Math, **1**, (1999), 29–40.
17. J. Herzberger, Über Matrixdarstellungen für iterationsverfahren bei nichlinearen Gleichungen, Computing, **12**, (1974), 215-222.
18. R.F. King, A family of fourth-order methods for nonlinear equations, SIAM. J. Numer. Anal., **10**, (1973), 876-879.
19. M.S. Petkovic, B. Neta, L. Petkovic, J. Džunič, Multipoint methods for solving nonlinear equations, Elsevier, 2013.
20. F.A. Potra, V. Ptak, Nondiscrete induction and iterative processes, Research Notes in Mathematics, Vol. 103, Pitman Publ., Boston, MA, 1984.
21. H. Ren, Q. Wu, W. Bi, New variants of Jarratt method with sixth-order convergence, Numer. Algorithms, **52**(4), (2009), 585–603.
22. W.C. Rheinboldt,An adaptive continuation process for solving systems of nonlinear equations, In: Mathematical models and numerical methods (A.N.Tikhonov et al. eds.) pub.3, (19), 129-142 Banach Center, Warsaw Poland.
23. J.F.Traub, Iterative methods for the solution of equations, Prentice Hall Englewood Cliffs, New Jersey, USA, 1964.