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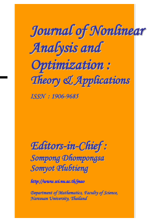

THE GENERALIZED VON NEUMMAN-JORDAN CONSTANT AND FIXED POINTS OF MULTIVALUED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. Recently, a new geometric constant called generalized von Neumman-Jordan constant was introduced. In this work, we give some sufficient conditions for the (DL)-condition and property (D) in terms of this new constant and many coefficients namely the weakly convergent sequence coefficient, the normal structure coefficient, the coefficient of weak orthogonality and the generalized García-Falset coefficient. As consequences, we obtain several sufficient conditions which imply the existence of fixed points for multivalued nonexpansive mappings and normal structure in Banach spaces. The obtained results generalize and unify some known results in the recent literature.

KEYWORDS : Multivalued nonexpansive mapping; Fixed point; Normal structure; Generalized von Neumman-Jordan constant; Weakly convergent sequence coefficient; Normal structure coefficient.
AMS Subject Classification: 47H10, 46B20

1. INTRODUCTION

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [24] and Nadler [26]. Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, game theory, convex optimization, differential inclusion and mathematical economics. Thus, it is natural to extend the known fixed point results for singlevalued mappings to the setting of multivalued mappings. Nevertheless, the fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than the corresponding theory of singlevalued nonexpansive mappings and many problems remain unsolved in it. For instance, the celebrated Kirk's theorem [23] which states the fixed point property for singlevalued nonexpansive mappings in reflexive Banach spaces with normal structure yields to a very natural question: Do reflexive Banach spaces

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with normal structure have the fixed point property for multivalued nonexpansive mappings? Until now, the answer is unknown.

The concept of normal structure plays an important role in metric fixed point theory for nonexpansive mappings. Since under various geometric properties of a Banach space often measured by different geometric constants, normal structure of the space is guaranteed, it is natural to study if those properties imply the fixed point theory for multivalued nonexpansive mappings. Dhompongsa et al. [6, 7] introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) and property (D) which imply the fixed point theory for multivalued nonexpansive mappings and normal structure in reflexive Banach spaces. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL)-condition or property (D).

Recently, many geometric constants for a Banach space have been investigated. Moreover, many recent studies have focused on sufficient conditions for the existence of fixed points of multivalued nonexpansive mappings and normal structure of Banach spaces in terms of these constants and some well known moduli and coefficients. For more details in this direction, we refer the reader to [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, 25, 29, 30] and the references mentioned therein.

Throughout this paper, we assume that X be a Banach space with the closed unit ball $B_X = \{x \in X \, : \, \|x\| \leq 1\}$ and the unit sphere $S_X = \{x \in X \, : \, \|x\| = 1\}$. The following two constants of a Banach space X,

$$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2\big(\|x\|^2 + \|y\|^2\big)} \, : \, x, y \in X, \, (x,y) \neq (0,0) \right\},$$

$$J(X) = \sup\big\{ \min\big(\|x+y\|, \|x-y\|\big) \, : \, x, y \in S_X \big\}$$

are called the von Neumann-Jordan [4] and James constants [15], respectively.

Recently, a new geometric constant $C_{NJ}^{(p)}(X)$ of a Banach space X was introduced which is related to the von Neumann-Jordan constant and can be used for much better characterization of a Banach space X.

In [5], Cui et al. defined the generalized von Neumann-Jordan constant by

$$C_{NJ}^{(p)}(X) = \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}\big(\|x\|^p + \|y\|^p\big)} \, : \, x, y \in X, \, (x,y) \neq (0,0) \right\},$$

where $1 \leq p < \infty$. It is shown that $1 \leq C_{NJ}^{(p)}(X) \leq 2$.

Recall that a Banach space X is said to be uniformly nonsquare [18], in the sense of James, if there exists a positive number $\delta < 2$ such that $\frac{\|x+y\|}{2} \leq \delta$ or $\frac{\|x-y\|}{2} \leq \delta$, whenever $x, y \in S_X$. It is known that every uniformly nonsquare space is reflexive. In [5] it was proved that X is uniformly nonsquare if and only if $C_{NJ}^{(p)}(X) < 2$ for any $1 \leq p < \infty$.

The main purpose of this paper is to investigate some sufficient conditions for the (DL)-condition and property (D) in terms of the generalized von Neumman-Jordan constant, the weakly convergent sequence coefficient, the normal structure coefficient, the coefficient of weak orthogonality and the generalized García-Falset coefficient, which enable us to present several sufficient conditions for the existence of fixed points of multivalued nonexpansive mappings and normal structure in Banach spaces. The obtained results generalize and unify a number of recent well known results in this subject.

2. PRELIMINARIES

We start with some concepts and results which will be used in what follows. For a widespread discussion, the reader is directed to [1, 17].

We recall that a Banach space X is said to have normal structure (weak normal structure, respectively) [2] if for every bounded closed (weakly compact, respectively) convex subset K in X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\big\{\|x_0 - y\| \,:\, y \in K\big\} < \operatorname{diam}(K),$$

where $\operatorname{diam}(K) = \sup\{\|x - y\| \,:\, x, y \in K\}$ denotes the diameter of K. For a reflexive Banach space X, normal structure and weak normal structure coincide.

The weakly convergent sequence coefficient $WCS(X) \in [1, 2]$ [1] is equivalently defined by

$$WCS(X) = \inf\left\{\frac{\lim_{n\neq m}\|x_n - x_m\|}{\limsup_n \|x_n\|}\right\},$$

where the infimum is taken over all weakly (not strongly) null sequences $\{x_n\}$ with $\lim_{n\neq m}\|x_n - x_m\|$ existing.

Let C be a nonempty bounded subset of X and E be a nonempty subset of X. The Chebyshev radius of C relative to E is defined by

$$r_E(C) = \inf\big\{r_x(C) \,:\, x \in E\big\},$$

where $r_x(C) = \sup\{\|x - y\| \,:\, y \in C\}$. We denote $r_C(C)$ by $r(C)$.

The normal structure coefficient $N(X) \in [1, 2]$ defined by Bynum [3] is the number

$$N(X) = \inf\left\{\frac{\operatorname{diam}(E)}{r(E)} \,:\, E \subset X \text{ bounded and convex and } \operatorname{diam}(E) > 0\right\}.$$

The WORTH property was introduced by Sims in [27]. A Banach space X has the WORTH property if

$$\lim_{n\longrightarrow\infty}\Big|\|x_n + x\| - \|x_n - x\|\Big| = 0$$

for all $x \in X$ and all weakly null sequences $\{x_n\}$. In [28], Sims defined the coefficient of weak orthogonality, which measures the “degree of WORTHwhileness". As in [19], we prefer to use its reciprocal, i.e. $\mu(X) \in [1, 3]$, which is defined as

$$\mu(X) = \inf\big\{\lambda \,:\, \limsup_{n\longrightarrow\infty}\|x_n + x\| \le \lambda \limsup_{n\longrightarrow\infty}\|x_n - x\|\big\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequences $\{x_n\}$ in X. It is worthwhile to mention that X has the WORTH property if and only if $\mu(X) = 1$

The generalized García-Falset coefficient $R(1, X) \in [1, 2]$, introduced by Domínguez Benavides [13], is defined as

$$R(1, X) = \sup\big\{\liminf_{n\longrightarrow\infty}\|x_n + x\|\big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$\limsup_{n\longrightarrow\infty}\big(\limsup_{m\longrightarrow\infty}\|x_n - x_m\|\big) \le 1.$$

Before going to the results, let us recall some concepts and results about multivalued mappings and ultrapowers of Banach spaces which will be needed in the sequel.

Let E be as above. We shall denote by $CB(X)$ the family of all nonempty bounded closed subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : E \longrightarrow CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \qquad x, y \in E,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) = \max\Big\{\sup_{x\in A}\inf_{y\in B}\|x - y\|, \sup_{y\in B}\inf_{x\in A}\|x - y\|\Big\}, \qquad A, B \in CB(X).$$

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r\big(E, \{x_n\}\big)$ and the asymptotic center $A\big(E, \{x_n\}\big)$ of $\{x_n\}$ in E are defined by

$$r\big(E, \{x_n\}\big) = \inf\Big\{\limsup_{n\longrightarrow\infty}\|x_n - x\| \ : \ x \in E\Big\}$$

and

$$A\big(E, \{x_n\}\big) = \Big\{x \in E \ : \ \limsup_{n\longrightarrow\infty}\|x_n - x\| = r\big(E, \{x_n\}\big)\Big\},$$

respectively. It is known that $A\big(E, \{x_n\}\big)$ is a nonempty weakly compact convex set whenever E is (see [17]).

The sequence $\{x_n\}$ is called regular with respect to E if $r\big(E, \{x_n\}\big) = r\big(E, \{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to E if $A\big(E, \{x_n\}\big) = A\big(E, \{x_{n_i}\}\big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform with respect to E if $\{x_n\}$ is regular and asymptotically uniform with respect to E.

The following two properties of Banach spaces were introduced and used to guarantee the existence of fixed points for multivalued nonexpansive mappings and normal structure in reflexive Banach spaces (see [6, 7]).

A Banach space X is said to satisfy property (D) [6] if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform with respect to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform with respect to E,

$$r\big(E, \{y_n\}\big) \leq \lambda\, r\big(E, \{x_n\}\big).$$

A Banach space X is said to satisfy the Domínguez-Lorenzo condition [7] if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular with respect to E,

$$r_C\big(A\big(E, \{x_n\}\big)\big) \leq \lambda\, r\big(E, \{x_n\}\big).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results show that property (D) and the (DL)-condition are stronger than weak normal structure and also imply the existence of fixed points for multivalued nonexpansive mappings (see [6, 7]).

Theorem 2.1. *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies (the (DL)-condition) property (D). Let $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.*

Theorem 2.2. *Let X be a Banach space satisfying (the (DL)-condition) property (D). Then X has weak normal structure.*

Let $\mathcal{F}$ be a filter on $\mathbb{N}$. A sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on $\mathbb{N}$ is called an ultrafilter if it is maximal with respect to set inclusion. An ultrafilter is called trivial if it is of the form $\{A \subset \mathbb{N} : i_0 \in A\}$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. Let $\ell_\infty(X)$ denotes the subspace of the product space $\prod_{n\in\mathbb{N}} X$ equipped with the norm $\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let

$$N_{\mathcal{U}} = \big\{(x_n) \in \ell_\infty(X) \ : \ \lim_{\mathcal{U}} \|x_n\| = 0\big\}.$$

The ultrapower of X, denoted by $\widetilde{X}$, is the quotient space $\frac{\ell_\infty(X)}{N_{\mathcal{U}}}$ equipped with the quotient norm. Write $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\big\|(x_n)_{\mathcal{U}}\big\| = \lim_{\mathcal{U}} \|x_n\|.$$

Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically.

3. MAIN RESULTS

We are now in a position to formulate and prove our main results.

Theorem 3.1. *Let $1 \le p < \infty$ and X be a Banach space such that*

$$C_{NJ}^{(p)}(X) < 1 + \left(\frac{WCS(X)}{2}\right)^p.$$

Then X has property (D).

Proof. Let E be a nonempty weakly compact convex subset of X. Denote $r = r\big(E, \{x_n\}\big)$ and $A = A\big(E, \{x_n\}\big)$. We can assume that $r > 0$. Let $\{x_n\} \subset E$ and $\{y_n\} \subset A$ are regular asymptotically uniform sequences with respect to E. Passing through a subsequence of $\{y_n\}$ if necessary, we can also assume that $\{y_n\}$ is weakly convergent to a point $y \in E$ and $d := \lim_{n,m\longrightarrow\infty,\, n\neq m} \|y_n - y_m\|$ exists. By using the convexity of A and again, passing through a subsequence of $\{x_n\}$ if necessary, we assume in addition that

$$\|x_n - y_{2n}\| \le r + \frac{1}{n}, \qquad \|x_n - y_{2n+1}\| \le r + \frac{1}{n}$$

and

$$\left\|x_n - \frac{1}{2}\big(y_{2n} + y_{2n+1}\big)\right\| \ge r - \frac{1}{n}$$

for all $n \in \mathbb{N}$. Consider

$$u_n = \frac{1}{r + \frac{1}{n}}\big(x_n - y_{2n}\big) \qquad \text{and} \qquad v_n = \frac{1}{r + \frac{1}{n}}\big(x_n - y_{2n+1}\big).$$

It is easy to see that $\lim_{n\longrightarrow\infty} \|u_n + v_n\| = 2$ and $\lim_{n\longrightarrow\infty} \|u_n - v_n\| = \frac{d}{r}$. Hence, we have

$$C_{NJ}^{(p)}(X) \ge \frac{2^p + \left(\frac{d}{r}\right)^p}{2^{p-1}(1+1)} = 1 + \frac{d^p}{2^p r^p}.$$

Now, we estimate d as follows:

$$\begin{aligned} d &= \lim_{n\neq m} \|y_n - y_m\| = \lim_{n\neq m} \big\|(y_n - y) - (y_m - y)\big\| \\ &\ge WCS(X) \limsup_{n} \|y_n - y\| \end{aligned}$$

$$\geq WCS(X) r\big(E, \{y_n\}\big).$$

Therefore, we obtain

$$r\big(E, \{y_n\}\big) \leq \frac{2\big(\sqrt[p]{C_{NJ}^{(p)}(X) - 1}\big)}{WCS(X)} r.$$

Since $C_{NJ}^{(p)}(X) < 1 + \big(\frac{WCS(X)}{2}\big)^p$, it follows that X satisfies property (D). □

Since $WCS(X) \leq 2$, if $C_{NJ}^{(p)}(X) < 1 + \big(\frac{WCS(X)}{2}\big)^p$, then $C_{NJ}^{(p)}(X) < 2$ for all $1 \leq p < \infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.1, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.2. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < 1 + \left(\frac{WCS(X)}{2}\right)^p,$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.3. *Let X be a Banach space such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < 1 + \left(\frac{WCS(X)}{2}\right)^p.$$

Then X has normal structure.

Theorem 3.4. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_E\big(A(E, \{x_n\})\big) \leq \frac{2\Big(\sqrt[p]{C_{NJ}^{(p)}(X) - 1}\Big)}{N(X)} r\big(E, \{x_n\}\big)$$

for all $1 \leq p < \infty$.

Proof. Denote $r = r\big(E, \{x_n\}\big)$ and $A = A\big(E, \{x_n\}\big)$. We can assume that $r > 0$. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. If $\operatorname{diam}(A) = 0$, then $r_E(A) = 0$ and hence we are done. So we assume that $\operatorname{diam}(A) > 0$. Let $\varepsilon > 0$ and $u, v \in A$ be such that $\|u - v\| \geq \operatorname{diam}(A) - \varepsilon > 0$. Convexity of A implies that $\frac{u+v}{2} \in A$. By the definition of A, we have

$$\limsup_{n\to\infty} \|x_n - u\| = \limsup_{n\to\infty} \|x_n - v\| = \limsup_{n\to\infty} \left\| x_n - \left(\frac{u+v}{2}\right) \right\| = r.$$

Since $\|u - v\| > 0$, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'} - u$ and $x_{n'} - v$ are not both zero. Thus, we have

$$\begin{aligned}\big(\operatorname{diam}(A) - \varepsilon\big)^p + \left\| 2\left(x_{n'} - \left(\frac{u+v}{2}\right)\right) \right\|^p &\leq \|u - v\|^p + \left\| 2\left(x_{n'} - \left(\frac{u+v}{2}\right)\right) \right\|^p \\ &\leq C_{NJ}^{(p)}(X)\big(2^{p-1}\big(\|x_{n'} - u\|^p \\ &\quad + \|x_{n'} - v\|^p\big)\big).\end{aligned}$$

By taking the upper limit as $n' \longrightarrow \infty$ throughout, we have

$$\big(\operatorname{diam}(A) - \varepsilon\big)^p + 2^p r^p \leq C_{NJ}^{(p)}(X)\big(2^{p-1}(r^p + r^p)\big),$$

from which it follows that

$$\big(\operatorname{diam}(A)-\varepsilon\big)^p \le 2^p\big(C_{NJ}^{(p)}(X)-1\big)r^p.$$

Because ε is arbitrarily small, we get

$$\big(\operatorname{diam}(A)\big)^p \le 2^p\big(C_{NJ}^{(p)}(X)-1\big)r^p,$$

which implies that

$$\operatorname{diam}(A) \le 2\Big(\sqrt[p]{C_{NJ}^{(p)}(X)-1}\Big)r. \tag{3.1}$$

Since A is a bounded convex subset of X with $\operatorname{diam}(A)>0$, it follows that

$$r_E(A) \le r(A) \le \frac{\operatorname{diam}(A)}{N(X)}. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$r_E(A) \le \frac{2\Big(\sqrt[p]{C_{NJ}^{(p)}(X)-1}\Big)}{N(X)}\,r.$$

□

Since $N(X)\le 2$, if $C_{NJ}^{(p)}(X) < 1+\big(\frac{N(X)}{2}\big)^p$, then $C_{NJ}^{(p)}(X)<2$ for all $1\le p<\infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.4, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.5. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1\le p<\infty$,*

$$C_{NJ}^{(p)}(X) < 1+\left(\frac{N(X)}{2}\right)^p,$$

and $T: E\to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.6. *Let X be a Banach space such that for some $1\le p<\infty$,*

$$C_{NJ}^{(p)}(X) < 1+\left(\frac{N(X)}{2}\right)^p.$$

Then X has normal structure.

Theorem 3.7. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_C\big(A(E,\{x_n\})\big) \le \left(\frac{\Big(2^pC_{NJ}^{(p)}(X)-\big(1+\frac{1}{\mu(X)}\big)^p\Big)^{\frac{1}{p}}}{1+\frac{1}{\mu(X)}}\right) r\big(E,\{x_n\}\big)$$

for all $1\le p<\infty$.

Proof. For convenience, we denote $r=r\big(E,\{x_n\}\big)$ and $A=A\big(E,\{x_n\}\big)$. We can assume that $r>0$. Since $\{x_n\}\subset E$ is bounded and E is a weakly compact set, we can also assume, by passing through a subsequence if necessary, that $\{x_n\}$ is weakly convergent to a point $x\in E$. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$.

Let $z \in A$. Then $\limsup_n \|x_n - z\| = r$. Denote $\mu = \mu(X)$. Since $(x_n - x) \xrightarrow{\omega} 0$ and by the definition of μ, we have

$$\begin{aligned}\limsup_n \|x_n - 2x + z\| &= \limsup_n \big\|(x_n - x) + (z - x)\big\| \\ &\leq \mu \limsup_n \big\|(x_n - x) - (z - x)\big\| \\ &= \mu r.\end{aligned}$$

Convexity of E implies that $\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \in E$ and thus we obtain

$$\limsup_n \left\| x_n - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \right) \right\| \geq r.$$

On the other hand, by the weak lower semicontinuity of the norm, we have

$$\liminf_n \left\| \left(1 - \frac{1}{\mu}\right)(x_n - x) - \left(1 + \frac{1}{\mu}\right)(z - x) \right\| \geq \left(1 + \frac{1}{\mu}\right)\|z - x\|.$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| < r + \varepsilon$.

(2) $\|x_N - 2x + z\| \leq \mu(r + \varepsilon)$.

(3) $\left\| x_N - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \right) \right\| \geq r - \varepsilon$.

(4) $\left\| \left(1 - \frac{1}{\mu}\right)(x_N - x) - \left(1 + \frac{1}{\mu}\right)(z - x) \right\| \geq \left(1 + \frac{1}{\mu}\right)\|z - x\|\left(\frac{r - \varepsilon}{r}\right)$.

Consider $u = \frac{1}{r+\varepsilon}(x_N - z) \in B_X$ and $v = \frac{1}{\mu(r+\varepsilon)}(x_N - 2x + z) \in B_X$. By applying the above estimates, we obtain

$$\begin{aligned}\|u + v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} + \frac{x_N - x}{\mu(r+\varepsilon)} + \frac{z - x}{\mu(r+\varepsilon)} \right\| \\ &= \left\| \left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)} \right)(x_N - x) - \left(\frac{1}{r+\varepsilon} + \frac{1}{\mu(r+\varepsilon)} \right)(z - x) \right\| \\ &= \left(\frac{1}{r+\varepsilon}\right)\left(1 + \frac{1}{\mu}\right)\left\| (x_N - x) - \left(\frac{1 - \frac{1}{\mu}}{1 + \frac{1}{\mu}} \right)(z - x) \right\| \\ &= \left(\frac{1}{r+\varepsilon}\right)\left(1 + \frac{1}{\mu}\right)\left\| x_N - \left(\frac{2}{\mu+1}x + \frac{\mu-1}{\mu+1}z \right) \right\| \\ &\geq \left(1 + \frac{1}{\mu}\right)\left(\frac{r - \varepsilon}{r + \varepsilon}\right), \\ \|u - v\| &= \left\| \frac{x_N - x}{r+\varepsilon} - \frac{z - x}{r+\varepsilon} - \frac{x_N - x}{\mu(r+\varepsilon)} - \frac{z - x}{\mu(r+\varepsilon)} \right\| \\ &= \left(\frac{1}{r+\varepsilon}\right)\left\| \left(1 - \frac{1}{\mu}\right)(x_N - x) - \left(1 + \frac{1}{\mu}\right)(z - x) \right\| \\ &\geq \left(1 + \frac{1}{\mu}\right)\left(\frac{\|z - x\|}{r}\right)\left(\frac{r - \varepsilon}{r + \varepsilon}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}C_{NJ}^{(p)}(X) &\geq \frac{\|u + v\|^p + \|u - v\|^p}{2^{p-1}(\|u\|^p + \|v\|^p)} \\ &\geq \frac{\left(1 + \frac{1}{\mu}\right)^p \left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p + \left(1 + \frac{1}{\mu}\right)^p \left(\frac{\|z-x\|}{r}\right)^p \left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p}{2^{p-1}(1 + 1)}.\end{aligned}$$

Letting $\varepsilon \longrightarrow 0^+$, we obtain

$$C_{NJ}^{(p)}(X) \geq \frac{\left(1+\frac{1}{\mu}\right)^p + \left(1+\frac{1}{\mu}\right)^p \left(\frac{\|z-x\|}{r}\right)^p}{2^p}.$$

This holds for arbitrary $z \in A$. Hence, we have

$$\sup_{z\in A} \|x-z\| \leq \left(\frac{\left(2^p C_{NJ}^{(p)}(X) - \left(1+\frac{1}{\mu(X)}\right)^p\right)^{\frac{1}{p}}}{1+\frac{1}{\mu(X)}}\right) r.$$

from which it follows that

$$r_E(A) \leq \left(\frac{\left(2^p C_{NJ}^{(p)}(X) - \left(1+\frac{1}{\mu(X)}\right)^p\right)^{\frac{1}{p}}}{1+\frac{1}{\mu(X)}}\right) r.$$

□

Remark 3.8. According to Theorem 3.7, if X has the WORTH property, then

$$r_C\big(A(E,\{x_n\})\big) \leq r\big(E,\{x_n\}\big) \sqrt[p]{C_{NJ}^{(p)}(X) - 1}$$

for all $1 \leq p < \infty$, since $\mu(X) = 1$.

Since $\mu(X) \geq 1$, if $C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{\mu(X)}\right)^p$, then $C_{NJ}^{(p)}(X) < 2$ for all $1 \leq p < \infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.7, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.9. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{\mu(X)}\right)^p,$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.10. *Let X be a Banach space such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{\mu(X)}\right)^p.$$

Then X has normal structure.

Theorem 3.11. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_C\big(A(E,\{x_n\})\big) \leq \frac{2^{p-1} C_{NJ}^{(p)}(X)}{\left(1+\frac{1}{R(1,X)}\right)^p} r\big(E,\{x_n\}\big)$$

for all $1 \leq p < \infty$.

Proof. For convenience, we denote $r = r\big(E,\{x_n\}\big)$ and $A = A\big(E,\{x_n\}\big)$. We can assume that $r > 0$. Since $\{x_n\} \subset E$ is bounded and E is a weakly compact set, we can also assume, by passing through a subsequence if necessary, that $\{x_n\}$ is weakly convergent to a point $x \in E$ and $d := \lim_{n\neq m} \|x_n - x_m\|$ exists. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. Observe that, since the norm is weak lower semicontinuity, we have

$$\liminf_n \|x_n - x\| \le \liminf_n \liminf_m \|x_n - x_m\| = \lim_{n \neq m} \|x_n - x_m\| = d.$$

Let $\varepsilon > 0$. Taking a subsequence if necessary, we can assume that $\|x_n - x\| < d + \varepsilon$ for all n.

Let $z \in A$, then $\limsup_n \|x_n - z\| = r$ and $\|x - z\| \le \liminf_n \|x_n - z\| \le r$. Denote $R = R(1, X)$. Since $(x_n - x) \xrightarrow{w} 0$ and by the definition of R, we have

$$R \ge \liminf_n \left\| \frac{x_n - x}{d + \varepsilon} + \frac{z - x}{r} \right\| = \liminf_n \left\| \frac{x_n - x}{d + \varepsilon} - \frac{x - z}{r} \right\|.$$

On the other hand, observe that the convexity of E implies that $\frac{R-1}{R+1}x + \frac{2}{R+1}z \in E$ and by the weak lower semicontinuity of the norm, we have

$$\begin{aligned}
&\liminf_n \left\| \frac{x_n - z}{r} + \frac{1}{R}\left(\frac{x_n - x}{d + \varepsilon} - \frac{x - z}{r} \right) \right\| \\
&= \liminf_n \left\| \left(\frac{1}{r} + \frac{1}{R(d + \varepsilon)} \right) x_n - \left(\frac{1}{R(d + \varepsilon)} + \frac{1}{Rr} \right) x - \left(\frac{1}{r} - \frac{1}{Rr} \right) z \right\| \\
&\ge \left\| \left(\frac{1}{r} - \frac{1}{Rr} \right) x + \frac{2}{Rr} z - \left(\frac{1}{r} + \frac{1}{Rr} \right) z \right\| \\
&= \left(\frac{1}{r} + \frac{1}{Rr} \right) \left\| \frac{R-1}{R+1} x + \frac{2}{R+1} z - z \right\| \\
&\ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right), \\
&\liminf_n \left\| \frac{x_n - z}{r} - \frac{1}{R}\left(\frac{x_n - x}{d + \varepsilon} - \frac{x - z}{r} \right) \right\| \\
&= \liminf_n \left\| \left(\frac{1}{r} - \frac{1}{R(d + \varepsilon)} \right) (x_n - x) - \left(\frac{1}{r} + \frac{1}{Rr} \right) (z - x) \right\| \\
&\ge \left(\frac{1}{r} + \frac{1}{Rr} \right) \|z - x\| \ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right).
\end{aligned}$$

For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

(1) $\|x_N - z\| \le r + \varepsilon$.

(2) $\left\| \frac{r(x_N - x)}{d + \varepsilon} - (x - z) \right\| \le R(r + \varepsilon)$.

(3) $\frac{1}{Rr} \left\| R(x_N - z) + \frac{r(x_N - x)}{d + \varepsilon} - (x - z) \right\| \ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right) \left(\frac{r - \varepsilon}{r} \right)$.

(4) $\frac{1}{Rr} \left\| R(x_N - z) - \left(\frac{r(x_N - x)}{d + \varepsilon} - (x - z) \right) \right\| \ge \left(1 + \frac{1}{R} \right) \left(\frac{r_E(A)}{r} \right) \left(\frac{r - \varepsilon}{r} \right)$.

In the ultrapower $\widetilde{X}$ of X, we consider

$$\widetilde{u} = \left(\frac{x_N - z}{r + \varepsilon} \right)_{\mathcal{U}} \in B_X \qquad \text{and} \qquad \widetilde{v} = \frac{1}{R(r + \varepsilon)} \left(\frac{r(x_N - x)}{d + \varepsilon} - (x - z) \right)_{\mathcal{U}} \in B_X.$$

By applying the above estimates, we obtain

$$\|\widetilde{u} + \widetilde{v}\| = \frac{1}{R(r + \varepsilon)} \left\| R(x_N - z) + \frac{r(x_N - x)}{d + \varepsilon} - (x - z) \right\|$$

$$\geq \left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right),$$

$$\|\widetilde{u}-\widetilde{v}\| = \frac{1}{R(r+\varepsilon)}\left\|R(x_N-z)-\left(\frac{r(x_N-x)}{d+\varepsilon}-(x-z)\right)\right\|$$

$$\geq \left(1+\frac{1}{R}\right)\left(\frac{r_E(A)}{r}\right)\left(\frac{r-\varepsilon}{r+\varepsilon}\right).$$

Therefore, by the definition of $C_{NJ}^{(p)}(\widetilde{X})$, we have

$$C_{NJ}^{(p)}(\widetilde{X}) \geq \frac{\|\widetilde{u}+\widetilde{v}\|^p + \|\widetilde{u}-\widetilde{v}\|^p}{2^{p-1}\left(\|\widetilde{u}\|^p + \|\widetilde{v}\|^p\right)}$$

$$\geq \frac{2\left(1+\frac{1}{R}\right)^p\left(\frac{r_E(A)}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p}{2^{p-1}(1+1)}$$

$$\geq \frac{\left(1+\frac{1}{R}\right)^p\left(\frac{r_E(A)}{r}\right)^p\left(\frac{r-\varepsilon}{r+\varepsilon}\right)^p}{2^{p-1}}.$$

Since the above inequality is true for every $\varepsilon > 0$ and $C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(\widetilde{X})$, we obtain

$$r_E(A) \leq \frac{2^{p-1}C_{NJ}^{(p)}(X)}{\left(1+\frac{1}{R}\right)^p}\, r.$$

□

Since $R(1, X) \geq 1$, if $C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p$, then $C_{NJ}^{(p)}(X) < 2$ for all $1 \leq p < \infty$, which implies that X is uniformly nonsquare, and consequently X is reflexive. Thus, by applying Theorems 2.1, 2.2 and 3.11, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.12. *Let E be a nonempty bounded closed convex subset of a Banach space X such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p,$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.13. *Let X be a Banach space such that for some $1 \leq p < \infty$,*

$$C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}\left(1+\frac{1}{R(1,X)}\right)^p.$$

Then X has normal structure.

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