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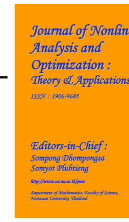

AN EXISTENCE RESULT FOR AN ELLIPTIC PROBLEM INVOLVING A FOURTH ORDER OPERATOR

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ABSTRACT. In this paper, we prove the existence of solutions for a $p-$Biharmonic in bounded domain, by applying the Bohnenblust-Karlin fixed point theorem. The regularity of a such solution is also established.

KEYWORDS : $p-$Biharmonic, Fixed point Theorem.
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1. INTRODUCTION

We consider in this paper the critical situation, which is devoted to the study of the $p-$Biharmonic problem

$$(\mathcal{P}) \begin{cases} \Delta_p^2 u = V(x) \mid u \mid^{p^*-2} u + f(x,u) \text{ in } \Omega, \\ u \in \mathcal{D}_0^{2,p}(\Omega), \ \frac{N}{2} > p, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the $p-$Biharmonic operator with $\Delta u = div\,(\nabla u)$ is the Laplace operator, $1 < p < \frac{N}{2}$, $p^* = Np/(N-2p), V \in L^\infty(\Omega), V > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function where $f(x,0) \neq 0$.

The nonlinear boundary value problem involving the $p-$Biharmonic operator appears in physics and related sciences such as quantum mechanics, surface diffusion on solids, flow in Hele-Shaw cells and also furnishes a model for studying traveling wave in suspension bridges (cf.[12, 15]).

There are many results relating to these problems which have been widely studied in bounded domains. For example we just refer to [2, 4, 9, 10, 12, 13, 14, 19]... This work is motivated by the papers [1] and [8]. Our problem aroused an interesting result because of the lack of compactness, so we could not use the standard variational methods, here by means of the point fixed due to Bohnenblust-Karlin, we shall prove the existence of solution.

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Let us record the following definition,

Definition 1.1. We say that u is a weak solution for problem $(\mathcal{P})$ if

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx - \int_\Omega V(x)|u|^{p^*-2}uvdx - \int_\Omega f(x,u)vdx = 0, \forall v \in \mathcal{D}_0^{2,p}(\Omega).$$

We recall that $\mathcal{D}_0^{2,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with the norm

$$\| u \| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

Our main theorem is stated below.

Theorem 1.1. *Under the standing hypothesis*
(F) $|f(x,t)| \leq C[1+|t|^{r-1}], C>0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$ *with* $r \in (1,p)$, *then problem* $(\mathcal{P})$ *has a weak solution. Furthermore, this solution belongs to* $L^\infty(\Omega)$.

We state the Bohnenblust-Karlin Theorem which provide a platform to establish the main result of the paper.

Theorem 1.2. *(cf.[5, 6, 17]) Let* D *be a nonempty subset of a Banach space* X*, which is bounded, closed and convex. Suppose that* $L : D \to 2^X \setminus \{0\}$ *be an upper semi-continuous set-valued mapping with convex and closed values such that* $L(D) \subset D$ *and* $L(D) = \bigcup_{x\in D} L(x)$ *is relatively compact. Then* L *has a fixed point.*

Recall (cf. [16]) that L is said to be a convex if the inclusion

$$\lambda L(x) + (1-\lambda)L(y) \subset L(\lambda x + (1-\lambda)y)$$

holds for all $x, y \in D$ and for every $\lambda \in [0,1]$.

We say that L has closed values if $L(x)$ is a closed set for every $x \in D$.

2. Proof of the main result

Consider the Sobolev space

$$X = \mathcal{D}_0^{2,p}(\Omega)$$

with the norm

$$\| u \| = \left(\int_\Omega |\Delta u|^p dx\right)^{\frac{1}{p}}.$$

Define two operators A and B from X into X^* by

$$A(u).v = \int_\Omega |\Delta u|^{p-2}\Delta u \Delta v dx,$$

$$B(u).v = \int_\Omega f(x,u)vdx + \int_\Omega V(x)|u|^{p^*-2}uvdx$$

where X^* is the dual of X.

Proof of Theorem *1.1.* We have the following properties,

(1) A is monotone, hemicontinuous, coercive.

In view of [7], we have the following inequality for $p \geq 2$,

$$|x-y|^p \leq (|x|^{p-2}x - |y|^{p-2}y).(x-y), \forall x, y \in \mathbb{R}^N.$$

Thus,

$$\begin{aligned} \langle A(u)-A(v),u-v\rangle &= \int_\Omega |\Delta u|^{p-2}\Delta u\Delta(u-v)dx - \int_\Omega |\Delta v|^{p-2}\Delta v\Delta(u-v)dx \\ &= \int_\Omega (|\Delta u|^{p-2}\Delta u - |\Delta v|^{p-2}\Delta v)(\Delta u-\Delta v)dx \\ &\geq \int_\Omega |\Delta u-\Delta v|^p dx = \| u-v \|^p, \end{aligned} \tag{2.1}$$

and then A is monotone. On the other hand, since A is the derivative operator of the functional $u \to \frac{1}{p}\int_\Omega |\Delta u|^p dx$ which is of class C^1, then the continuity of the operator A holds, so it is hemicontinuous.
Moreover, it is clear that A is coercive since $A(u).u = \| u \|^p$.

(2) The operator B is compact.
Let $(u_n)_n$ be a bounded sequence in X. Up to a subsequence denoted also by $(u_n)_n$, we have

$$u_n \rightharpoonup u \; in \; X,$$

by the compact embedding $\mathcal{D}^{2,p}(\Omega)$ into $L^p(\Omega)$, we have

$$u_n(x) \to u(x) \; a.e. \; in \; \Omega.$$

Since f is Carathéodory function which also verifies the condition (F),

$$f(x,u_n)u_n \to f(x,u)u, \;\; a.e \; in \; \Omega.$$

By using Hölder's inequality and Sobolev's embedding and according to Dominated convergence theorem, we obtain

$$B(u_n) \to B(u).$$

Remark 2.1. Let $(u_n)_n \subset X$ and $u \in X$ such that

$$u_n \rightharpoonup u, \;\; A(u_n) \to A(u),$$

then $A(u_n).u_n \to A(u).u$, which yields $\| u_n \|^p \to \| u \|^p$. Because X is uniformly convex so it follows that $u_n \to u$.

(3) In our next step, let $D \subset X$ be a bounded closed convex. Define the operator L by

$$L(v) = \{u : \;\; A(u) = B(v)\}.$$

It has a closed. Indeed, let $v_n \to v$ in X, $u_n \in L(v)$ and $u_n \to u$, so it would like to show that $u \in L(v)$.

We know that A and B are demicontinuous operators which imply that

$$A(u_n) \rightharpoonup A(u), B(v_n) \rightharpoonup B(v).$$

As we have $A(u_n) = B(v_n)$, so it yields $A(u) = B(v)$ (due to the uniqueness of the limit) then $u \in L(v)$.

Next it will be shown that $L(D) = \bigcup_{v\in D} L(v)$ is relatively compact. Let $(u_n)_n \subset \bigcup_{v\in D} L(v)$ and $v_n \in D$ with $A(u_n) = B(v_n)$. Since D is bounded domain and B is compact, hence $B(D)$ is relatively compact. Afterwards, there is $h \in X^*$ such that

$$A(u_n) = B(u_n) \to h,$$

whence, $A(u_n)$ is bounded, which means that $(u_n)_n$ is a bounded sequence, so we may choose a subsequence denoted also by $(u_n)_n$ where $u_n \rightharpoonup u$. As the operator A is monotone, we have

$$\langle A(v) - A(u_n), v - u_n \rangle \geq 0, \forall v \in X.$$

Therefore,

$$\langle A(v) - h, v - u \rangle \geq 0,$$

in view of proposition of Minty (Proposition 2.2) in [18], we get

$$A(v) = h.$$

From Remark 2.1, it follows that $u_n \to u$ in X.

Now, let $\mathbb{B}_R$ the ball of radius R, we are to prove that $L(\mathbb{B}_R) \subset \mathbb{B}_R$. Suppose that $A(u) = B(v)$ and $\| v \| \leq R$, then

$$\begin{aligned} \| u \|^p &= \int_\Omega |\Delta u|^p dx = \int_\Omega V(x)|v|^{p^*-2} vu dx + \int_\Omega f(x,v) vu dx \\ &\leq c_1 \| v \|^{p^*-1} \| u \| + c_2 \| u \| \| v \|^{r-1}, \end{aligned} \tag{2.2}$$

with c_1 and c_2 are two positive constants. Therefore,

$$\| u \|^{p-1} \leq c_1 \| v \|^{p^*-1} + c_2 \| v \|^{r-1}.$$

Because $r \leq p \leq p^*$, we may find $R > 0$ such that

$$\begin{aligned} c_1 \| v \|^{p^*-1} + c_2 \| v \|^{r-1} &\leq c_1 R^{p^*-1} + c_2 R^{r-1} \\ &\leq R^{p-1}, \end{aligned} \tag{2.3}$$

from which we obtain

$$\| u \| \leq R.$$

We can see that all the assumptions of Bohnenblust-Karlin Theorem are satisfied, hence L has a fixed point which is a solution of the problem $(\mathcal{P})$.

In the sequel, one proceeds as in [3], so we sketch the regularity property of this solution. Let u be a solution of $(\mathcal{P})$. We set

$$\Lambda_\lambda = \{x \in \Omega \ : \ u(x) \geq \lambda\}.$$

For $k > 0$ fixed, putting $\omega_k = u - k$ if $u(x) \geq k$ and $\omega_k = 0$ else. From Cavalieri's principle we have

$$\int_{-\infty}^{\infty} |\Lambda_\lambda| d\lambda = \int_k^{\infty} |\Lambda_\lambda| d\lambda = \int_\Omega \omega_k dx.$$

We point out that when $k > 1$ is greater enough, we have

$$|u|^{p^*-1} = 0, \ a.e.in \ [|u| \geq k],$$

since $u \in L^{p^*}(\Omega)$.

Using the Hölder inequality, for $k > k_0 > 0$, we entail that

$$\begin{aligned} \| \omega_k \|^p &\leq \int_\Omega V(x)|u|^{p^*-1} \omega_k dx + \int_\Omega | f(x, \omega_k) u | \omega_k dx \\ &\leq c_1 \int_{[|u| \leq 1] \cap \Lambda_k} |u|^{p^*-1} \omega_k dx + c_2 \int_{[|u| \geq 1] \cap \Lambda_k} \omega_k dx \end{aligned}$$

$$\leq \quad c_3(|\Lambda_k|^{1-\frac{1}{p^*}})(\int_\Omega |\omega_k|^{p^*} dx)^{\frac{1}{p^*}}, \qquad (2.4)$$

moreover, we have

$$\| \omega_k \|^p \quad \geq \quad S(\int_\Omega |\omega_k|^{p^*} dx)^{\frac{p}{p^*}}, \qquad (2.5)$$

where S is the best Sobolev constant for the embedding

$$\mathcal{D}_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega),$$

defined by

$$S = \inf_{u \in \mathcal{D}^{2,p}(\Omega), u \neq 0} \frac{\int_\Omega |\Delta u|^p dx}{\left(\int_\Omega |u|^{p^*} dx\right)^{\frac{p}{p^*}}},$$

so we get

$$\left(\int_\Omega |\omega_k|^{p^*} dx\right)^{\frac{1}{p^*}} \leq c\left(|\Lambda_k|^{(1-\frac{1}{p^*})(\frac{1}{p-1})}\right).$$

By the Cavalieri's principle [11] and the last inequality, for $k \geq k_0$

$$\begin{aligned} \int_k^\infty |\Lambda_\lambda| d\lambda &= \int_\Omega \omega_k dx \\ &\leq |\Lambda_k|^{1-\frac{1}{p_*}}(\int_\Omega |\omega_k|^{p^*} dx)^{\frac{1}{p^*}} \\ &\leq c(|\Lambda_k|^{1+\frac{1}{p^*}\frac{p^*-1}{p-1}}). \qquad (2.6) \end{aligned}$$

Accordingly, since

$$1 + \frac{1}{p^*}\frac{p^*-1}{p-1} > 1,$$

then easly we get

$$|\Lambda_k| = 0$$

and thus there is $M > 0$ such that

$$|u|_\infty \leq M.$$

$\square$

References

1. G.A. Afrouzi and T.A. Roushan, Existence of solutions for elliptic systems with critical Sobolev exponent in $\mathbb{R}^N$, International Journal of nonlinear Science 11/4 (2011) 484-494.
2. M. Allaoui, A. El Amrouss, A. Ourraoui, Three solutions for A quasi-linear elliptic problem. Appl. Math. E-Notes, 3(2013), 51-59.
3. C. O. Alves , A. El Hamidi, Existence of solution for a anisotropic equation with critical exponent , Differential and Integral Equations, 2008, Vol.21 (1-2), pp.25-40.
4. C.O. Alves, J. M . Ó and O.H. Miyagaki, On a class of singular biharmonic problems involving critical exponents, J. Math. Anal. Appl. 277 (2003) 12-26.
5. C. Avramescu, A fixed point theorem for multivalued mappings, Ele Jour of Qualitative Theory of Diff Equ, 2004, No. 17, 1-10.
6. H . F. Bohnenblust, S. Karlin, On a theorem of Ville. In: Contributions to the Theory of Games, vol. I, pp. 155-160. Princeton University Press, Princeton (1950).
7. Ph. Clement, J. Fleckinger, E. Mitidieri, F. de Thelin, Existence of positive solutions for a nonvariational quasilinear elliptic systems, J. Differential Equations 166 (2000), 455-477.
8. A. Djellit, S. Tas. On some nonlinear elliptic systems. Nonl. Anal. 59 (2004), 695-706.
9. P. Drabek, M. Otani, Global bifurcation result for the p-biharmonic operator, Electronic J. of Differential Equations, 48 (2001), 1–19.

10. A. R. El Amrouss, S. El Habib, N. Tsouli, Existence of solutions for an eigenvale problem with weight, Electron. J. Differential Equations, 45 (2010), 1–10
11. H. Eves, Two Surprising Theorems on Cavalieri Congruence, The College Mathematics Journal, volume 22, number 2, March (1991), pages 118-124.
12. A. Ferrero, G. Warnault, On a solutions of second and fourth order elliptic with power type nonlinearities, Nonlinear Anal.T.M.A 70 (2009)2889-2902.
13. F. Gazzola, H. C. Grunau, Radial entire solutions for supercritical biharmonic equations, Math.Ann.334(2006), 905-936.
14. S. Heiderkhani, Three solutions for a class of (p1,...,pn)-biharmonic systems via variational methods, Thai J. Math . 10 (2012) Number 3, 497-515.
15. A. Lazer, P. McKenna, Large amplitude periodic oscillations in suspension bridges: some new connections with, Nonlinear analysis. SIAM Rev. 32, 537-578 (1990).
16. D.T. Luc, Theory of vector optimization, Springer-Verlag, Berlin (1989).
17. D.R. Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1974.
18. R. E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Mathematical Surveys and Monographs, 49. American Mathematical Society, Providence, RI, 1997.
19. M. Talbi, N. Tsouli, On the Spectrum of the Weighted p-Biharmonic Operator with Weight, Mediterranean Journal of Mathematics April 2007, Volume 4, Issue 1, pp 73-86.