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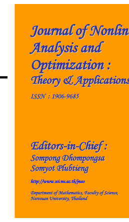

COUPLED FIXED POINT THEOREMS FOR GENERALIZED α-ψ-CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED METRIC-TYPE SPACES

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ABSTRACT. In this paper, we state some coupled fixed point theorems for generalized α-ψ-contractive mappings in partially ordered metric-type spaces. In addition, some particular cases and consequences of our theorems are given. Moreover, we give some examples to illustrate the obtained results.

KEYWORDS: Coupled fixed point; metric-type space; α-ψ-contractive mapping.
AMS Subject Classification: Primary 47H10, 54H25; Secondary 54D99, 54E99.

1. INTRODUCTION

The notion of a metric-type space was introduced by Khamsi in [5] as follows.

Definition 1.1 ([5], Definition 2.7)**.** Let X be a non-empty set, $K \geq 1$ be a real number and $D : X \times X \longrightarrow [0, \infty)$ be a mapping satisfying the following.

(i) $D(x, y) = 0$ if and only if $x = y$;
(ii) $D(x, y) = D(y, x)$ for all $x, y \in X$;
(iii) For all $x, y_1, y_2, \dots, y_n, z \in X$, we have

$$D(x, z) \leq K[D(x, y_1) + D(y_1, y_2) + \dots + D(y_n, z)].$$

Then D is called a *metric-type* on X and (X, D, K) is called a *metric-type space*.

Remark 1.2. (X, d) is a metric space if and only if $(X, d, 1)$ is a metric-type space.

Some other authors in [2], [3] and [4] considered another metric-type space, where the condition (3) in Definition 1.1 is replaced by

$$D(x, y) \leq K[D(x, z) + D(z, y)]$$

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for all $x, y, z \in X$ and proved several other fixed point and common fixed point results in this metric-type space. In this paper, we consider the metric-type space in the sense of Definition 1.1.

The convergence and the completeness in the metric type-spaces were defined as follows.

Definition 1.3 ([5], Definition 2.8)**.** Let (X, D, K) be a metric-type space and $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is said to *converge* to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} D(x_n, x) = 0$.

(ii) $\{x_n\}$ is said to be *Cauchy* if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.

(iii) (X, D, K) is said to be *complete* if every Cauchy sequence is a convergent sequence.

Remark 1.4. On the metric-type space, we always use the topology induced by its convergence.

In [1], Bhaskar and Lakshmikantham introduced the notions of mixed monotone property and coupled fixed point for contractive mapping $F : X \times X \longrightarrow X$, where X is a partially ordered metric space as follows.

Definition 1.5 ([1], Definition 1.1)**.** Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ be a mapping. Then F is said to *have the mixed monotone property* if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y),$$

and

$$y_1, y_2 \in X,\ y_1 \preceq y_2 \text{ implies } F(x, y_1) \succeq F(x, y_2).$$

Definition 1.6 ([1], Definition 1.2)**.** An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F : X \times X \longrightarrow X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Moreover, in [1], the authors proved some coupled fixed point theorems for a mixed monotone mapping, see [1, Theorem 2.1], [1, Theorem 2.2] and [1, Theorem 2.4]. Afterwards, in [7], the authors established coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces which extend the results of [1]. For more details on coupled fixed point theory, we also refer the reader to [8, 11, 14] and references therein.

Recently, Samet *et al.* [15] introduced the α-ψ-contractive and the α-admissible mapping with $\alpha : X \times X \longrightarrow [0, \infty)$ and proved fixed point theorems for mappings in complete metric spaces. After that, some authors studied fixed point results for a new α-ψ-contractive and various classes of mappings which are based on α-admissible mappings, see for example [6, 12, 13] and references therein. Most recently, Mursaleen *et al.* [9] introduced the notions of α-admissible mapping with $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ and α-ψ-contractive as follows.

Denote by Ψ the family of non-decreasing functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ satisfying:

(i) $\psi^{-1}(0) = 0$;

(ii) $\psi(t) < t$ for all $t > 0$;

(iii) $\lim_{r\to t^+} \psi(r) < t$ for all $t > 0$.

Lemma 1.7 ([9], Lemma 3.1)**.** *If $\psi : [0,\infty) \longrightarrow [0,\infty)$ is non-decreasing and right continuous, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \geq 0$ if and only if $\psi(t) < t$ for all $t > 0$.*

Definition 1.8 ([10], Definition 3.2)**.** Let $(X, d, \preceq)$ be a partially ordered metric space and $F : X \times X \longrightarrow X$ be a mapping. Then F is said to be *α-ψ-contractive* if there exist two functions $\alpha : X^2 \times X^2 \longrightarrow [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha\big((x,y),(u,v)\big)\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2} \leq \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$.

Definition 1.9 ([9], Definition 3.3)**.** Let $F : X \times X \longrightarrow X$ and $\alpha : X^2 \times X^2 \longrightarrow [0,\infty)$ be two mappings. Then F is said to be *α-admissible* if

$$\alpha\big((x,y),(u,v)\big) \geq 1 \text{ implies } \alpha\Big(\big(F(x,y),F(y,x)\big),\big(F(u,v),F(v,u)\big)\Big) \geq 1$$

for all $x, y, u, v \in X$.

Furthermore, in [9], the authors established some coupled fixed point results on partially ordered metric spaces which are generalizations of the main results in [1], see [9, Theorem 3.4], [9, Theorem 3.5] and [9, Theorem 3.6].

The aim of this paper is to state some coupled fixed point theorems for generalized α-ψ-contractive mappings in partially ordered metric-type spaces. In addition, some particular cases and consequences of our theorems are given. Moreover, we give some examples to illustrate the obtained results.

2. MAIN RESULTS

We start with an example about a non-continuous metric-type as follows.

Example 2.1. Let $X = \Big\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\Big\}$ and $D : X \times X \longrightarrow [0,\infty)$ be defined by

$$D(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \text{ and } x, y \in \{0,1\} \\ |x-y| & \text{if } x, y \in \Big\{0, \frac{1}{n}, \frac{1}{m}\Big\}, \text{ where } n, m \geq 2 \\ \frac{1}{3} & \text{if } x, y \in \Big\{1, \frac{1}{n}\Big\}, \text{ where } n \geq 2. \end{cases}$$

Then, D is a non-continuous metric-type with $K = 3$.

Proof. For all $x, y \in X$, we have $D(x,y) \geq 0$, $D(x,y) = 0$ if and only if $x = y$ and $D(x,y) = D(y,x)$. For all $x, y_1, \ldots, y_k, y \in X, k \geq 1$, we will show that

$$D(x,y) \leq 3[D(x,y_1) + D(y_1,y_2) + \ldots + D(y_k,y)]. \tag{2.1}$$

Put

$$\sigma = D(x,y_1) + D(y_1,y_2) + \ldots + D(y_k,y).$$

We only consider three following cases.

Case 1. $D(x,y) = D(0,1) = 1$ or $D(x,y) = D\big(1, \frac{1}{n}\big) = \frac{1}{3}$ for $n \geq 2$. Then $\sigma \geq \frac{1}{3}$.

Case 2. $D(x,y) = D\big(0, \frac{1}{n}\big) = \frac{1}{n}$ for $n \geq 2$. Then $\sigma \geq \frac{1}{3}$ if there exists $i \in \{1, \dots, k\}$ such that $y_i = 1$ and $\sigma \geq \frac{1}{n}$ if $y_i \neq 1$ for all $i = 1, \dots, k$.

Case 3. $D(x,y) = D\big(\frac{1}{n}, \frac{1}{m}\big) = \Big|\frac{1}{n} - \frac{1}{m}\Big|$. Then $\sigma \geq \frac{1}{3}$ if there exists $i \in \{1, \dots, k\}$ such that $y_i = 1$ and $\sigma \geq \Big|\frac{1}{n} - \frac{1}{m}\Big|$ if $y_i \neq 1$ for all $i = 1, \dots, k$.

From the above cases, we conclude that (2.1) holds. This proves that D is a metric-type on X with $K = 3$.

Now, we have $\lim\limits_{n\to\infty} D\big(\frac{1}{n}, 0\big) = \lim\limits_{n\to\infty} \frac{1}{n} = 0$. However, $\lim\limits_{n\to\infty} D\big(\frac{1}{n}, 1\big) = \frac{1}{3} \neq 1 = D(0,1)$. This proves that D is non-continuous. □

Next, we introduce the notion of a generalized α-ψ-contractive mapping in a partially ordered metric-type space as follows.

Definition 2.2. Let $(X, D, K, \preceq)$ be a partially ordered metric-type space and $F : X \times X \longrightarrow X$ be a mapping. Then F is said to be *generalized α-ψ-contractive* if there exist two functions $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha\big((x,y),(u,v)\big)\frac{D\big(F(x,y),F(u,v)\big) + D\big(F(y,x),F(v,u)\big)}{2} \leq \psi\Big(\frac{M(x,y,u,v)}{2}\Big) \tag{2.2}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, where

$$\begin{aligned} M(x,y,u,v) = \max\Big\{ & D(u,F(x,y)) + D(v,F(y,x)), D(x,F(x,y)) + D(y,F(y,x)), \\ & D(x,u) + D(y,v), \frac{D(u,F(u,v)) + D(v,F(v,u))}{2K}, \\ & \frac{D(x,F(u,v)) + D(y,F(v,u))}{2K}\Big\}. \end{aligned}$$

Our first result is the following.

Theorem 2.3. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *F is generalized α-ψ-contractive;*
(ii) *F is α-admissible;*
(iii) *F is continuous;*
(iv) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$ and*

$$\alpha\big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\big) \geq 1.$$

Then F has a coupled fixed point.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$ and

$$\alpha\big((x_0,y_0),\big(F(x_0,y_0),F(y_0,x_0)\big)\big) \geq 1.$$

Let $x_1, y_1 \in X$ be such that $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Let $x_2, y_2 \in X$ be such that $F(x_1, y_1) = x_2$ and $F(y_1, x_1) = y_2$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n)$$

for all $n \in \mathbb{N}$. We will show that

$$x_n \preceq x_{n+1}, \quad y_n \succeq y_{n+1} \tag{2.3}$$

for all $n \in \mathbb{N}$ by the mathematical induction.

Let $n = 0$. We have $x_0 \preceq F(x_0, y_0) = x_1$ and $y_0 \succeq F(y_0, x_0) = y_1$. Thus, (2.3) holds for $n = 0$. Now, suppose that (2.3) holds for some fixed $n \in \mathbb{N}$. Then, since $x_n \preceq x_{n+1}$, $y_n \succeq y_{n+1}$ and the mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \succeq F(x_n, y_{n+1}) \succeq F(x_n, y_n) = x_{n+1}$$

and

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \preceq F(y_n, x_{n+1}) \preceq F(y_n, x_n) = y_{n+1}.$$

From the above, we have $x_{n+1} \preceq x_{n+2}$ and $y_{n+1} \succeq y_{n+2}$. Therefore, by the mathematical induction, we conclude that (2.3) holds for all $n \in \mathbb{N}$.

If there exists some $n \in \mathbb{N}$ such that $x_{n+1} = x_n$ and $y_{n+1} = y_n$, then $F(x_n, y_n) = x_n$ and $F(y_n, x_n) = y_n$, that is, F has a coupled fixed point. Now, we assume that $x_{n+1} \neq x_n$ or $y_{n+1} \neq y_n$ for all $n \in \mathbb{N}$. Since F is α-admissible and

$$\alpha\big((x_0, y_0), (x_1, y_1)\big) = \alpha\big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\big) \geq 1,$$

we get $\alpha\big(\big(F(x_0, y_0), F(y_0, x_0)\big), \big(F(x_1, y_1), F(y_1, x_1)\big)\big) \geq 1$. Thus,

$$\alpha\big((x_1, y_1), (x_2, y_2)\big) \geq 1.$$

By the mathematical induction, we have

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \tag{2.4}$$

for all $n \in \mathbb{N}$. Since F is generalized α-ψ-contractive and using (2.3), (2.4), we get

$$\begin{aligned}
&\frac{D(x_n, x_{n+1}) + D(y_n, y_{n+1})}{2} \\
= \; & \frac{D\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) + D\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)}{2} \\
\leq \; & \alpha\big((x_{n-1}, y_{n-1}), (x_n, y_n)\big) \times \\
& \quad \times \frac{D\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) + D\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big)}{2} \\
\leq \; & \psi\Big(\frac{M(x_{n-1}, y_{n-1}, x_n, y_n)}{2}\Big)
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
& M(x_{n-1}, y_{n-1}, x_n, y_n) \\
= \; & \max\Big\{D(x_n, F(x_{n-1}, y_{n-1})) + D(y_n, F(y_{n-1}, x_{n-1})), \\
& \qquad D(x_{n-1}, F(x_{n-1}, y_{n-1})) + D(y_{n-1}, F(y_{n-1}, x_{n-1})), \\
& \qquad D(x_{n-1}, x_n) + D(y_{n-1}, y_n), \frac{D(x_n, F(x_n, y_n)) + D(y_n, F(y_n, x_n))}{2K}, \\
& \qquad \frac{D(x_{n-1}, F(x_n, y_n)) + D(y_{n-1}, F(y_n, x_n))}{2K}\Big\} \\
= \; & \max\Big\{D(x_n, x_n) + D(y_n, y_n)), D(x_{n-1}, x_n) + D(y_{n-1}, y_n), \\
& \qquad D(x_{n-1}, x_n) + D(y_{n-1}, y_n), \frac{D(x_n, x_{n+1}) + D(y_n, y_{n+1})}{2K}, \\
& \qquad \frac{D(x_{n-1}, x_{n+1}) + D(y_{n-1}, y_{n+1})}{2K}\Big\}
\end{aligned}$$

$$\begin{aligned} &\le \max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1}),\\ &\qquad \frac{D(x_{n-1},x_n)+D(y_{n-1},y_n)+D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2}\Big\}\\ &= \max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\}. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned} &\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2}\\ &\le \psi\Big(\frac{\max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\}}{2}\Big). \end{aligned} \tag{2.7}$$

If there exists some $n \ge 1$ such that

$$\max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\} = D(x_n,x_{n+1})+D(y_n,y_{n+1}),$$

then (2.7) becomes

$$\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2} \le \psi\Big(\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2}\Big).$$

It is a contradiction to $\psi(t) < t$ for all $t > 0$. Therefore,

$$\max\Big\{D(x_{n-1},x_n)+D(y_{n-1},y_n), D(x_n,x_{n+1})+D(y_n,y_{n+1})\Big\} = D(x_{n-1},x_n)+D(y_{n-1},y_n)$$

for all $n \ge 1$. Then, (2.7) becomes

$$\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2} \le \psi\Big(\frac{D(x_{n-1},x_n)+D(y_{n-1},y_n)}{2}\Big). \tag{2.8}$$

Repeating the above process, we get

$$\frac{D(x_n,x_{n+1})+D(y_n,y_{n+1})}{2} \le \psi^n\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big) \tag{2.9}$$

for all $n \ge 1$. For $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \ge n(\varepsilon)} \psi^n\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big) < \frac{\varepsilon}{2K}. \tag{2.10}$$

Let $n, m \in \mathbb{N}$ be such that $m > n > n(\varepsilon)$. Then, by using (2.10), we have

$$\begin{aligned} &\frac{D(x_n,x_m)+D(y_n,y_m)}{2}\\ &\le K\sum_{k=n}^{m-1} \frac{D(x_k,x_{k+1})+D(y_k,y_{k+1})}{2}\\ &\le K\sum_{k=n}^{m-1} \psi^k\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big)\\ &\le K\sum_{n \ge n(\varepsilon)} \psi^n\Big(\frac{D(x_0,x_1)+D(y_0,y_1)}{2}\Big) < \frac{\varepsilon}{2}. \end{aligned} \tag{2.11}$$

It implies that $D(x_n,x_m)+D(y_n,y_m) < \varepsilon$. Therefore,

$$D(x_n,x_m) \le D(x_n,x_m)+D(y_n,y_m) < \varepsilon$$

and

$$D(y_n,y_m) \le D(x_n,x_m)+D(y_n,y_m) < \varepsilon.$$

This implies $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in (X, D, K). Since X is a complete metric-type space, we have $\{x_n\}$ and $\{y_n\}$ are convergent in (X, D, K). Then there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y \tag{2.12}$$

Since F is continuous and $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$, taking the limit as $n \to \infty$, we get

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(y, x),$$

that is, $F(x, y) = x$ and $F(y, x) = y$. Therefore, F has a coupled fixed point. □

In the next theorem, we omit continuous hypothesis of F.

Theorem 2.4. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space, where D is continuous in each variable and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *F is generalized α-ψ-contractive;*
(ii) *F is α-admissible;*
(iii) *If $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha\big((x_n, y_n), (x, y)\big) \geq 1$;*
(iv) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $\quad y_0 \succeq F(y_0, x_0)$ and*
$$\alpha\big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\big) \geq 1.$$

Then F has a coupled fixed point.

Proof. Following the proof of Theorem 2.3, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \tag{2.13}$$

and

$$\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1 \tag{2.14}$$

for all $n \in \mathbb{N}$. By using (2.13), (2.14) and hypothesis (3), we get

$$\alpha\big((x_n, y_n), (x, y)\big) \geq 1 \tag{2.15}$$

for all $n \in \mathbb{N}$. Since F is generalized α-ψ-contractive and using (2.15), we get

$$\begin{aligned}
&\frac{D(F(x,y),x) + D(F(y,x),y)}{2} \\
\leq\ & K\frac{D(F(x,y),F(x_n,y_n)) + D(F(y,x),F(y_n,x_n))}{2} \\
&\quad +K\frac{D(F(x_n,y_n),x) + D(F(y_n,x_n),y)}{2} \\
=\ & K\frac{D(F(x,y),F(x_n,y_n)) + D(F(y,x),F(y_n,x_n))}{2} \\
&\quad +K\frac{D(x_{n+1},x) + D(y_{n+1},y)}{2} \\
\leq\ & K\alpha\big((x_n,y_n),(x,y)\big)\frac{D(F(x,y),F(x_n,y_n)) + D(F(y,x),F(y_n,x_n))}{2} \\
&\quad +K\frac{D(x_{n+1},x) + D(y_{n+1},y)}{2}
\end{aligned}$$

$$\begin{aligned}
&\le K\psi\Big(\frac{M(x_n,y_n,x,y)}{2}\Big)+K\frac{D(x_{n+1},x)+D(y_{n+1},y)}{2}\\
&\le K\frac{M(x_n,y_n,x,y)}{2}+K\frac{D(x_{n+1},x)+D(y_{n+1},y)}{2}, \qquad (2.16)
\end{aligned}$$

where

$$\begin{aligned}
&M(x_n,y_n,x,y)\\
=\ &\max\Big\{D(x,F(x_n,y_n))+D(y,F(y_n,x_n)),D(x_n,F(x_n,y_n))+D(y_n,F(y_n,x_n)),\\
&\quad D(x_n,x)+D(y_n,y),\frac{D(x,F(x,y))+D(y,F(y,x))}{2K},\\
&\quad \frac{D(x_n,F(x,y))+D(y_n,F(y,x))}{2K}\Big\}\\
=\ &\max\Big\{D(x,x_{n+1})+D(y,y_{n+1}),D(x_n,x_{n+1})+D(y_n,y_{n+1}),\\
&\quad D(x_n,x)+D(y_n,y),\frac{D(x,F(x,y))+D(y,F(y,x))}{2K},\\
&\quad \frac{D(x_n,F(x,y))+D(y_n,F(y,x))}{2K}\Big\}. \qquad (2.17)
\end{aligned}$$

Letting $n\to\infty$ in (2.17), using (2.12) and the continuity in each variable property of D, we get

$$\lim_{n\to\infty} M(x_n,y_n,x,y) = \frac{D(x,F(x,y))+D(y,F(y,x))}{2K}. \qquad (2.18)$$

Letting $n\to\infty$ in (2.16), using (2.13) and (2.18), we obtain

$$D(x,F(x,y))+D(y,F(y,x)) \le \frac{D(x,F(x,y))+D(y,F(y,x))}{2}.$$

It implies $D(x,F(x,y))+D(y,F(y,x))=0$. Hence, $D(x,F(x,y))=D(y,F(y,x))=0$. Therefore, $F(x,y)=x$ and $F(y,x)=y$. Thus, F has a coupled fixed point. □

In the following theorem, we will prove the uniqueness of the coupled fixed point. If $(X,\preceq)$ is a partially ordered set, then we endow the product $X\times X$ with the partially ordered relation as follows.

$$(x,y)\preceq(u,v) \Longleftrightarrow x\preceq u,\quad y\succeq v$$

for all $(x,y),(u,v)\in X\times X$.

Theorem 2.5. *In addition to the hypothesis of Theorem 2.3 or Theorem 2.4, suppose that for every $(x,y),(s,t)$ in $X\times X$, there exists (u,v) in $X\times X$ such that (u,v) is comparable to (x,y), (s,t) and*

$$\alpha\big((x,y),(u,v)\big)\ge 1,\quad \alpha\big((s,t),(u,v)\big)\ge 1.$$

Then F has a unique coupled fixed point.

Proof. Following the proof of Theorem 2.3 and Theorem 2.4, F has a coupled fixed point. Suppose that (x,y) and (s,t) are two coupled fixed points of F. By the assumption, there exists (u,v) in $X\times X$ such that (u,v) is comparable to (x,y) and (s,t) and

$$\alpha\big((x,y),(u,v)\big)\ge 1,\quad \alpha\big((s,t),(u,v)\big)\ge 1. \qquad (2.19)$$

We define two sequences $\{u_n\}$ and $\{v_n\}$ as follows

$$u_0=u,\quad v_0=v,\quad u_{n+1}=F(u_n,v_n),\quad v_{n+1}=F(v_n,u_n)$$

for all $n \in \mathbb{N}$.

Since (u, v) is comparable to (x, y), we may assume that $(x, y) \preceq (u, v) = (u_0, v_0)$. By using the mathematical induction and the mixed monotone property of F, we can show that $x \preceq u_n$ and $y \succeq v_n$ for all $n \in \mathbb{N}$.

If $u_n = x$ and $v_n = y$ for all $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} u_n = x$ and $\lim_{n\to\infty} v_n = y$. Now, we assume that $u_n \neq x$ or $v_n \neq y$ for some $n \in \mathbb{N}$. Since F is α-admissible and using (2.19), we have

$$\alpha\big((F(x,y), F(y,x)), (F(u,v), F(v,u))\big) \geq 1.$$

Since $u_0 = u$ and $v_0 = v$, we get

$$\alpha\big((F(x,y), F(y,x)), (F(u_0,v_0), F(v_0,u_0))\big) \geq 1.$$

Thus, $\alpha\big((x,y),(u_1,v_1)\big) \geq 1$. Therefore, by the mathematical induction, we obtain

$$\alpha\big((x,y),(u_n,v_n)\big) \geq 1 \tag{2.20}$$

for all $n \in \mathbb{N}$. Since F is generalized α-ψ-contractive and (2.20), we get

$$\begin{aligned}
&\frac{D(x,u_{n+1}) + D(y,v_{n+1})}{2} \\
= \;& \frac{D(F(x,y),F(u_n,v_n)) + D(F(y,x),F(v_n,u_n))}{2} \\
\leq \;& \alpha\big((x,y),(u_n,v_n)\big)\frac{D(F(x,y),F(u_n,v_n)) + D(F(y,x),F(v_n,u_n))}{2} \\
\leq \;& \psi\Big(\frac{M(x,y,u_n,v_n)}{2}\Big)
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
& M(x,y,u_n,v_n) \\
= \;& \max\Big\{D(u_n,F(x,y)) + D(v_n,F(y,x)), D(x,F(x,y)) + D(y,F(y,x)), \\
& \quad D(x,u_n) + D(y,v_n), \frac{D(u_n,F(u_n,v_n)) + D(v_n,F(v_n,u_n))}{2K}, \\
& \quad \frac{D(x,F(u_n,v_n)) + D(y,F(v_n,u_n))}{2K}\Big\} \\
= \;& \max\Big\{D(u_n,x) + D(v_n,y), D(x,x) + D(y,y), \\
& \quad D(x,u_n) + D(y,v_n), \frac{D(u_n,u_{n+1}) + D(v_n,v_{n+1})}{2K}, \\
& \quad \frac{D(x,u_{n+1}) + D(y,v_{n+1})}{2K}\Big\} \\
= \;& \max\Big\{D(x,u_n) + D(y,v_n), \frac{D(u_n,u_{n+1}) + D(v_n,v_{n+1})}{2K}, \\
& \quad \frac{D(x,u_{n+1}) + D(y,v_{n+1})}{2K}\Big\} \\
\leq \;& \max\Big\{D(x,u_n) + D(y,v_n), \frac{D(x,u_n) + D(y,v_n) + D(x,u_{n+1}) + D(y,v_{n+1})}{2}, \\
& \quad D(x,u_{n+1}) + D(y,v_{n+1})\Big\} \\
= \;& \max\Big\{D(x,u_n) + D(y,v_n), D(x,u_{n+1}) + D(y,v_{n+1})\Big\}.
\end{aligned} \tag{2.22}$$

From (2.21) and (2.22), we have

$$\begin{aligned}&\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\\ \leq\ &\psi\Big(\frac{\max\big\{D(x,u_n)+D(y,v_n),D(x,u_{n+1})+D(y,v_{n+1})\big\}}{2}\Big).\end{aligned}\tag{2.23}$$

If there exists some $n\in\mathbb{N}$ such that

$$\max\big\{D(x,u_n)+D(y,v_n),D(x,u_{n+1})+D(y,v_{n+1})\big\}=D(x,u_{n+1})+D(y,v_{n+1}),$$

then (2.23) becomes

$$\begin{aligned}\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2} &\leq \psi\Big(\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\Big)\\ &< \frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}.\end{aligned}\tag{2.24}$$

It is a contradiction. Therefore,

$$\max\big\{D(x,u_n)+D(y,v_n),D(x,u_{n+1})+D(y,v_{n+1})\big\}=D(x,u_n)+D(y,v_n)$$

for all $n\in\mathbb{N}$, then (2.23) becomes

$$\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\leq\psi\Big(\frac{D(x,u_n)+D(y,v_n)}{2}\Big).$$

Repeating the above process, we get

$$\frac{D(x,u_{n+1})+D(y,v_{n+1})}{2}\leq\psi^n\Big(\frac{D(x,u_1)+D(y,v_1)}{2}\Big)\tag{2.25}$$

for $n\geq 1$. Letting $n\to\infty$ in (2.25) and using Lemma 1.7, we get

$$\lim_{n\to\infty}\big(D(x,u_{n+1})+D(y,v_{n+1})\big)=0.$$

This implies that $\lim\limits_{n\to\infty}D(x,u_{n+1})=\lim\limits_{n\to\infty}D(y,v_{n+1})=0$. Thus, $\lim\limits_{n\to\infty}u_n=x$ and $\lim\limits_{n\to\infty}v_n=y$. Therefore, from the above, we have

$$\lim_{n\to\infty}u_n=x,\quad \lim_{n\to\infty}v_n=y.\tag{2.26}$$

Similarly, we can show that

$$\lim_{n\to\infty}u_n=s,\quad \lim_{n\to\infty}v_n=t.\tag{2.27}$$

From (2.26) and (2.27), we conclude that $x=s$ and $y=t$. Hence, F has a unique coupled fixed point. □

Since every metric space (X,d) is a metric-type space $(X,d,1)$, from Theorem 2.3, Theorem 2.4 and Theorem 2.5, we get two following corollaries.

Corollary 2.6. *Let $(X,d,\preceq)$ be a partially ordered and complete metric space and $F:X\times X\longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *There exist two functions $\alpha:X^2\times X^2\longrightarrow[0,\infty)$ and $\psi\in\Psi$ such that*

$$\alpha\big((x,y),(u,v)\big)\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2}\quad\leq\quad\psi\Big(\frac{N(x,y,u,v)}{2}\Big)$$

for all $x,y,u,v\in X$ with $x\succeq u$ and $y\preceq v$, where

$$\begin{aligned}N(x,y,u,v)\ =\ \max\Big\{&d(u,F(x,y))+d(v,F(y,x)),d(x,F(x,y))+d(y,F(y,x)),\\ &d(x,u)+d(y,v),\frac{d(u,F(u,v))+d(v,F(v,u))}{2},\end{aligned}$$

$$\frac{d(x,F(u,v))+d(y,F(v,u))}{2}\Big\};$$

(ii) *F is α-admissible;*
(iii) *Suppose either*
  (a) *F is continuous or*
  (b) *If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\alpha\big((x_n, y_n), (x_{n+1}, y_{n+1})\big) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha\big((x_n, y_n), (x, y)\big) \geq 1$;*
(iv) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$ and*

$$\alpha\big((x_0, y_0), \big(F(x_0, y_0), F(y_0, x_0)\big)\big) \geq 1.$$

Then F has a coupled fixed point.

Corollary 2.7. *In addition to the hypothesis of Corollary 2.6, suppose that for every $(x,y),(s,t)$ in $X \times X$, there exists (u,v) in $X \times X$ such that (u,v) is comparable to (x,y), (s,t) and*

$$\alpha\big((x,y),(u,v)\big) \geq 1, \quad \alpha\big((s,t),(u,v)\big) \geq 1.$$

Then F has a unique coupled fixed point.

Remark 2.8. We see that [9, Theorem 3.4] and [9, Theorem 3.5] are two direct consequences of Corollary 2.6, [9, Theorem 3.6] is a direct consequence of Corollary 2.7.

By using similar arguments as in the proofs of [15, Theorem 3.4], [15, Theorem 3.5] and [15, Theorem 3.6], from Theorem 2.3, Theorem 2.4 and Theorem 2.5, we get following results.

Proposition 2.9. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *There exists $\lambda \in [0,1)$ such that*

$$\begin{aligned} & D\big(F(x,y),F(u,v)\big) + D\big(F(y,x),F(v,u)\big) \\ \leq\ & \lambda \max\Big\{D(u,F(x,y)) + D(v,F(y,x)), D(x,F(x,y)) + D(y,F(y,x)), \\ & D(x,u)+D(y,v), \frac{D(u,F(u,v))+D(v,F(v,u))}{2K}, \\ & \frac{D(x,F(u,v))+D(y,F(v,u))}{2K}\Big\}, \end{aligned}$$

for all $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$;
(ii) *F is continuous;*
(iii) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$.*

Then F has a coupled fixed point.

Proposition 2.10. *Let $(X, D, K, \preceq)$ be a partially ordered and complete metric-type space where D is continuous in each variable and $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property such that*

(i) *There exists $\lambda \in [0,1)$ such that*

$$\begin{aligned} & D\big(F(x,y),F(u,v)\big) + D\big(F(y,x),F(v,u)\big) \\ \leq\ & \lambda \max\Big\{D(u,F(x,y)) + D(v,F(y,x)), D(x,F(x,y)) + D(y,F(y,x)), \\ & D(x,u)+D(y,v), \frac{D(u,F(u,v))+D(v,F(v,u))}{2K}, \end{aligned}$$

$$\frac{D(x, F(u,v)) + D(y, F(v,u))}{2K}\Big\},$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$;

(ii) *X has the following properties: If $\{x_n\}$ is a non-decreasing sequence in X such that $\lim_{n\to\infty} x_n = x$ and $\{y_n\}$ is a non-increasing sequence in X such that $\lim_{n\to\infty} y_n = y$, then $x_n \preceq x$ and $y_n \succeq y$ for all $n \in \mathbb{N}$;*

(iii) *There exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$.*

Then F has a coupled fixed point.

Proposition 2.11. *In addition to the hypothesis of Corollary 2.9 or Corollary 2.10, suppose that for every $(x, y), (s, t)$ in $X \times X$, there exists (u, v) in $X \times X$ such that (u, v) is comparable to (x, y) and (s, t). Then F has a unique coupled fixed point.*

Finally, in order to support the useability of our results, let us introduce some following examples.

Example 2.12. Let $X = \{1, 2, 3\}$ with the partially ordered relation as follows.

$$x \succeq y \text{ if and only if } x \geq y \text{ and } x, y \in \{1, 2\}.$$

Define a function $D : X \times X \longrightarrow [0, \infty)$ such that

$$D(1,1) = D(2,2) = D(3,3) = 0,$$

$$D(1,2) = D(2,1) = D(1,3) = D(3,1) = 1,$$

$$D(2,3) = D(3,2) = 4.$$

Then, (X, D, K) is a complete metric-type space with $K = 2$. Consider a mapping $F : X \times X \longrightarrow X$ by

$$F(1,1) = F(2,2) = F(2,1) = F(1,2) = 1,$$

$$F(3,3) = F(3,1) = F(1,3) = F(2,3) = F(3,2) = 2.$$

Define a function $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ by

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 & \text{if } x = y = u = v = 1, \\ \dfrac{1}{2} & \text{if otherwise.} \end{cases}$$

Then, for all $(x, y), (u, v) \in X \times X$ with $x \succeq u, y \preceq v$, we have

$$\begin{aligned} & \alpha\big((x,y),(u,v)\big)\frac{D\big(F(x,y), F(u,v)\big) + D\big(F(y,x), F(v,u)\big)}{2} \\ = \; & \alpha\big((x,y),(u,v)\big)\frac{D(1,1) + D(1,1)}{2} \\ = \; & 0 \\ \leq \; & \psi\Big(\frac{M(x,y,u,v)}{2}\Big). \end{aligned}$$

Therefore, (2.2) holds for all $\psi \in \Psi$, and also the hypothesis of Theorem 2.3 are fulfilled. Therefore, there exists a coupled fixed point of F. In this case, $(1, 1)$ is a coupled fixed point of F.

The following example show that Corollary 2.6 is proper generalization of some results in [9].

Example 2.13. Let $X = \{0, 1, 2\}$ with the usual order $\leq$ on $\mathbb{R}$ and d be defined by

$$d(0,0) = d(1,1) = d(2,2) = 0, d(1,2) = d(2,1) = 4,$$
$$d(0,1) = d(1,0) = d(0,2) = d(2,0) = 2.$$

Define a mapping $F : X \times X \longrightarrow X$ as follows

$$F(0,1) = F(1,1) = F(2,1) = 1,$$
$$F(0,0) = F(1,0) = F(2,0) = 2,$$
$$F(0,2) = F(1,2) = F(2,2) = 0.$$

Consider a function $\psi(t) = \dfrac{t}{2}$ for all $t \geq 0$ and a function $\alpha : X^2 \times X^2 \longrightarrow [0, \infty)$ such that

$$\alpha\big((x,y),(u,v)\big) = \begin{cases} 1 & \text{if } x = y = u = v = 1, \\ \dfrac{3}{10} & \text{if otherwise.} \end{cases}$$

Then (X, d) is a complete metric space. For all $(x, y), (u, v) \in X \times X$ with $x \preceq u, y \succeq v$, we put

$$\sigma_1 = (u, F(x,y)) + d(v, F(y,x)), \quad \sigma_2 = d(x, F(x,y)) + d(y, F(y,x)),$$
$$\sigma_3 = \frac{d(u, F(u,v)) + d(v, F(v,u))}{2}, \quad \sigma_4 = \frac{d(x, F(u,v)) + d(y, F(v,u))}{2},$$
$$\sigma_5 = d(x,u) + d(y,v), \quad N = \max\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\},$$
$$L = \frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2}.$$

Then, we have the following table.

u	v	x	y	L	σ_4	σ_3	σ_2	σ_1	σ_5	N
0	0	0	0	0	2	2	4	4	0	4
0	0	0	1	2	3	2	6	4	2	6
0	0	0	2	1	1	2	0	2	2	2
0	1	0	1	0	3	3	6	6	0	6
0	1	0	2	1	1	3	0	4	4	4
0	2	0	2	0	0	0	0	0	0	0
1	0	0	0	2	2	3	4	6	2	6
1	0	0	1	4	1	3	6	2	4	6
1	0	0	2	3	3	3	0	4	4	4
1	1	0	1	2	1	0	6	4	2	6
1	1	0	2	3	3	0	0	6	6	6
1	2	0	2	2	2	3	0	2	2	3
1	0	1	0	0	3	3	6	6	0	6
1	0	1	1	2	2	3	0	2	2	3
1	0	1	2	1	4	3	6	4	2	6
1	1	1	1	0	0	0	0	0	0	0
1	1	1	2	1	2	0	6	4	4	6
1	2	1	2	0	3	3	6	4	0	6
2	0	0	0	1	1	0	4	2	2	4
2	0	0	1	3	2	0	6	6	4	6
2	0	0	2	2	2	0	0	4	4	4
2	1	0	1	1	2	3	6	6	2	6
2	1	0	2	2	2	3	0	6	6	6
2	2	0	2	1	1	2	0	2	2	2

2	0	1	0	1	2	0	6	2	4	6
2	0	1	1	3	3	0	0	6	6	6
2	0	1	2	2	3	0	6	4	6	6
2	1	1	1	1	1	3	0	4	4	4
2	1	1	2	2	1	3	6	2	8	8
2	2	1	2	1	2	2	6	6	4	6
2	0	2	0	0	0	0	0	0	0	0
2	0	2	1	2	1	0	6	4	2	6
2	0	2	2	1	1	0	4	2	2	4
2	1	2	1	0	3	3	6	6	0	6
2	1	2	2	1	3	3	4	4	4	4
2	2	2	2	0	2	2	4	4	0	4

Now, let $(x, y, u, v) = (1, 0, 0, 1)$, we have

$$\alpha\big((x,y),(u,v)\big)\frac{d\big(F(x,y),F(u,v)\big)+d\big(F(y,x),F(v,u)\big)}{2}$$

$$= \frac{3}{10}.4 = \frac{6}{5} > 1 = \psi(2) = \psi\Big(\frac{d(1,0)+d(0,1)}{2}\Big).$$

Therefore, [9, Theorem 3.4] and [9, Theorem 3.5] are not applicable to F, (X, d), α and ψ. Otherwise, the above calculations show that assumption (1) of Corollary 2.6 holds. Moreover, the assumptions of Corollary 2.6 are fulfilled. Therefore, there exists a coupled fixed point of F. In this case, $(1, 1)$ is a coupled fixed point of F.

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