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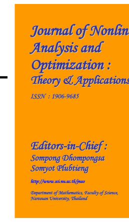

THE CHEBYSHEV WAVELETS OPERATIONAL MATRIX OF INTEGRATION AND PRODUCT OPERATION MATRIX FOR STURM-LIOUVILLE PROBLEM

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ABSTRACT. In this research paper, we present the Chebyshev wavelets operational matrix of integration and product operation matrix. These matrices have been applied to find a solution for Sturm-Liouville problem. We have provided numerical examples to indicate that operational matrix of integration and product operation matrix are applicable for Chebyshev wavelets.

KEYWORDS: Chebyshev wavelets; Operational matrix; Product operational matrix; Sturm-Liouville problem.
AMS Subject Classification: 34Bxx; 34B24; 65T60.

1. INTRODUCTION

In recent years, wavelets have been applied in many different fields of science and engineering. For example, Haar wavelet operational matrix has been extensively used in system analysis [1,2], system identification [3,4], optimal control [5,6], and numerical solution of integral and differential equations [7-15]. Moreover the application of Legendre wavelets [16,17], Hybrid functions [18,19] has received special attention among researchers. In this paper, we have presented Chebyshev wavelets operational matrix and product operation matrix to find a solution for Sturm-Liouville problem. A linear Sturm-Liouville operator has the form:

$$Ky(t) := Ly(t) = \lambda r(t), \tag{1.1}$$

where

$$L := -\frac{d}{dt}[p(t)\frac{d}{dt}] + q(t), \quad t \in I := [a, b].$$

Related to with differential equation (1) are the separated homogeneous boundary conditions $ay(0) + by'(0) = 0$ and $cy(1) + dy'(1) = 0$, in which a, b, c and d, are arbitrary constants. The values of λ for which the boundary value problem has a

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nontrivial solutions are named eigenvalues. To simplify the issue, we will assume that $p(t), p'(t), q(t)$, and $r(t)$ are continuous and $p(t) > 0$ and $r(t) > 0$ for all $t \in I$, for simplicity.
This paper consist of the following section: in section 2, we briefly review basic definitions fractional calculus. In section 3, we demonstrate how we can derive Haar wavelet operational matrix of fractional order integration. We have provided some illustrative examples in section 4 to demonstrate the application of operational matrix of integration for Haar wavelets.
Wavelets are a family of functions which are formed from delation and translation of a single function called the mother wavelet. When the the delation parameter a and translation parameter b vary continuously, the result will be following family of continuous wavelets as [8],

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}}\psi(\frac{t-b}{a}), \quad a, b \in R, \quad a \neq 0.$$

If we limit the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$ and $b_0 > 0$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}(a_0^k t - nb_0)$$

in which $\psi_{k,n}$ form a wavelet basis for $L^2(R)$. In particular when $a_0 = 2$ and $b_0 = 1$ forms as orthogonal basis. Chebyshev wavelets $\psi_{n,m}$, an the interval $[0, 1)$ are defined as [8]:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}}\widetilde{T}_m(2^{k+1}t - 2n + 1) & \frac{n-1}{2^k} \leq t < \frac{n}{2^k} \\ 0 & otherwise, \end{cases} \tag{1.2}$$

where

$$\widetilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0, \\ \frac{2}{\sqrt{\pi}}T_m(t) & m > 0, \end{cases}$$

and $m = 0, 1, ..., M-1$, $n = 0, 1, ..., 2^k,$, k is any positive integer and $T_m(t)$ are Chebyshev polynomial of the first kind of degree m which are orthogonal with respect to the weight function $\omega(t) = \frac{1}{\sqrt{1-t^2}}$ on the interval $[-1, 1]$ and $T_m(t)$ can be determined by the following recurrence formula:
$T_0(t) = 1$, $T_1(t) = t$, $T_{m+}(t) = 2tT_m(t) - T_{m-1}(t)$, $m = 1, 2, , ...$
The set of Chebyshev wavelets are an orthogonal set with respect to the weight function $\omega_n(t) = \omega(2^{k+1}t - 2n + 1)$.
A function $f(t)$ defined over $[0, 1)$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{nm}\psi_{nm}, \tag{1.3}$$

where $f_{nm} =< f(t), \psi_{nm} >$ in which $< \cdot >$ denotes the inner product. If the infinite series in Eq.(3) is shortened, then Eq.(3) can be written as:

$$f(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} f_{nm}\psi_{nm} = F^T\psi(t), \tag{1.4}$$

where F and ψ are $2^{k-1}M \times 1$ matrics given by

$$F = [f_{10}, f_{11}, ..., f_{1M-1}, f_{20}, f_{21}, ..., f_{2M-1}, ..., f_{2^k0}, f_{2^k1}, ..., f_{2^kM-1}]^T \tag{1.5}$$

$$\psi(t) = [\psi_{10}, \psi_{11}, ..., \psi_{1M-1}, \psi_{20}, \psi_{21}, ..., \psi_{2M-1}, ..., \psi_{2^k0}, \psi_{2^k1}, ..., \psi_{2^kM-1}]^T. \tag{1.6}$$

2. OPERATIONAL MATRIX OF INTEGRATION

The integration of the vector $\psi(t)$ can be determined as

$$\int_0^t \psi(r)dr = P\psi(t), \tag{2.1}$$

where P is a $(2^k M) \times (2^k M)$ matrix given by:

$$P = \begin{pmatrix} C & S & S & . & . & . & S \\ 0 & C & S & . & . & . & S \\ 0 & 0 & C & . & . & . & S \\ . & . & . & . & & & \\ . & . & . & & . & & \\ . & . & . & & & . & S \\ 0 & 0 & 0 & . & . & . & C \end{pmatrix} \tag{2.2}$$

where s and c are $M \times M$ matrices given by

$$S = \frac{\sqrt{2}}{2^k}\begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ -\frac{1}{3} & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ -\frac{1}{15} & 0 & 0 & . & . & . & 0 \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ -\frac{1}{M(M-2)} & 0 & 0 & . & . & . & 0 \end{pmatrix} \tag{2.3}$$

$$C = \frac{1}{2^k}\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & . & . & . & 0 & 0 & 0 \\ -\frac{1}{8\sqrt{2}} & 0 & \frac{1}{8} & 0 & . & . & . & 0 & 0 & 0 \\ -\frac{1}{6\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & & & . & . & . \\ . & . & . & . & & . & & . & . & . \\ . & . & . & . & & & . & . & . & . \\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & . & . & . & -\frac{1}{4(M-3)} & 0 & -\frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & . & . & . & 0 & -\frac{1}{4(M-2)} & 0 \end{pmatrix} \tag{2.4}$$

3. CHEBYSHEV WAVELETS PRODUCT OPERATION MATRIX

The following property of the product of two Chebyshev wavelet function vectors will also be applied:

$$\psi(t)\psi^T(t)F \simeq \widetilde{F}\psi(t). \tag{3.1}$$

In this formula, F is given in Eq.(5), $\psi(t)$ can be obtained similarly to Eq.(6) and $\widetilde{F}$ is a $2^k M \times 2^k M$ matrix. Using $\psi(t)$, for $M = 3$ and $k = 1$ we obtain

$$\psi\psi^T = \frac{1}{\sqrt{\pi}}\begin{pmatrix} 2\psi_{10} & 2\psi_{11} & 2\psi_{12} & 0 & 0 & 0 \\ 2\psi_{11} & 2\psi_{10}+\sqrt{2}\psi_{12} & \sqrt{2}\psi_{11} & 0 & 0 & 0 \\ 2\psi_{12} & \sqrt{2}\psi_{11} & 2\psi_{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\psi_{20} & 2\psi_{21} & 2\psi_{22} \\ 0 & 0 & 0 & 2\psi_{21} & 2\psi_{20}+\sqrt{2}\psi_{22} & \sqrt{2}\psi_{21} \\ 0 & 0 & 0 & 2\psi_{22} & \sqrt{2}\psi_{21} & 2\psi_{20} \end{pmatrix}. \tag{3.2}$$

Therefore the 6×6 matrix $\widetilde{F}$ in Eq.(11) can be written as

$$\widetilde{F} = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}. \tag{3.3}$$

where B_i, $i = 1, 2$, are 3×3 matrices given by

$$B_i = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 2f_{i0} & 2f_{i1} & 2f_{i2} \\ 2f_{i1} & 2f_{i0} + \sqrt{2}f_{i2} & \sqrt{2}f_{i1} \\ 2f_{i2} & \sqrt{2}f_{i1} & 2f_{i0} \end{pmatrix}. \tag{3.4}$$

In general case, $\widetilde{F}$ is a $2^k M \times 2^k M$ matrix in the form

$$\widetilde{F} = \begin{pmatrix} B_1 & 0 & . & . & . & 0 \\ 0 & B_2 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & B_{2^k} \end{pmatrix}, \tag{3.5}$$

where B_i, $i = 1, 2, ..., 2^k$ are similar to those in Eq.(13).

4. EXAMPLES

Example 4.1. Consider the Sturm-Liouville problem

$$-y''(t) - \lambda y(t) = 0, \tag{4.1}$$

subject to

$$y(0) = -1, \qquad y'(0) = 4. \tag{4.2}$$

The exact solution of system(16-17) is

$$y(t, \lambda) = -\cos(\sqrt{\lambda}t) + \frac{4}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t). \tag{4.3}$$

Now, we solve the same problem using chebyshev wavelets, with $M = 3$ and $k = 1$. Let us suppose

$$y''(t) = Y^T \psi(t), \tag{4.4}$$

where

$$Y = [y_{10}, y_{11}, y_{12}, y_{20}, y_{21}, y_{22}]^T, \tag{4.5}$$

$$\psi = [\psi_{10}, \psi_{11}, \psi_{12}, \psi_{20}, \psi_{21}, \psi_{22}]^T. \tag{4.6}$$

From (19) and (17), we get

$$y'(t) = \int_0^t y''(s)ds + y'(0) = Y^T P\psi(t) + 4, \tag{4.7}$$

and

$$y(t) = \int_0^t y'(s)ds + y(0) = Y^T P^2\psi(t) + 4t - 1. \tag{4.8}$$

Now, if we assume

$$D = [0 \ \frac{\sqrt{\pi}}{2\sqrt{2}} \ 0 \ \sqrt{\pi} \ \frac{\sqrt{\pi}}{2\sqrt{2}} \ 0]^T, \tag{4.9}$$

then

$$y(t) = (Y^T P^2 + D^T)\psi(t). \tag{4.10}$$

Substituting Eqs.(19) and (24-25) into Eq.(16), we obtain

$$-Y^T\psi(t) - \lambda(Y^T P^2 + D^T)\psi(t) = 0. \tag{4.11}$$

Therefore

$$-\psi^T(t)Y - \lambda\psi^T(t)P^{2^T}Y = -\lambda\psi^T(t)D. \tag{4.12}$$

Consequently

$$(I + \lambda P^{2^T})Y = -\lambda D. \tag{4.13}$$

Hence, we can get the same $y(t)$ as the exact solution to the problem.

Example 4.2. (*Airy equation*) Consider the following Airy differential equation

$$y''(t) - ty(t) = 0, \tag{4.14}$$

plus initial conditions

$$y(0) = 1, \qquad y'(0) = 0. \tag{4.15}$$

The exact solution of Eq.(29-30) is demonstrated by

$$y(t) = 1 + \sum_{n=1}^{\infty} \frac{t^{3n}}{(3n)(3n-1)(3n-3)(3n-4)...(3)(2)}. \tag{4.16}$$

Now, we can solve the same problem by applying chebyshev wavelets, with $M = 3$ and $k = 1$.
We suppose that the unknown function $y''(t)$ is given by

$$y''(t) = Y^T\psi(t), \tag{4.17}$$

where

$$Y = [y_{10}, y_{11}, y_{12}, y_{20}, y_{21}, y_{22}]^T, \tag{4.18}$$

Therefore

$$y'(t) = \int_0^t y''(s)ds + y'(0) = Y^T P\psi(t), \tag{4.19}$$

and

$$y(t) = \int_0^t y'(s)ds + y(0) = (Y^T P^2 + D)\psi(t), \tag{4.20}$$

where

$$D_1 = [\frac{\sqrt{\pi}}{2}\ 0\ 0\ \frac{\sqrt{\pi}}{2}\ 0\ 0]^T. \tag{4.21}$$

To addition, we can denote t as

$$t = [\frac{\sqrt{\pi}}{2}\ \frac{\sqrt{\pi}}{8\sqrt{2}}\ 0\ \frac{3\sqrt{\pi}}{8}\ \frac{\sqrt{\pi}}{8\sqrt{2}}\ 0]^T\psi(t) = H^T\psi(t). \tag{4.22}$$

Now, if we substitute Eqs.(32) and (35-37) into Eq.(29), we obtain

$$Y^T\psi(t) - H^T\psi(t)(Y^T P^2 + D_1^T)\psi(t) = 0. \tag{4.23}$$

Therefore

$$\psi^T(t)Y - H^T\psi(t)\psi^T(t)P^{2^T}Y = H^T\psi(t)\psi(t)^T D_1. \tag{4.24}$$

Now, from Eq.(11) we have

$$\psi^T Y - \psi^T\widetilde{H}P^{2^T}Y = \psi^T\widetilde{H}D_1, \tag{4.25}$$

or

$$(I - \widetilde{H}P^{2^T})Y = \widetilde{H}D_1, \tag{4.26}$$

where $\widetilde{H}$ can be calculated similarly to Eq.(13). Thus, we can get the same $y(t)$ as the exact solution.

Example 4.3. (*Quantum mechanical harmonic oscillator problem*) Consider The Quantum mechanical harmonic oscillator problem

$$-y''(t) + (t^2 - \lambda)y(t) = 0, \qquad t \in (-\infty, \infty). \tag{4.27}$$

The singular eigenvalue problem (42) possesses the exact analytical solutions of the form

$$y_n^\infty(t) = A_n e^{\frac{-x^2}{2}} H_n(t), \qquad \lambda_n^\infty = 2n + 1, \qquad n = 0, 1, 2, ... \tag{4.28}$$

where $H_n(t)$ indicates the Hermit polynomials and A_n are normalization constants. Now, suppose the following system (42) on a truncated domain $0 \leq t \leq l$ for all $l > 0$:

$$-y''(t) + (t^2 - \lambda)y(t) = 0, \tag{4.29}$$

subject to

$$y(0) = \sqrt{\pi}, \quad y'(0) = 0, \tag{4.30}$$

featuring boundary values (Dirichlet boundary conditions)

$$y(-l) = y(l) = 0. \tag{4.31}$$

In the same way the previous examples can be set as Setting

$$y''(t) = Y^T \psi(t). \tag{4.32}$$

And using initial conditions, we get

$$y'(t) = \int_0^t y''(s)ds + y'(0) = Y^T P \psi(t), \tag{4.33}$$

and

$$y(t) = \int_0^t y'(s)ds + y(0) = (Y^T P^2 + D_1)\psi(t), \tag{4.34}$$

where

$$D_2 = [\frac{\pi}{4}\ 0\ 0\ \frac{\pi}{4}\ 0\ 0]^T. \tag{4.35}$$

We can also express t^2 as

$$t^2 = [\frac{11\sqrt{\pi}}{64}\ 0\ \frac{\sqrt{\pi}}{64\sqrt{2}}\ 0\ \frac{\sqrt{\pi}}{16\sqrt{2}}\ 0]^T \psi(t) = H_1^T \psi(t). \tag{4.36}$$

Substituting Eqs.(45) and (48-50) into Eq.(44), we obtain

$$-Y^T \psi(t) + (H_1^T \psi(t) - \lambda)(Y^T P^2 + D_2^T)\psi(t) = 0. \tag{4.37}$$

Therefore

$$-\psi^T(t)Y + H_1^T \psi(t)\psi^T(t)P^{2^T} Y + H_1^T \psi(t)\psi^T(t)D_2 - \lambda\psi^T(t)P^{2^T} Y - \lambda\psi^T(t)D_2 = 0 \tag{4.38}$$

Table1 : Eigenvalues of Eq.(44)

n	l	λ	$\lvert\lambda-\lambda_n^\infty\rvert$	λ_n^∞
0	2	1.08231643774281	0.08231643774281	1
	π	1.39521365408429	0.39521365408429	
	4.5	1.27710535363838	0.27710535363838	
1	2	1.08231643774291	1.91768356225709	3
	π	3.16650693566247	0.16650693566247	
	4.5	3.29162140783956	0.29162140783656	

Now, from Eq.(11), we will get

$$-\psi^T(t)Y-\psi^T(t)\widetilde{H}_1P^{2^T}Y+\psi^T(t)\widetilde{H}_1D_2-\lambda\psi^T(t)P^{2^T}Y=\lambda\psi^T(t)D_2. \quad (4.39)$$

Thus

$$(I+\widetilde{H}_1P^{2^T}-\widetilde{H}_1D_2+\lambda P^{2^T})Y=-\lambda D_2, \quad (4.40)$$

where $\widetilde{H}_1$ can be calculated in the same way as Eq.(13). Equation (53) is a set of algebraic equation which can be solved for Y with parameter λ.
In table 1, by applying Dirichlet boundary conditions $y(-l)=y(l)=0$, we present the approximate eigenvalues of system (44) for different values of l. The obtained eigenvalues are comparable to the exact eigenvalues of the harmonic oscillator: $\lambda_n^\infty=2n+1$ for $n=0$ and $n=1$.

5. CONCLUSION

We presented Chebyshev wavelets operational matrix of integration and product operation matrix. These matrices have been applied to find a solution for Sturm-Liouville problem. The approximate examples used in this paper consequently display the efficiency of the present method. Also, the examples provided and all approximate calculations in the present study have been performed on a *PC*, applying programs written in Mathematica.

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