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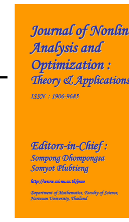

LOCAL CONVERGENCE FOR JARRATT-LIKE ITERATIVE METHODS IN BANACH SPACE UNDER WEAK CONDITIONS

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ABSTRACT. We study the method considered in Sharma and Arora(2014), for solving systems of nonlinear equations, modified suitably to include the nonlinear equations in Banach spaces. Our conditions are weaker than the conditions used in earlier studies. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

KEYWORDS : Jarratt-like method; radius of convergence; local convergence; restricted convergence domains.
AMS Subject Classification:65D10, 65D99, 65J20, 49M15, 74G20, 41A25

1. Introduction

In this study, we consider the problem of approximating the solution x^* of nonlinear equation

$$H(x) = 0 \tag{1.1}$$

where $H : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a continuous differentiable operator in the sense of Fréchet between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ and Ω is a convex set. We consider the following method considered in [15] for increasing the order of convergence of iterative methods to solve (1.1).

$$\begin{aligned} y_n &= x_n - \frac{2}{3}H'(x_n)^{-1}H(x_n) \\ z_n &= x_n - [\frac{23}{8}I - H'(x_n)^{-1}H'(y_n)(3I - \frac{9}{8}H'(x_n))^{-1}H'(y_n)] \\ x_{n+1} &= z_n - (\frac{5}{2}I - \frac{3}{2}H'(x_n)^{-1}H'(y_n))H'(x_n)^{-1}H(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in \Omega$ is an initial point. Let $U(a, \rho) := \{x \in \mathcal{B}_1 : \|x - a\| < \rho\}$ and let $\bar{U}(a, \rho)$ be its closure.

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Due to its wide applications, finding solution for the equation (1.1) and improving the order of convergence of iterative method for solving (1.1) is an important problem in mathematics. In [15] the existence of the Fréchet derivative of H of order up to the sixth was used for the convergence analysis of method (1.2) although only the first derivative appears in the method for the special case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^m$. This assumption on the higher order Fréchet derivatives of the operator H restricts the applicability of method (1.2). For example consider the following:

EXAMPLE 1.1. Let $X = C[0,1], D = \bar{U}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [2, 3, 13] defined by

$$x(s) = \int_0^1 G(s,t)\frac{x(t)^2}{2}dt,$$

where the kernel G is the Green's function defined on the interval $[0,1] \times [0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s \\ s(1-t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\frac{x(t)^2}{2}dt.$$

Notice that

$$\| \int_0^1 G(s,t)dt\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)x(t)dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}\|x-y\|.$$

One can see that, higher order derivatives of F do not exist in this example.

Our goal is to weaken the assumptions in [15] and apply the method for solving equation (1.1) in Banach spaces, so that the applicability of the method (1.2) can be extended. The technique introduced in this study can be applied to other iterative methods [1–17].

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result. Special cases and numerical examples are given in the last section.

2. Local Convergence analysis

The local convergence analysis of method (1.2) that follows is based on some scalar functions and parameters. Let function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and non-decreasing with $w_0(0) = 0$. Define the parameter r_0 by

$$r_0 = \sup\{t \ge 0 : w_0(t) < 1\}. \tag{2.1}$$

Let $w : [0, r_0) \longrightarrow [0, +\infty)$, $v : [0, r_0) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions with $w(0) = 0$. Define functions $g_i, h_i, i = 1, 2$ on the interval $[0, r_0)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 v(\theta t)d\theta}{1 - w_0(t)},$$

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)} \\ &+ \left[\frac{15}{8}\frac{w_0(t)+w_0(g_1(t)t)}{1-w_0(t)} \right. \\ &+ \left. \frac{9}{8}\frac{w_0(t)+w_0(g_1(t)t)\int_0^1 v(\theta g_1(t)t)d\theta}{(1-w_0(t))^2}\right]\frac{\int_0^1 v(\theta t)d\theta}{1-w_0(t)} \end{aligned}$$

and

$$h_i(t) = g_i(t) - 1, i = 1, 2.$$

Suppose that

$$v(0) < 3. \tag{2.2}$$

We have that $h_1(0) = \frac{v(0)}{3} - 1 < 0$(by (2.2) and $h_1(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$, $h_2(0) = -1 < 0$ and $h_2(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$. It follows from the intermediate value theorem that functions h_i have zeros in the interval $(0, r_0)$. Denote by r_i the smallest such zeros of functions h_i, respectively. Define parameter $\bar{r}_0$ by

$$\bar{r}_0 = \max\{t \in [0, r_0] : w_0(g_2(t)t) < 1\}. \tag{2.3}$$

Define functions g_3 and h_3 on the interval $[0, \bar{r}_0)$ by

$$\begin{aligned} g_3(t) &= \left[\frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1-w_0(g_2(t)t)} \right. \\ &+ \frac{(w_0(t)+w_0(g_2(t)t))\int_0^1 v(\theta g_2(t)t)d\theta}{(1-w_0(t))(1-w_0(g_2(t)t))} \\ &+ \left. \frac{3}{2}\frac{(w_0(t)+w_0(g_1(t)t)\int_0^1 v(\theta g_2(t)t)d\theta}{(1-w_0(t))^2}\right] g_2(t) \end{aligned}$$

and

$$h_3(t) = g_3(t) - 1.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \longrightarrow +\infty$ as $t \longrightarrow \bar{r}_0^-$. Denote by r_3 the smallest zero of h_3 in the interval $(0, \bar{r}_0)$. Define the radius of convergence r by

$$r = \min\{r_i\} \;\; i = 1, 2, 3. \tag{2.4}$$

Then, for each $t \in [0, r)$

$$0 \le g_i(t) < 1, \tag{2.5}$$

$$0 \le w_0(t) < 1, \tag{2.6}$$

and

$$0 \le w_0(g_2(t)t) < 1. \tag{2.7}$$

Next, the local convergence analysis of method (1.2) is shown using the preceding notation.

THEOREM 2.1. Let $H : \Omega \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose:

there exist $x^* \in \Omega$ and function $w_0 : [0, +\infty) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w_0(0) = 0$ such that for each $x \in \Omega$

$$H(x^*) = 0, \quad H'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1), \tag{2.8}$$

and

$$\|H'(x^*)^{-1}(H'(x) - H'(x^*)\| \leq w_0(\|x - x^*\|); \tag{2.9}$$

Let $\Omega_0 = \Omega \cap U(x^*, r_0)$. There exist functions $w : [0, r_0) \longrightarrow [0, +\infty), v : [0, r_0) \longrightarrow [0, +\infty)$ continuous and nondecreasing with $w(0) = 0$ such that for each $x, y \in \Omega_0$

$$\|H'(x^*)^{-1}(H'(x) - H'(y)\| \leq w(\|x - y\|), \tag{2.10}$$

$$\|H'(x^*)^{-1}H'(x)\| \leq v(\|x - y\|), \tag{2.11}$$

and

$$\bar{U}(x^*, r) \subseteq \Omega, \tag{2.12}$$

where the convergence radii r_0 and r are given by (2.1) and (2.4), respectively. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (1.2) is well defined in $U(x^*, r)$, stays in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.13}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.15}$$

where the functions $g_i, i = 1, 2, 3$ are defined previously. Furthermore, if there exists $R^* \geq r$ satisfies

$$\int_0^1 w_0(\theta R) d\theta < 1, \tag{2.16}$$

then the limit point x^* is the only solution of equation $H(x) = 0$ in $\Omega_1 = \Omega \cap \bar{U}(x^*, R)$.

Proof. Estimates (2.13)-(2.15) shall be shown using mathematical induction on the integer k Let $x \in U(x^*, r)$. Using (2.4), (2.6), (2.8) and (2.9), we have that

$$\|H'(x^*)^{-1}(H'(x) - H'(x^*)\| \leq w_0(\|x - x^*\|) \leq w_0(r) < 1. \tag{2.17}$$

Hence by (2.17) and the Banach Lemma on invertible operators [2, 4, 7] we get that $H'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|H'(x)^{-1}H'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{2.18}$$

In particular, (2.18) holds for $x = x_0$, since $x_0 \in U(x^*, r)$ and points y_0, z_0 and x_1 are well defined by method (1.2) for $n = 0$. We can write by (2.8) that

$$H(x_0) = H(x_0) - H(x^*) = \int_0^1 H'(x^* + \theta(x_0 - x^*))(x_0 - x^*) d\theta. \tag{2.19}$$

Notice that $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, so $x^* + \theta(x_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$. In view of (2.11) and (2.19), we obtain that

$$\|H'(x^*)^{-1}H'(x_0)\| \leq \int_0^1 v(\theta\|x_0 - x^*\|) d\theta \|x_0 - x^*\|. \tag{2.20}$$

Using (2.4), (2.5) (for $i = 1$), (2.8), (2.10), (2.18), (2.20) and method (1.2), we have in turn that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|x_0 - x^* - H'(x_0)^{-1}H'(x_0)\| + \frac{1}{3}\|H'(x_0)^{-1}H'(x^*)\| \\
&\leq \|H'(x_0)^{-1}H'(x^*)\|\|\int_0^1 H'(x^*)^{-1}(H'(x^*+\theta(x_0-x^*)) - H'(x_0))(x_0-x^*)d\theta\| \\
&\quad + \frac{1}{3}\|H'(x_0)^{-1}H'(x^*)\|\|H'(x^*)^{-1}H(x_0)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)} \\
&\quad + \frac{1}{3}\frac{\int_0^1 v(\theta\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0-x^*\|)} \\
&= g_1(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \qquad (2.21)
\end{aligned}$$

which shows (2.13) for $n = 0$ and $y_0 \in B(x^*, r)$. By the second substep of method (1.2) for $n = 0$, we can write

$$\begin{aligned}
z_0 - x^* &= x_0 - x^* - [\frac{23}{8}I - H'(x_0)^{-1}H'(y_0)(3I - \frac{9}{8}H'(x_0)^{-1}H'(y_0))]H'(x_0)^{-1}H(x_0) \\
&= x_0 - x^* - H'(x_0)H(x_0) + \left[\frac{15}{8}H'(x_0)^{-1}(H'(y_0) - H'(x^*)) + (H'(x^*) - H'(x_0))\right. \\
&\quad \left. + \frac{9}{8}H'(x_0)^{-1}H'(y_0)H'(x_0)^{-1}(H'(x_0) - H'(y_0))\right] H'(x_0)^{-1}H(x_0). \qquad (2.22)
\end{aligned}$$

Using (2.4), (2.5) (for $i = 2$), (2.18), (2.20) (for $y_0 = x_0$), (2.21) and (2.22) we get in turn that

$$\begin{aligned}
\|z_0 - x^*\| &\leq \frac{\int_0^1 w((1-\theta)\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-w_0(\|x_0x-^*\|} \\
&\quad + \left[\frac{15}{8}\frac{w_0(\|x_0-x^*\|)+w_0(\|y_0-x^*\|)}{1-w_0(\|x_0-x^*\|)}\right. \\
&\quad \left. + \frac{9}{8}\frac{(w_0(\|x_0-x^*\|)+w_0(\|y_0-x^*\|))\int_0^1 v(\theta\|y_0-x^*\|)d\theta}{(1-w_0(\|x_0-x^*\|))^2}\right] \\
&\quad \times \frac{\int_0^1 v(\theta\|x_0-x^*\|)d\theta\|x_0x-^*\|}{1-w_0(\|x_0-x^*\|)} \\
&\leq g_2(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \qquad (2.23)
\end{aligned}$$

which shows (2.14) for $n = 0$ and $z_0 \in U(x^*, r)$. Notice also that (2.18) holds for $x = z_0$. Next, by the third substep of method (1.2) for $n = 0$, we can write

$$\begin{aligned}
x_1 - x^* &= z_0 - x^* - (\frac{5}{2}I - \frac{3}{2}H'(x_0)^{-1}H'(y_0))H'(x_0)^{-1}H(z_0) \\
&= z_0 - x^* - H'(z_0)^{-1}H(z_0) + H'(z_0)^{-1}(H'(x_0) - H'(z_0))H'(x_0)^{-1}H(x_0) \\
&\quad + \frac{3}{2}H'(x_0)^{-1}(H'(x_0) - H'(y_0))H'(x_0)^{-1}H(z_0). \qquad (2.24)
\end{aligned}$$

Then, using (2.4), (2.5) (for $i = 3$) (2.18) (for $x = x_0$ and $x = z_0$), (2.21), (2.23) and (2.24), we obtain in turn that

$$\|x_1 - x^*\| \leq \frac{\int_0^1 w((1-\theta)\|z_0-x^*\|)d\theta\|z_0-x^*\|}{1-w_0(\|z_0-x^*\|)}$$

$$
\begin{aligned}
&+\frac{(w_0(\|x_0-x^*\|)+w_0(\|z_0-x^*\|))\int_0^1 v(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\|}{(1-w_0(\|x_0-x^*\|))(1-w_0(\|z_0-x^*\|))}\\
&+\frac{3}{2}\frac{(w_0(\|x_0-x^*\|)+w_0(\|y_0-x^*\|))\int_0^1 v(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\|}{(1-w_0(\|x_0-x^*\|))^2}\\
\leq\ & g_3(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < r, \qquad (2.25)
\end{aligned}
$$

which shows (2.15) for $n=0$ and $x_1 \in U(x^*, r)$. The induction for estimates (2.13)-(2.15) can be finished by replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates. Then, from the estimates

$$\|x_{k+1}-x^*\| \leq c\|x_k-x^*\| < r, \qquad (2.26)$$

where $c = g_3(\|x_0-x^*\|) \in [0,1)$, we conclude that $\lim_{k\longrightarrow\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Finally to show the uniqueness part, let $y^* \in \Omega_2$ with $H(y^*) = 0$. Define the linear operator $T = \int_0^1 H'(x^*+\theta(y^*-x^*))d\theta$. Using (2.8), we obtain that

$$
\begin{aligned}
\|H'(x^*)^{-1}(T-H'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^*-y^*\|)d\theta \\
&\leq \int_0^1 w_0(\theta R^*)d\theta < 1,
\end{aligned} \qquad (2.27)
$$

Hence, we have that $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = H(y^*) - H(x^*) = T(y^*-x^*)$, we conclude that $x^* = y^*$. □

REMARK 2.2. (a) In the case when $w_0(t) = L_0 t, w(t) = Lt$ and $\Omega_0 = \Omega$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (2.7)-(2.9). Notice that the convergence radius for Newton's method given independently by Rheinboldt [14] and Traub [17] is given by

$$\rho = \frac{2}{3L} < r_A.$$

As an example, let us consider the function $H(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = B(0,1)$. Then, we have that $L_0 = e-1 < L = e$, so $\rho = 0.24252961 < r_A = 0.324947231$.

Moreover, the new error bounds [2] are:

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L_0\|x_n-x^*\|}\|x_n-x^*\|^2,$$

whereas the old ones [5, 7]

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L\|x_n-x^*\|}\|x_n-x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r_3 to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2–7].

(c) The results can be also be used to solve equations where the operator H' satisfies the autonomous differential equation [2–4]:

$$H'(x) = P(H(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator. Since $H'(x^*) = P(H(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $H(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [15]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } \; n = 0, 1, 2, \ldots.$$

(e) In view of (2.4) and the estimate

$$\begin{aligned}\|H'(x^*)^{-1}H'(x)\| &= \|H'(x^*)^{-1}(H'(x) - H'(x^*)) + I\| \\ &\leq 1 + \|H'(x^*)^{-1}(H'(x) - H'(x^*))\| \leq 1 + w_0(\|x - x^*\|)\end{aligned}$$

condition (2.6) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t)$$

or

$$v(t) = 1 + w_0(r_0),$$

since $t \in [0, r_0)$.

(f) Let us choose $\alpha = 1$ and $\varphi(x, y) = y - F'(y)^{-1}F(y)$. Then, we have in (2.22) with x_k replaced by y_k

$$\|\varphi(x_k, y_k) - x^*\| \leq \frac{\int_0^1 w((1-\theta)\|y_k - x^*\|)d\theta\|y_k - x^*\|}{1 - w_0(g_1(\|x_k - x^*\|)\|x_k - x^*\|)},$$

so we can choose $p = 1$ and

$$\psi(t) = \frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta g_1(t)}{1 - w_0(t)}.$$

(h) Condition $v(0) < 3$ can be dropped as follows. Define $R_0 = g_1(r)r$ and replace (2.12) by

$$\bar{U}(x^*, R_1) \subseteq D, \tag{2.28}$$

where $R_1 = \max\{R_0, r\}$. We must also replace (2.13) by

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_0 - x^*\| \leq R_1. \tag{2.29}$$

Then, the conclusions of Theorem 2.1 hold in this setting without the restrictive condition $v(0) < 3$.

3. Numerical Examples

We present two examples in this section. We choose $\alpha = 1, p = 1$ and ψ as in Remark 2.2 (g) in both examples.

EXAMPLE 3.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function H on D for $w = (x,y,z)^T$ by

$$H(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$H'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using (2.5)–(2.7), we can choose $w_0(t) = L_0 t, w(t) = e^{\frac{1}{L_0}} t, v(t) = e^{\frac{1}{L_0}}, L_0 = e - 1$.

Then, the radius of convergence r is given by

$$r_1 = 0.1544,\ r_2 = 0.0340,\ r_3 = 0.0128 = r.$$

EXAMPLE 3.2. Returning back to the motivational example given at the introduction of this study, we can choose (see also Remark 2.2 (5) for function v)$w_0(t,s) = w(t,s) = \frac{t+s}{16}$ and $v_0(t) = 1 + w_0(r_0) \backsimeq 1.4142$. Then, the radius of convergence r is given by

$$r_1 = 5.6384,\ r_2 = 0.8761,\ r_3 = 0.7651 = r.$$

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