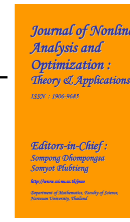
Journal of **N**onlinear **A**nalysis and **O**ptimization
Vol. 7, No. 2, (2016), 1-16
ISSN : 1906-9685
http://www.math.sci.nu.ac.th



OPTIMALITY CONDITIONS FOR WEAKLY EFFICIENT SOLUTION OF VECTOR EQUILIBRIUM PROBLEM WITH CONSTRAINTS IN TERMS OF SECOND-ORDER CONTINGENT DERIVATIVES

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ABSTRACT. In this paper, we present second-order necessary and sufficient optimality conditions for weakly efficient solution of a vector equilibrium problem with constraints (in short, VEPC) in terms of second-order contingent derivative and second-order asymptotic contingent derivative. With this purpose, we impose the objective functions, either all them are twice Fréchet differentiable at optimal point or the Fréchet derivatives are calm at optimal point or the profile mappings has the cone-Aubin properties. Besides, we also can invoke constraint qualifications of the Kurcyusz - Robinson - Zowe (KRZ) type. Our paper point out new improvements from the known results of Gutierrez, Jiménez and Novo (2010) and Khanh and Tung (2015); see [8], [10] in cases of single-valued optimization and give some discusses about it.

KEYWORDS : Second-order optimality conditions; Second-order contingent derivatives; Kurcyusz-Robinson-Zowe type constraint qualification; Weakly efficient solutions.
AMS Subject Classification: 90C29, 90C46, 90C48, 49J52

1. INTRODUCTION

The vector equilibrium problems generalize many well-known problems in the optimization theory as vector complementarity problems, vector saddle point problems, vector optimization problems and vector variational inequality problems. On second-order optimality conditions involving the above problems have been widely investigated by many researchers, see, for instance, Aubin and Frankowska [1]; Jiménez and Novo [2], [6], [7]; Guerraggio et al. [3], [4]; Luu [5]; Gutierrez et al. [8]; Khanh et al. [9], [10]; Clarke [13]; Morgan and Romaniello [15]; Su [16], [17], [18] and the references therein. On using set-valued radial second order directional derivatives, Gutierrez, Jiménez and Novo [8] obtained second order necessary optimality conditions in primal forms through second order derivatives and second order sufficient optimality conditions in dual forms with "envelope- like effect"

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Article history : Received 2 December 2015 Accepted 2 January 2018.

through Fritz John type Lagrange multiplier rule. On using multivalued second order contingent derivatives, Khanh and Tung [10] recived Karush-Kuhn-Tucker (KRZ) second order optimality conditions for the multivalued vector optimization problems with attention to the "envelope- like effect". Luu [5] obtained higher order necessary and sufficient optimality conditions for strict local pareto minima via the higher order Studniarski's derivatives. In case the functions considered in $C^{1,1}$, the authors Guerraggio and Luc [2], [3] have established optimality conditions for vector optimization problems in terms of the second-order contingent cones. Khanh and Tuan [9] have dealt with necessary and sufficient optimality conditions for both weak efficiency and firm efficiency in multivalued optimization problems in terms of the second order Hadamard directional derivatives.

We can apply the obtained result in Khanh and Tung [10] to the (local) weak efficiency of vector equilibrium problem with constraints. Therefore, we need discuss to improve the obtained result in [10] for (local) weakly efficient solution in case of single-valued is very necessary. Motivated by this arguments, in this papers we consider the vector equilibrium problem with constraints VEPC with datas are single-valued, in which Fréchet derivatives at an optimal point are calm at that point or objective functions are twice Fréchet differentiable at optimal point that. We use the second-order contingent derivatives for functions to establish the necessary and sufficient optimality conditions for (local) weakly efficient solution to the problem VEPC with attention to the "envelope-like effect".

In this article, the following vector equilibrium problem with constraints VEPC is considered: let X, Y and Z be real Banach spaces, C be nonempty subset of X, $Q \subset Y$ be a closed convex cone with its interior nonempty, which defines a partial order on Y, where cone Q is not necessarily pointed, and let $S \subset Z$ be a convex cone in Z. Given a bifunction $F : X \times X \to Y$ and a constraints function $g : X \to Z$ such that $F(x, x) = 0 \;\; \forall x \in X$. Our problem here is finding $\overline{x} \in K$ satisfying

$$F(\overline{x}, x) \notin -intQ \quad \forall x \in K, \tag{VEPC}$$

where

$$K = \{x \in C \, : \, g(x) \in -S\}$$

is called a feasible set of problem VEPC. A vector $\overline{x}$ solved (VEPC) is called a weakly efficient solution of VEPC. If there exists a neighborhood U of $\overline{x}$ such that

$$F(\overline{x}, x) \notin -intQ \quad \forall x \in K \cap U,$$

then vector $\overline{x}$ is called a local weakly efficient solution of VEPC. If $\overline{x}$ is a local weakly efficient solution or a weakly efficient solution of VEPC then we write $\overline{x}$ is a (local) weakly efficient solution of VEPC.

The remainder of this paper is organized as follows. After some preliminaries and definitions, Sect. 3 deals with the second-order necessary optimality conditions for efficient solutions of problem VEPC in terms of contingent derivatives. Besides, we also give some discusses. In Sect. 4, we present the second-order sufficient optimality conditions using Fritz John type Lagrange multiplier rule for efficient solutions of problem VEPC.

2. Preliminaries and Definitions

From now on, if not otherwise stated, $X = \mathbb{R}^n, Y = \mathbb{R}^m$ and $Z = \mathbb{R}^p$, where $\mathbb{R}$ ($\mathbb{N}$, respectively) denotes the space of all real (natural, respectively) numbers. For $A \subset X$, as usual, we denote $intA, clA$ and bdA instead of the interior, closure and boundary of A, respectively. The cone generated by A is given as $coneA = \{ta \, : \, t \geq$

$0,\ a \in A\}$. The dual cone of A is defined as $A^+ = \{\xi \in X^* \mid \langle \xi, q\rangle \geq 0 \quad \forall\, a \in Q\,\}$, where $\langle\, .\, ,\, .\,\rangle$ denotes the coupling between the space X and the dual space X^* of X, B_X denotes the open unit ball of X, and $B_X(x,\delta)$ denotes the open ball centered at $x \in X$ and radius $\delta > 0$ (similarly for other spaces). We also denote by $S_{g(\overline{x})} := cone(S + g(\overline{x}))$, where $\overline{x} \in X$ such that $g(\overline{x}) \in -S$. Then it is not difficult to see that $S^+_{g(\overline{x})} = \{\eta \in S^+ \,:\, \langle \xi, g(\overline{x})\rangle = 0\}$. Let $F : X \to 2^Y$ be a set-valued mapping from X into 2^Y, where 2^Y indicates the family of all subsets of Y. The effective domain, graph and epigraph of a set-valued mapping F are given respectively as

$$\begin{aligned}
&\mathrm{dom}F = \{x \in X \,|\, F(x) \neq \emptyset\},\\
&\mathrm{graph}F = \{(x,y) \in X \times Y \,|\, y \in F(x)\},\\
&\mathrm{epi}F = \{(x,y) \in X \times Y \,:\, x \in \mathrm{dom}F,\ y \in F(x) + Q\}.
\end{aligned}$$

We denote by $F(A) = \bigcup\limits_{a \in A} F(a)$ and the profile mapping $F_+ : X \to 2^Y$ is defined as $F_+(x) = F(x) + Q$ $(\forall\, x \in X)$. If F is a single-valued mapping then we write f_+ instead of $f + Q$. Let the mappings $f : X \to Y,\ g : X \to Z$, let us define a new profile mapping $(f_+, g_+)(x) = (f(x)+Q)\times(g(x)+S)$ $(\forall\, x \in X)$, where Z is partially ordered by S, and moreover $(f,g)(x) = (f(x), g(x))$ $(\forall\, x \in X)$. Recall (see [10]) that f_+ is said to be $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ if and only if there exists neighborhoods U of $\overline{x}$, V of $f(\overline{x})$, and $L > 0$ satisfying

$$(f_+(x) \cap V) \subset f_+(x') + Q + L\|x - x'\|\, clB_Y \quad \forall\, x, x' \in U.$$

The profile mapping g_+ has property $S-$ Aubin at $(\overline{x}, g(\overline{x}))$ is similarly defined.

Next, let us provide the definitions about the contingent sets, which will be needed in this paper

Definition 2.1. ([8, 10, 17]) Let M be a subset of X and let $\overline{x}, u \in X$.
(i) The contingent cone (resp., adjacent cone and interior tangent cone) of M at $\overline{x}$ is

$$T(M,\overline{x}) = \{x \in X : \exists\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n x_n \in M\ \forall\ n \in \mathbb{N}\},$$

$$\Big(resp., A(M,\overline{x}) = \{x \in X : \forall\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n x_n \in M\ \forall\ n \in \mathbb{N}\},$$

$$IT(M,\overline{x}) = \{x \in X : \forall\, t_n \to 0^+, \forall x_n \to x \text{ such that } \overline{x} + t_n x_n \in M\ \forall n \text{ large}\}\Big).$$

(ii) The second-order contingent set (resp., adjacent set and interior tangent set) of M at $\overline{x}$ in direction u is

$$T^2(M,\overline{x},u) = \{x \in X : \exists\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x}+t_n u+\frac{1}{2}t_n^2 x_n \in M\ \forall\ n \in \mathbb{N}\},$$

$$\Big(resp., A^2(M,\overline{x},u) = \{x \in X : \forall\, t_n \to 0^+, \exists x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in M\ \forall\ n \in \mathbb{N}\},$$

$$IT^2(M,\overline{x},u) = \{x \in X : \forall\, t_n \to 0^+, \forall x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in M\ \forall n \text{ large}\}\Big).$$

(iii) The asymptotic second-order contingent set of M at $\overline{x}$ in direction u is

$$T^{''}(M,\overline{x},u) = \{x \in X : \exists\,(t_n, r_n) \to (0^+,0^+),\ \frac{t_n}{r_n} \to 0^+,\ \exists x_n \to x \text{ such that } \overline{x} + t_n u + \frac{1}{2}t_n r_n x_n \in M\ \forall\ n \in \mathbb{N}\}.$$

We say that $M \subset X$ is a second-order derivative at $(\overline{x}, u)$ if and only if

$$T^2(M, \overline{x}, u) = A^2(M, \overline{x}, u).$$

It is well known that if $\overline{x} \notin clM$ then all the above tangent sets are null. Moreover, if $u \notin T(M, \overline{x})$ then all the above second-order tangent sets are null.

Proposition 2.2. *([12], Proposition 2.2) Let $M \subset X$ be a convex set, $\overline{x} \in M$ and $v \in T(M, \overline{x})$, then*

$$T^2(M, \overline{x}, v) + T(T(M, \overline{x}), v) \subset T^2(M, \overline{x}, v). \tag{2.1}$$

Additionally, if $0 \in T^2(M, \overline{x}, v)$ (in particular, when M is polyhedral), then

$$T^2(M, \overline{x}, v) = T(T(M, \overline{x}), v).$$

We suppose that $A^2(M, \overline{x}, v) \neq \emptyset$. As X is reflexible space, making use of Proposition 2.1 (iv) [10], it follows that $clIT^2(M, \overline{x}, v) = A^2(M, \overline{x}, v)$ and

$$A^2(M, \overline{x}, v) + T(T(M, \overline{x}), v) \subset A^2(M, \overline{x}, v). \tag{2.2}$$

Definition 2.3. ([1, 17]) Let $f : X \to Y$ be a single-valued mapping and let $\overline{x} \in X$, $(u, v) \in X \times Y$.
(i) The contingent derivative of f (resp., f_+) at a point $\overline{x}$ is defined as

$$\text{graph}\Big(D_c f(\overline{x})\Big) = T(\text{graph}(f), (\overline{x}, f(\overline{x})))$$
$$\big(resp., \text{graph}\Big(D_c(f_+)(\overline{x}, f(\overline{x}))\Big) = T(\text{epi}(f), (\overline{x}, f(\overline{x}))\big).$$

(ii) The second-order contingent derivative of f (resp., f_+) at a point $\overline{x}$ in direction (u, v) is defined as

$$\text{graph}\Big(D_c^2 f(\overline{x}, f(\overline{x}), u, v)\Big) = T^2(\text{graph}(f), (\overline{x}, f(\overline{x})), (u, v))$$
$$\big(resp., \text{graph}\Big(D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)\Big) = T^2(\text{epi}(f), (\overline{x}, f(\overline{x})), (u, v))\big).$$

The second-order adjacent derivatives $D_c^{b,2} f(\overline{x}, f(\overline{x}), u, v)$ and $D_c^{b,2}(f_+)(\overline{x}, f(\overline{x}), u, v)$ are similar, with A^2 replacing T^2.
(iii) The second-order asymptotic contingent derivative of f (resp., f_+) at a point $\overline{x}$ in direction (u, v) is defined as

$$\text{graph}\Big(D_c^{''} f(\overline{x}, f(\overline{x}), u, v)\Big) = T^{''}(\text{graph}(f), (\overline{x}, f(\overline{x})), (u, v))$$
$$\big(resp., \text{graph}\Big(D_c^{''}(f_+)(\overline{x}, f(\overline{x}), u, v)\Big) = T^{''}(\text{epi}(f), (\overline{x}, f(\overline{x})), (u, v))\big).$$

By definitions, it can easily be seen that for each $(u, v, w) \in X \times Y \times Z$, we obtain

$$D_c^2 f(\overline{x}, f(\overline{x}), u, v)(x) + Q \subset D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \quad \forall x \in X, \tag{2.3}$$

which means that

$$dom\big(D_c^2 f(\overline{x}, f(\overline{x}), u, v)\big) \subset dom\big(D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)\big).$$

The case of the profile map (f_+, g_+), one has the following inclusion holds for all $x \in X$

$$\begin{aligned} D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x) \subset & D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \\ & \times D_c^2(g_+)(\overline{x}, g(\overline{x}), u, w)(x). \end{aligned} \tag{2.4}$$

Definition 2.4. ([14]) A mapping $f : X \longrightarrow Y$ is said to be m-calm at $\overline{x}$ if there exist a neighborhood U of $\overline{x}$ and $L > 0$ such that

$$\|f(x) - f(\overline{x})\| \leq L\|x - \overline{x}\|^m \quad \forall\, x \in U,$$

where $\| \,.\, \|$ denotes a norm in Banach spaces.

Of course, if f is m-calm then f is continuous at that point. When $m = 1$, $1-$ calmness is called simply calmness. If f is 1-calm then it also can be called "calm" or "stable" (see [2]). If, in addition, $||f(x) - f(x')|| \leq L||x - x'|| \quad \forall\, x, x' \in U$, then we say that f is Lipschitz around $\overline{x}$. If for each $\overline{x} \in X$ there exists a neighborhood U of $\overline{x}$ such that f is Lipschitz around $\overline{x}$, we will say that f is locally Lipschitz on X. If f is Lipschitz around $\overline{x}$, making use of the obtained result in [2], then one gets f is steady at $\overline{x}$, which yields that f is calm at $\overline{x}$.

Finally, let us denote by $t_n \to 0^+$ instead of a sequence of positive numbers with limit 0 and for each $\overline{x} \in K$, the mapping $f = F_{\overline{x}} : X \to Y$.

Proposition 2.5. *([9], Lemma 2.3) Let $\overline{x} \in X$ and assume, in addition, that S closed convex. If $g(\overline{x}) \in -S$ and $\lim\limits_{t_n \longrightarrow 0^+} \frac{z_n - g(\overline{x})}{t_n} \in -int\, S_{g(\overline{x})}$ then $z_n \in -S$ for n large enough.*

It is not hard to see that Proposition 2.5 is still holds if the closedness of cone S is deleted.

3. Second-order Necessary Optimality Conditions

In this subsection, we establish some second-order necessary optimality conditions in dual and primal forms for (local) weakly efficient solution of VEPC in terms of second-order contingent (or adjacent) derivatives.

Proposition 3.1. *Let $\overline{x} \in C$ be a (local) weakly efficient solution to the problem VEPC. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (-S)$, we have*

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x) \cap IT(-Q, v) \times -int(S_{g(\overline{x})}) = \emptyset \tag{3.1}$$

for all $x \in IT^2(C, \overline{x}, u)$. Furthermore:

(i) If, in addition, (f, g) has Fréchet derivative $(\triangledown f(\overline{x}), \triangledown g(\overline{x}))$ which is stable at $\overline{x}$ then for all $x \in A^2(C, \overline{x}, v)$, (3.1) is fulfilled.

(ii) If, in addition, f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and one of the following two conditions is satisfied

(I) g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$;

(II) g satisfies (i).

Then one has

$$D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x) \\ \cap IT(-Q, v) \times -int(S_{g(\overline{x})}) = \emptyset \;\; \forall\, x \in A^2(C, \overline{x}, u). \tag{3.2}$$

Proof. We fixed $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u)$. Assume, to the contrary, that there exists $x \in IT^2(C, \overline{x}, u)$ such that the left-hand side of (3.1) is nonempty. On finds $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$ such that $y \in IT(-Q, v)$ and $z \in -int(S_{g(\overline{x})})$. By definitions it holds that

$$(x, y, z) \in T^2(epi(f, g), (\overline{x}, (f, g)(\overline{x})), u, (v, w)),$$

which is equivalent to $\exists t_n \to 0^+$, $\exists x_n \to x$ and $\exists (y_n, z_n) \to (y, z)$ satisfying

$$f(\overline{x}) + t_n v + \frac{1}{2}t_n^2 y_n \in f_+(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n),$$

$$g(\overline{x}) + t_n w + \frac{1}{2}t_n^2 z_n \in g_+(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n).$$

Taking $s_n \in S$ and $z'_n := g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) + s_n - t_n w$ such that $z_n = \frac{z'_n - g(\overline{x})}{\frac{1}{2}t_n^2} \to z \in -int(S_{g(\overline{x})})$ as $2^{-1}t_n^2 \to 0^+$. Making use of Proposition 2.5, we get $z'_n \in -S$ for n sufficiently large, which yields that $g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) \in -S$ for n sufficiently large. Note that $x \in IT^2(C, \overline{x}, u)$, it implies that

$$\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n \in K \quad \text{for n large enough.} \tag{3.3}$$

By a similar argument as in the proof of Proposition 3.1 [10], we deduce that

$$f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) \in -intQ \quad \text{for n large enough.} \tag{3.4}$$

Combining (3.3)-(3.4), it yields that $\overline{x} \in K$ is not a (local) weakly efficient solution for VEPC. In view of the initial assumptions, it follows that (3.1) holds for all $x \in IT^2(C, \overline{x}, u)$.

Case (i). If, in addition, (f, g) has Fréchet derivative $(\nabla f(\overline{x}), \nabla g(\overline{x}))$ which is stable at $\overline{x}$, then by an argument analogous to that used for the above results, with A^2 replacing IT^2, we deduce that, for the preceding sequence t_n, there exists $x'_n \to x$ such that

$$\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n \in C \quad \forall n \in \mathbb{N}.$$

We put $z''_n := g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n) + s_n - t_n w$. According to the initial assumption it follows that $\nabla g(\overline{x})$ is continuous at $\overline{x}$ and hence g Lipschitz around $\overline{x}$. Then there exists $L_g > 0$ such that

$$\begin{aligned}
\lim_{n \longrightarrow \infty} \left\| \frac{z'_n - g(\overline{x})}{2^{-1}t_n^2} - \frac{z''_n - g(\overline{x})}{2^{-1}t_n^2} \right\| &= \lim_{n \longrightarrow \infty} \left\| \frac{g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) - g(\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n)}{2^{-1}t_n^2} \right\| \\
&\leq \lim_{n \longrightarrow \infty} L_g \left\| \frac{(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) - (\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n)}{2^{-1}t_n^2} \right\| \\
&= \lim_{n \longrightarrow \infty} L_g \|x_n - x'_n\| \\
&\leq \lim_{n \longrightarrow \infty} L_g (\|x_n - x\| + \|x - x'_n\|) = 0.
\end{aligned}$$

Hence, $\frac{z''_n - g(\overline{x})}{2^{-1}t_n^2} \longrightarrow z \in -int(S_{g(\overline{x})})$, since $\frac{z'_n - g(\overline{x})}{2^{-1}t_n^2} \longrightarrow z$. A consequence is

$$\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n \in K \quad \text{for n large enough.} \tag{3.5}$$

Again choosing sequence $(q_n)_{n \geq 1} \subset Q$ such that

$$y_n = \frac{f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x_n) + q_n - t_n v}{2^{-1}t_n^2} \longrightarrow y \in IT(-Q, v).$$

By repeating the above proofs, with f replacing g, we obtain as follows (note that $IT(-Q, v) = IT(-intQ, v)$)

$$y'_n := \frac{f(\overline{x} + t_n u + \frac{1}{2}t_n^2 x'_n) + q_n - t_n v}{2^{-1}t_n^2} \longrightarrow y \in IT(-intQ, v).$$

Therefore for n large enough,

$$t_n v + \frac{1}{2} t_n^2 y_n' \in -intQ,$$

which implies that

$$f(\overline{x} + t_n u + \frac{1}{2} t_n^2 x_n') \in -intQ \quad \text{for sufficiently large n.}$$

This together with (3.5), it yields that $\overline{x}$ is not a (local) weakly efficient solution to the VEPC. From here we conclude that (3.2) holds for all $x \in A^2(C, \overline{x}, u)$.

Case (ii). Let us denote by $M = \{x \in X \,:\, g(x) \in -S\}$, and assume that f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$. Then for every $x \in A^2(C, \overline{x}, u)$, two cases can occur as follows:

Case 1. $x \in IT^2(M, \overline{x}, u)$ then by direct using result (I) of Proposition 3.2 ([10], p. 74), with $\{f\}$ replacing F, yields that $D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \cap IT(-Q, v) = \emptyset$. Thus condition (3.2) is valid.

Case 2. $x \notin IT^2(M, \overline{x}, u)$ and let us may be assumed to the contrary that there exists $y \in D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$ sao cho $y \in -int(S_{g(\overline{x})})$. By definitions, we have $(x, y) \in A^2(epi(g), (\overline{x}, g(\overline{x})), u, w)$ and this leads to the following result: $\forall\, t_n \to 0^+$, $\exists\, (x_n, y_n) \to (x, y)$ and $\exists\, s_n \in S$ such that for all $n \geq 1$,

$$g(\overline{x}) + t_n w + \frac{1}{2} t_n^2 y_n \in g_+(\overline{x} + t_n w + \frac{1}{2} t_n^2 x_n).$$

As g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$, thus $\exists U, \exists V$ (neighborhoods of $\overline{x}$ and $g(\overline{x})$, resp.,) and $\exists L > 0$ such that

$$g_+(x) \cap V \subset g(x') + L||x - x'||cl\, B_Z + S \quad \forall\, x, x' \in U.$$

For every $x_n' \longrightarrow x$, there exists $N > 0$ such that, for all $n \geq N$, $\overline{x} + t_n u + \frac{1}{2} t_n^2 x_n, \overline{x} + t_n u + \frac{1}{2} t_n^2 x_n' \in U$, $g(\overline{x}) + t_n w + \frac{1}{2} t_n^2 y_n \in V$, and moreover

$$g_+(\overline{x} + t_n u + \frac{1}{2} t_n^2 x_n) \cap V \subset g(\overline{x} + t_n u + \frac{1}{2} t_n^2 x_n') + \frac{1}{2} t_n^2 ||x_n - x_n'||cl\, B_Z + S.$$

Consequently, for all $n \geq N$,

$$g(\overline{x}) + t_n w + \frac{1}{2} t_n^2 y_n \in g(\overline{x} + t_n w + \frac{1}{2} t_n^2 x_n') + \frac{1}{2} t_n^2 ||x_n - x_n'||cl\, B_Z + S,$$

which is equivalent to, for sufficiently large n, there exists $b_n \in clB_Z$ and $s_n' \in S$ such that

$$y_n - \|x_n - x_n'\| b_n = \frac{g(\overline{x} + t_n w + \frac{1}{2} t_n^2 x_n') + s_n - t_n w - g(\overline{x})}{2^{-1} t_n^2} \longrightarrow y.$$

By repeating the preceding proofs, we obtain

$$g(\overline{x} + t_n w + \frac{1}{2} t_n^2 x_n') \in -S \quad \text{for n large enough,}$$

or for large n, $\overline{x} + t_n w + \frac{1}{2} t_n^2 x_n' \in M$. By the definition of IT^2, $x \in IT^2(M, \overline{x}, u)$, and this is a contradiction. Finally, we consider g has Fréchet differentiable $\nabla g(\overline{x})$ which is stable at $\overline{x}$, then by repeating the proof of cases (i) and (ii), we arrive at the contradiction.

As was to be shown. □

Proposition 3.2. *Consider problem VEPC with X, Y and Z are real Banach spaces and $\overline{x} \in K$ is a (local) weakly efficient solution of VEPC. Assume, in addition, that (f, g) has Fréchet derivative $(\nabla f(\overline{x}), \nabla g(\overline{x}))$ which is stable at $\overline{x}$. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (-cl\, S_{g(\overline{x})})$, we have*

$$D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v, w))(x) \cap IT(-Q, v) \times IT^2(-S, g(\overline{x}), w) = \emptyset$$
$$\forall\ x \in A^2(C, \overline{x}, u).$$

Proof. Repeat the proof of Proposition 3.1 in the case (i), with $IT^2(-S, g(\overline{x}), w)$ replacing $-int(S_{\overline{x}})$ and $A^2(C, \overline{x}, u)$ replacing $IT^2(C, \overline{x}, u)$, we conclude.

As was to be shown. □

Note 3.3. (i) Proposition 3.1 is still true if X and Y are real Banach spaces. Because in the proof we only use to the result of Proposition 2.5 in sense that Z is an finite-dimensional space. Moreover, if we replace the profile mapping (f_+, g_+) with a single-valued mapping (f, g), then the statements in Propositions 3.1, 3.1 (i) and 3.2 are still not changed.

(ii) If both f and g are twice Fréchet differentiable at $\overline{x}$, then (i) in Proposition 3.1 and the statement in Proposition 3.2 are still valid. Since in this case $\nabla f(\overline{x})$ and $\nabla g(\overline{x})$ are stable at $\overline{x}$.

(iii) If f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$ then by making use of Proposition 3.2 of Khanh (2015) et al. ([10], p. 74), we deduce that for all $x \in A^2(C, \overline{x}, u)$,

$$D_c^2(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D_c^{b,2}(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$$
$$\cap\, IT(-Q, v) \times IT^2(-S, g(\overline{x}), w) = \emptyset.$$

In particular, under suitable assumptions, with $u = 0, v = 0, w = 0$, we obtain the first-order necessary optimality conditions in the primal forms respectively as follows:

$$D_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}))(x) \cap \big(-int(Q \times S_{g(\overline{x})})\big) = \emptyset;$$

$$D_c(f_+)(\overline{x}, f(\overline{x}))(x) \times D_c(g_+)(\overline{x}, g(\overline{x}))(x) \cap \big(-int(Q \times S_{g(\overline{x})})\big) = \emptyset.$$

Notice that in many well known second-order necessary conditions, such a critical direction w is not mentioned. For example, w is only in $-cone(S + g(\overline{x}))$; see [9], [10], etc.

(iv) Since $IT^2(C, \overline{x}, u) \subset A^2(C, \overline{x}, u)$, hence Proposition 3.2 improves Proposition 3.1 (i) in ([10], p. 73) in the case of single-valued optimization. It should be noted here that $-int(S_{g(\overline{x})})$ and $IT^2(-S, g(\overline{x}), w)$ in Propositions 3.1 and 3.2 play an important role in establishing necessary optimality conditions in the dual form, see, for instance, Theorems 3.1-3.3 below.

Theorem 3.1. *Consider problem VEPC with X, Y and Z are real Banach spaces and $\overline{x} \in K$ is a (local) weakly efficient solution of VEPC. Assume, in addition, that (f, g) has Fréchet derivative $(\nabla f(\overline{x}), \nabla g(\overline{x}))$ which is stable at $\overline{x}$. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (P)$, the following assertions are holds:*

(i) If $P = -S$ and suppose, furthermore, that $dim(Z) < +\infty$ then for all $x \in A^2(C, \overline{x}, u)$ and for all $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v, w))(x)$, there exist $(\lambda, \eta) \in$

$Q^+ \times N(-S, g(\overline{x})) \setminus \{(0,0)\}$ *satisfying*

$$\langle \lambda, v \rangle = 0;$$

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \ \geq \ 0.$$

(ii) If $P = -cl\, S_{g(\overline{x})}$ *and* Z *is reflexible Banach space, then for all* $x \in A^2(C, \overline{x}, u)$ *and for all* $(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$, *there exist* $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x})) \setminus \{(0,0)\}$ *satisfying*

$$\langle \lambda, v \rangle = \langle \eta, w \rangle = 0;$$

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \ \geq \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a \rangle .$$

Proof. (i) We observe Proposition 3.1 (i) see that $\forall\ x \in A^2(C, \overline{x}, u)$,

$$D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x) \cap IT(-Q, v) \times -int(S_{g(\overline{x})}) = \emptyset.$$

Therefore, for all $(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$, it yields that

$$(y,z) \notin IT(-Q, v) \times -int(S_{g(\overline{x})}).$$

By the standart separation theorem, one finds $(\lambda, \eta) \in (Y \times Z)^* \setminus \{0\}$ satisfying

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \geq \langle \lambda, a \rangle + \langle \eta, b \rangle \quad \forall\, a \in IT(-Q, v)\ \forall\, b \in -int(S_{g(\overline{x})}). \tag{3.6}$$

It can be seen that $\langle \lambda, a \rangle \leq 0$ for all $a \in IT(-Q, v) = -int(Q_v)$. Since λ is a continuous linear mapping on Y, hence $\langle \lambda, a \rangle \leq 0$ for all $a \in -cl\, int(Q_v) = -cl\, cone(Q+v)$. This leads to $\lambda \in Q^+$ and $\langle \lambda, v \rangle = 0$. Similarly, one obtains $\eta \in N(-S, g(\overline{x}))$. Again taking the closures of $IT(-Q, v)$ and $-int(S_{g(\overline{x})})$ in (3.6), and then taking $a = b = 0$, we obtain the result.

(ii) In the similar way as above, one obtains the following inequality

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \geq \langle \lambda, a \rangle + \langle \eta, b \rangle \quad \forall\, a \in IT(-Q, v)\ \forall\, b \in IT^2(-S, g(\overline{x}), w),$$

which yields that $\lambda \in Q^+,\ \langle \lambda, v \rangle = 0$. By repeating the proof in Theorem 3.1 of Khanh (2015) et al. ([10], p. 78) and obtain the remains results, and the claim follows. □

An example is provided to illustrate for the obtained results, which can be stated as follows.

Example 3.4. Let $X = Y = \mathbb{R}^2, Z = \mathbb{R}$, $C = Q = \mathbb{R}^2_+, S = \mathbb{R}_+$, $\overline{x} = (0,0)$, and the mappings f, g be given respectively as

$f(x,y) = (y^2 - x^2, y - x)$ for all $(x,y) \in X.$,

$g(x,y) = x - y$ for all $(x,y) \in X$.

Then the feasible set of VEPC is $K = \{(x,y) \ : \ y \geq x \geq 0\}$. It is clear to verify that $\overline{x} = (0,0)$ is a weakly efficient solution to the VEPC. One gets (f,g) is Fréchet differentiable at $\overline{x}$ and its Fréchet derivatives $\left(\nabla f(\overline{x}) = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \nabla g(\overline{x}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$, which does stable at $\overline{x}$. For every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (-S)$, by directly calculating, we obtain the result $u = (a,a), v = w = 0$ for all $a \in \mathbb{R}$. Two cases can occur as follows:

Case 1. Consider $a = 0$ and this implies that $A^2(C, \overline{x}, u) = \mathbb{R}^2_+$. For all $x \in \mathbb{R}^2_+$, for all $(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$, it follows that $y_1 \geq 0$, $y_2 \geq x_2 - x_1$ and $z \geq x_1 - x_2$. We pick $(\lambda, \eta) = ((1,0),0) \in Q^+ \times N(-S, g(\overline{x})) \setminus \{0\}$, then $\langle \lambda, y \rangle + \langle \eta, z \rangle \geq y_1 \geq 0$.

Case 2. Consider $a \neq 0$ and this implies that $A^2(C, \overline{x}, u) = \mathbb{R}^2$. For all $x \in \mathbb{R}^2$, for all $(y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$, it follows that $y_1 \geq 0$, $y_2 \geq x_2 - x_1$ and $z \geq x_1 - x_2$. Let (λ, η) be given as in case 1 and the desired conclusion follows.

As $g(\overline{x}) = 0$ thus $-cl(S_{g(\overline{x})}) = -S$. $dim(Z) = 1$ yields Z is reflexible. In this sense, we have $A^2(-S, g(\overline{x}), w) = -S$ and $\eta \in N(-S, g(\overline{x}))$ yields $\eta \in S^+$ and this leads to

$$\sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a \rangle \leq 0.$$

From the assumption (i) it leads to (ii) be completely checked.

Theorem 3.2. *Under the assumptions of Theorem 3.1 and assume, furthermore, that there exist $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (P)$ such that $D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}, u))$ is a convex set. Then the following assertions hold:*

(i) If $P = -S$ and $dim(Z) < +\infty$ then there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0, 0)$ satisfying $\langle \lambda, v \rangle = 0$ and

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \; \geq \; 0$$

$$\forall \, x \in A^2(C, \overline{x}, u), \quad \forall \, (y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x).$$

Moreover, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{ z \in Z \, : \, (y, z) \in cone\big(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}, u)) \big) \Big\} + S_{g(\overline{x})} = Z.$$

(ii) If $P = -cl\, S_{g(\overline{x})}$ and Z is reflexible Banach space, then there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0, 0)$ satisfying $\langle \lambda, v \rangle = \langle \eta, w \rangle = 0$ and

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \; \geq \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a \rangle \, ,$$

$$\forall \, x \in A^2(C, \overline{x}, u), \quad \forall \, (y, z) \in D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x).$$

Moreover, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{ z \in Z \, : \, (y, z) \in cone\big(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}, u)) - \{0\} \times A^2(-S, g(\overline{x}), w) \big) \Big\} \, + \, S_{g(\overline{x})} = Z.$$

Proof. (i) By taking into account Proposition 3.1 (i), we get

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(A^2(C, \overline{x}), u) \cap \big(- int(Q_v) \big) \times \big(- int(S_{g(\overline{x})}) \big) = \emptyset.$$

By using a separation theorem, one finds $(\lambda, \eta) \in (Y \times Z)^* \setminus \{0\}$ such that

$$\inf_{(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}), u)} \Big(\langle \lambda, y \rangle + \langle \eta, z \rangle \Big) \geq \sup_{(a,b) \in \big(-int(Q_v) \big) \times \big(-int(S_{g(\overline{x})}) \big)} \Big(\langle \lambda, a \rangle + \langle \eta, b \rangle \Big).$$

Similar to the proof of Theorem 3.1 (i), we ensure that $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x})) \setminus \{(0, 0)\}$ satisfying $\langle \lambda, v \rangle = 0$ and

$$\inf_{(y,z) \in D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}), u)} \Big(\langle \lambda, y \rangle + \langle \eta, z \rangle \Big) \geq 0,$$

which leads to a conclusion.

Let us next show that $\lambda \neq 0$ under the qualification condition of the KRZ type. In fact, if it were not so, then we have $\eta \in N(-S, g(\overline{x})) \setminus \{0\}$ and moreover

$$\langle \eta, z\rangle \geq 0 \quad \forall\, (y,z) \in cone\big(D^2_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}, u))\big).$$

It is not difficult to see that

$$\langle \eta, z_0\rangle \geq 0 \quad \forall\, z_0 \in Z.$$

A consequence is $\eta = 0$ and this is a contradiction. So, we have shown that $\lambda \neq 0$.

(ii) It is processed similar as in the proof of (i), and the claim follows. □

Theorem 3.3. *Let $X, Y, Z, C, K, \overline{x}$ and f be given as in Theorem 3.1. Then, for every $u \in X$, $v \in D_c(f_+)(\overline{x}, f(\overline{x}))(u) \cap (-bdQ)$, and $w \in D_c(g_+)(\overline{x}, g(\overline{x}))(u) \cap (P)$, the following assertions are holds:*
(i) If $P = -S$ and suppose, furthermore, that $dim(Z) < +\infty$, f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$ and (I) or (II) in Proposition 3.1 (ii) is fulfilled, then for all $x \in A^2(C, \overline{x}, u)$ and for all $(y,z) \in D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D^{b,2}_c(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$, there exist $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = 0 \;\; \textit{and} \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \;\geq\; 0.$$

In particular, for (u,v,w) such that

$$\Big(D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v), D^{b,2}_c(g_+)(\overline{x}, g(\overline{x}), u, w)\Big)(A^2(C, \overline{x}), u)$$

is convex, there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = 0 \;\; \textit{and} \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \;\geq\; 0$$

for all (x,y,z) mentioned in (i) above.

Furthermore, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{z \in Z \,:\, (y,z) \in cone\big(D^2_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(A^2(C, \overline{x}, u))\big)\Big\} + S_{g(\overline{x})} = Z.$$

(ii) If $P = -cl\, S_{g(\overline{x})}$ and suppose, furthermore, that Z is reflexible Banach space, f_+ is $Q-$ Aubin at $(\overline{x}, f(\overline{x}))$, g_+ is $S-$ Aubin at $(\overline{x}, g(\overline{x}))$, then for all $x \in A^2(C, \overline{x}, u)$ and for all $(y,z) \in D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v)(x) \times D^{b,2}_c(g_+)(\overline{x}, g(\overline{x}), u, w)(x)$, there exist $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = \langle \eta, w\rangle = 0 \;\; \textit{and} \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \;\geq\; \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle\,.$$

In particular, for (u,v,w) such that

$$\Big(D^2_c(f_+)(\overline{x}, f(\overline{x}), u, v), D^{b,2}_c(g_+)(\overline{x}, g(\overline{x}), u, w)\Big)(A^2(C, \overline{x}), u)$$

is convex, there exists a common $(\lambda, \eta) \in Q^+ \times N(-S, g(\overline{x}))$ with $(\lambda, \eta) \neq (0,0)$ satisfying

$$\langle \lambda, v\rangle = \langle \eta, w\rangle = 0 \;\; \textit{and} \;\; \langle \lambda, y\rangle + \langle \eta, z\rangle \;\geq\; \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle$$

for all (x,y,z) mentioned in (ii) above.

Furthermore, $\lambda \neq 0$ if the following qualification condition of the KRZ type is satisfied:

$$\Big\{z \in Z \ : \ (y,z) \in cone\big(D_c^2(f_+,g_+)(\overline{x},(f,g)(\overline{x}),u,(v,w))(A^2(C,\overline{x},u)) - \{0\} \times A^2(-S,g(\overline{x}),w)\big)\Big\} + S_{g(\overline{x})} = Z.$$

Proof. Case (i): It is proved similarly as in preceding Theorems 3.1 (i) and 3.2 (i). Case (ii) is a direct consequence of Theorem 3.2 ([10], p. 78), with (f_+, g_+) replacing (F_+, G_+), where F, G are set-valued mappings. The proof is completed. □

Note 3.5. (i) The result in article [10] of Khanh and Tung (2015) is extended from the result is well known in references of [10]. However, the case $w \in -S$ is not mentioned in article [10]. Consider the case $w \in -cl\, S_{g(\overline{x})}$, we provide assumptions, which involving f and g, such as f and g are Fréchet differentiable at $\overline{x}$ whose its Fréchet derivatives at $\overline{x}$ stable at that point, then Theorems 3.1 (ii) and 3.2 (ii) in our article are better than Theorem 3.1 of Khanh (2015) et al. ([10], p. 77) in case $F = \{f\}$. It also can be seen as a good improvement one of the results in this article when we considering problem VEPC with data is single-valued functions.

(ii) Since $IT^2(C,\overline{x},u) \subset A^2(C,\overline{x},u)$, hence in single-valued optimization, Theorem 3.3 improves Theorem 3.2 in [10] in the case, either g has Fréchet derivative $\nabla g(\overline{x})$ which is stable at $\overline{x}$, or g is twice Fréchet differentiable at $\overline{x}$.

4. Second-order Sufficient Optimality Conditions

In this subsection, we establish second-order sufficient optimality conditions for local weakly efficient solution for VEPC, where Q and S are closed cones (possibly nonconvex with empty interior). We set

$$u^{\perp} = \{w \in X \ : < u, w >= 0\};$$

$$\Delta(\overline{x}) = \Big\{(\lambda,\eta) \in Y \times Z \ : \ \lambda_0 \nabla f(\overline{x}) + \eta_0 \nabla g(\overline{x}) = 0, \quad \lambda \in Q^+, \ \ \eta \in N(-S,g(\overline{x})), \ \ (\lambda,\eta) \neq (0,0)\Big\}.$$

Theorem 4.1. *Let $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q be pointed $(Q \cap (-Q) = \{0\})$. Suppose that $\nabla f(\overline{x})$ and $\nabla g(\overline{x})$ are calm at $\overline{x}$ and all the following conditions are fulfilled:*

(i) For all $u \in T(C,\overline{x}) \cap \{u \in X \ : \ \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $D_c f(\overline{x})(u) \cap (-intQ) = \emptyset$ and $D_c g(\overline{x})(u) \cap T(-S, g(\overline{x})) \neq \emptyset$;

(ii) For all $u \in T(C,\overline{x}) \cap \{u \in X \ : \ \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $v \in D_c f(\overline{x})(u) \cap (-bdQ)$ and $w \in D_c g(\overline{x})(u) \cap T(-S, g(\overline{x}))$, we have

(a) $\forall\, x \in T^2(C,\overline{x},u)$, $\forall\,(y,z) \in D_c^2(f,g)(\overline{x},(f,g)(\overline{x}),u,(v,w))(x)$ such that $z \in T^2(-S,g(\overline{x}),w)$, there exists $(\lambda,\eta) \in \Delta(\overline{x})$ satisfying

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \ 0;$$

(b) $\forall\, x \in T^{''}(C,\overline{x},u) \cap u^{\perp} \setminus \{0\}$, $\forall\,(y,z) \in D_c^{''}(f,g)(\overline{x},(f,g)(\overline{x}),u,(v,w))(x)$ such that $z \in T^{''}(-S,g(\overline{x}),w)$, there exists $(\lambda,\eta) \in \Delta(\overline{x})$ satisfying

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \ 0.$$

Then $\overline{x}$ is a local weakly efficient solution to the VEPC.

Proof. Assume to the contrary, that $\overline{x} \in K$ is not a local weakly efficient solution to the VEPC. Then there exists sequence $(x_n)_{n\geq 1} \subset K \setminus \{\overline{x}\} \subset C \setminus \{\overline{x}\}$ such that $x_n \to \overline{x}$ and

$$f(x_n) - f(\overline{x}) \in -intQ \quad \forall n \geq 1; \tag{4.1}$$

$$g(x_n) - g(\overline{x}) \in -S - g(\overline{x}) \quad \forall n \geq 1. \tag{4.2}$$

Making use of Lemma 4.1 (i) ([10], p. 83), we get $t_n = \|x_n - \overline{x}\| \to 0^+$ and

$$u_n := \frac{x_n - \overline{x}}{t_n} \to u \in T(C, \overline{x}) \cap \{u \in X \,:\, \|u\| = 1\}.$$

We pick $v = \nabla f(\overline{x})(u), w = \nabla g(\overline{x})(u)$. It is not difficult to see that $v \in D_c f(\overline{x})(u) \cap (-Q)$ and $w \in T(-S, g(\overline{x})) \cap D_c g(\overline{x})(u)$. Thus we have shown that $u \in T(C, \overline{x}) \cap \{u \in X \,:\, \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $v \in D_c f(\overline{x})(u)$ and $D_c g(\overline{x})(u) \cap T(-S, g(\overline{x})) \neq \emptyset$. By the hypotheses of (i), $v \notin -intQ$, which yields that $v \in D_c f(\overline{x})(u) \cap (-bdQ)$.
In other words, we pick sequence $(w_n)_{n\geq 1}$, where

$$w_n = \frac{x_n - \overline{x} - t_n u}{2^{-1} t_n^2}, \quad \forall n \geq 1.$$

Two cases can occur as follows:

(I) $(w_n)_{n\geq 1}$ is bounded. As $dim(X) < +\infty$, thus (taking a subsequence if necessary) there exists the limit of sequence $(w_n)_{n\geq 1}$, and assuming that $w_n \to x \in X$. We setting

$$(y_n, z_n) := \frac{(f,g)(x_n) - (f,g)(\overline{x}) - t_n(v,w)}{2^{-1} t_n^2}, \ n \geq 1.$$

In view of the proof of Proposition 2 ([8], p. 204), we deduce that $(y_n, z_n)_{n\geq 1}$ is bounded, and therefore, there exists a subsequence, denoted in the same way $(y_n, z_n)_{n\geq 1}$ converging to some (y, z). By definitions, we have $x \in T^2(C, \overline{x}, u)$ because $x_n \in C$ $(\forall n)$, $z \in T^2(-S, g(\overline{x}), w)$ is due to $g(x_n) \in -S$ $(\forall n)$, and moreover $(y, z) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$. On the other hand, by hypotheses (a), there exists $(\lambda, \eta) \in \Delta(\overline{x})$ such that

$$\langle \lambda, y \rangle + \langle \eta, z \rangle \ > \ 0. \tag{4.3}$$

It follows from Equation (16) ([8], p. 210) that

$$\langle \lambda, v \rangle = 0, \ \ \langle \eta, w \rangle = 0.$$

Furthermore,

$$(4.1) \Longleftrightarrow \frac{f(x_n) - f(\overline{x})}{2^{-1} t_n^2} \in -intQ \ \ \forall n \geq 1,$$

which is equivalent to

$$y_n + \frac{2\,v}{t_n} \in -intQ \ \ \forall n \geq 1.$$

Because $\langle \lambda, v \rangle = 0$ and $\lambda \in Q^+ \cap Y^*$, hence

$$\lim_{n \longrightarrow +\infty} \langle \lambda, y_n \rangle = \langle \lambda, y \rangle \leq 0. \tag{4.4}$$

Similar to (4.1), one has

$$\frac{g(x_n) - g(\overline{x}) - t_n w}{2^{-1} t_n^2} + \frac{2w}{t_n} \in -S_{g(\overline{x})} \ \ \forall n \geq 1,$$

which is equivalent to

$$z_n + \frac{2\,w}{t_n} \in -S_{g(\overline{x})} \ \ \forall n \geq 1.$$

Because $\eta \in N(-S, g(\overline{x}))$ yields that $\eta \in (S_{g(\overline{x})})^+$. This together with the fact that $\langle \eta, w\rangle = 0$, and by taking limit, we have $\langle \eta, z\rangle \leq 0$. Therefore (see Eq. 4.4) $\langle \lambda, y\rangle + \langle \eta, z\rangle \leq 0$, which conflicts with (4.3).

(II) $(w_n)_{n\geq 1}$ is unbounded: Let us may assume that $\|w_n\| \to +\infty$ and

$$W_n = \frac{w_n}{\|w_n\|} \to x_1 \in X \cap \{u \in X \ \|u\| = 1\}.$$

For the preceding sequence t_n, we choose a new sequence $r_n = t_n\|w_n\|\forall n$, then easy to check that $r_n \to 0^+$, $\frac{t_n}{r_n} \to 0^+$ and moreover

$$x_n = \overline{x} + t_n u + \frac{1}{2} r_n t_n W_n \quad \forall n \geq 1.$$

Considering sequences

$$(y_n', z_n') := \frac{(f,g)(x_n) - (f,g)(\overline{x}) - t_n(v,w)}{2^{-1}t_n r_n} \to (y_1, z_1).$$

In the similar way as in (I) (can seen in the proof case (ii) in Theorem 3 ([8], p. 217-218), we conclude that $x_1 \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$ and moreover, $(y_1, z_1) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x_1)$ and $z_1 \in T^{''}(-S, g(\overline{x}), w)$. Therefore there exists $(\lambda_1, \eta_2) \in \Delta(\overline{x})$ satisfying

$$\langle \lambda_1, y_1\rangle + \langle \eta_1, z_1\rangle \ > \ 0.$$

Similarly to the preceding proof, $\langle \lambda_1, v\rangle = 0$, $\langle \eta_1, w\rangle = 0$, $y_n' + \frac{2v}{r_n} \in -intQ \ \ \forall n \geq 1$, $z_n' + \frac{2w}{r_n} \in -S_{g(\overline{x})} \ \ \forall n \geq 1$, which leads to a contradiction. From there we conclude that the proof of Theorem 4.1 is complete. □

Corollary 4.1. *Let* $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q *be pointed* $(Q \cap (-Q) = \{0\})$. *Suppose that* $\nabla f(\overline{x})$ *and* $\nabla g(\overline{x})$ *are calm at* $\overline{x}$ *and* $(v,w) = \Big(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)\Big)$. *Then* $\overline{x}$ *is a local weakly efficient solution to the VEPC if*

(i) $\forall x \in T^2(C, \overline{x}, u)$, $\forall (y,z) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ *such that* $z \in T^2(-S, g(\overline{x}), w)$, *there exists* $(\lambda, \eta) \in \Delta(\overline{x})$ *satisfying* $\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \ 0$;

(ii) $\forall x \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$, $\forall (y,z) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ *such that* $z \in T^{''}(-S, g(\overline{x}), w)$, *there exists* $(\lambda, \eta) \in \Delta(\overline{x})$ *satisfying* $\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \ 0$.

Proof. It is directly inferred from the proof of Theorem 4.1 and the claim follows. □

Note 4.2. (i) Because $\eta \in S_{g(\overline{x})}$ and $\langle \eta, w\rangle = 0$, hence $\eta \in \Big(cone(cone(S + g(\overline{x})) + w)\Big)^+$. Furthermore, it is well known that

$$A^2(-S, g(\overline{x}), w) \subset cl\Big(cone(cone(-S - g(\overline{x})) - w)\Big) = -cl\Big(cone(cone(S + g(\overline{x})) + w)\Big),$$

which yields that $\sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle \leq 0$. From the fact that $\langle \lambda, y\rangle + \langle \eta, z\rangle > 0$, it implies that

$$\langle \lambda, y\rangle + \langle \eta, z\rangle \ > \sup_{a \in A^2(-S, g(\overline{x}), w)} \langle \eta, a\rangle \,. \tag{4.5}$$

When $(-S)$ is second-order derivative at $(g(\overline{x}), \nabla g(\overline{x})u)$, i.e., $T^2(-S, g(\overline{x}), w) = A^2(-S, g(\overline{x}), w)$ for all $u \in X$, the necessary optimality conditions in above Theorem 4.1 only differs from (4.5) in the substitution of " $>$ " for " $\geq$ ". In this case, it will be said that second-order sufficient optimality conditions are very close to the second-order optimality necessary conditions.

(ii) The set $\Delta(\overline{x})$ in Theorem 4.1 is also called the set of all Fritz John type multipliers.

Theorem 4.2. *Let* $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q *be pointed* $(Q \cap (-Q) = \{0\})$. *Suppose that* f *and* g *are twice Fréchet differentiable at* $\overline{x}$ *and all the following conditions are fulfilled:*
(i) For all $u \in T(C, \overline{x}) \cap \{u \in X : \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $D_c f(\overline{x})(u) \cap (-intQ) = \emptyset$ *and* $D_c g(\overline{x})(u) \cap T(-S, g(\overline{x})) \neq \emptyset$;
(ii) For all $u \in T(C, \overline{x}) \cap \{u \in X : \nabla f(\overline{x})(u) \in -Q\} \setminus \{0\}$, $v \in D_c f(\overline{x})(u) \cap (-bdQ)$ *and* $w \in D_c g(\overline{x})(u) \cap T(-S, g(\overline{x}))$, *we have*

(a) $\forall\, x \in T^2(C, \overline{x}, u)$ *for which* $(\nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ *such that* $\nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \in T^2(-S, g(\overline{x}), w)$, *there exists* $(\lambda, \eta) \in \Delta(\overline{x})$ *satisfying*

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0;$$

(b) $\forall\, x \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$ *for which* $(\nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$ *such that* $\nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \in T^{''}(-S, g(\overline{x}), w)$, *there exists* $(\lambda, \eta) \in \Delta(\overline{x})$ *satisfying*

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0.$$

Then $\overline{x}$ *is a local weakly efficient solution to the VEPC.*

Proof. It is well-known that, when f and g are twice Fréchet differentiable at $\overline{x}$, then for $(v, w) = (\nabla f(\overline{x})(u), \nabla g(\overline{x})(u))$, one has (see [8])

$$D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x) = \{(\nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u))\}.$$

Repeat the proof Theorem 4.1, and the claim follows. □

Corollary 4.3. *Let* $\overline{x} \in C$, $f = F(\overline{x}, .)$, Q *be pointed* $(Q \cap (-Q) = \{0\})$. *Suppose that* f *and* g *are twice Fréchet differentiable at* $\overline{x}$ *and* $(v, w) = \Big(\nabla f(\overline{x})(u), \nabla g(\overline{x})(u)\Big)$. *Then* $\overline{x}$ *is a local weakly efficient solution to the VEPC if*

(i) $\forall\, x \in T^2(C, \overline{x}, u)$ *for which*

$$(\nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)) \in D_c^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$$

such that $\nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \in T^2(-S, g(\overline{x}), w)$, *there exists* $(\lambda, \eta) \in \Delta(\overline{x})$ *satisfying*

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0;$$

(ii) $\forall\, x \in T^{''}(C, \overline{x}, u) \cap u^{\perp} \setminus \{0\}$ *for which*

$$(\nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u),\ \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)) \in D_c^{''}(f,g)(\overline{x}, (f,g)(\overline{x}), u, (v,w))(x)$$

such that $\nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \in T^{''}(-S, g(\overline{x}), w)$, *there exists* $(\lambda, \eta) \in \Delta(\overline{x})$ *satisfying*

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u) \right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u) \right\rangle > 0.$$

Proof. Repeat the proof Theorem 4.2 and we arrive at the conclusion. □

Note 4.4. Note that the condition

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u)\right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)\right\rangle > 0$$

is implied by the following condition

$$\left\langle \lambda, \nabla f(\overline{x})(x) + \nabla^2 f(\overline{x})(u,u)\right\rangle + \left\langle \eta, \nabla g(\overline{x})(x) + \nabla^2 g(\overline{x})(u,u)\right\rangle > \sup_{a\in T^2(-S,g(\overline{x}),w)} \langle \eta, a\rangle .$$

Note 4.5. In case the objective functions are considered with vector values, the difference between Theorem 4.1 [10] and our results lies in $\Delta(\overline{x})$ and $\Delta^2(x)$, where

$$\Delta^2(x) = \Big\{(y,z) \,:\, (y,z) \in D_c^2(f,g)(\overline{x},(f,g)(\overline{x}),u,(v,w))(x),\ (x,y,z) \in (u,v,w)^{\perp}\Big\}.$$

5. Acknowledgements

The author is grateful to the reviewers for useful suggestions which improve the contents of this article.This research was funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grand number 101.01-2017.301.

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