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CONVERGENCE THEOREMS OF MONOTONE (α, β)-NONEXPANSIVE MAPPINGS FOR NORMAL-S ITERATION IN ORDERED BANACH SPACES WITH CONVERGENCE ANALYSIS

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ABSTRACT. In this work, we prove some theorems of existence of fixed points for a monotone (α, β)-nonexpansive mapping in a uniformly convex ordered Banach space. Also, we prove some weak and strong convergence theorems of normal-S iteration under some control condition. Finally, we give two numerical examples to illustrate the main result in this paper.

KEYWORDS:Ordered Banach space; fixed point; monotone (α, β)-nonexpansive mapping; normal S-iteration

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1. Introduction

Let E be an ordered Banach space with the partial order $\leq$. A mapping $T : E \to E$ said to be *monotone* if $Tx \leq Ty$ for all $x, y \in E$ with $x \leq y$ and *monotone nonexpansive* if T is monotone and

$$\|Tx - Ty\| \leq \|x - y\|,$$

for all $x, y \in E$ with $x \leq y$.

In 2015, Dehaish and Khamsi [1] consider Mann's iteration $\{x_n\}$ for a monotone nonexpansive mapping $T : C \to C$ defined by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n,$$

for each $n \geq 1$, where $\{\beta_n\}$ in $(0, 1)$ for finding some order fixed points of monotone nonexpansive mappings in uniformly convex ordered Banach spaces for prove some weak convergence theorems. The results of Dehaish and Khamsi , they gave the control condition $\{\beta_n\}$ in $[a, b]$ with $a > 0$ and $b < 1$, but their results do not entail $\beta_n = \frac{1}{n+1}$

Thus, to improve the results mentioned above, in 2016, Song et al. [2] they proved some weak convergence theorems of Mann's iteration satisfies the following condition:

$$\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty.$$

Clearly, this control condition $\{\beta_n\}$ contains $\beta_n = \frac{1}{n+1}$ as a special case.

In 2016, Song et al. [3] considered the convergence theorems of Mann's iteration for a monotone α-nonexpansive mapping T in an ordered Banach space E.

In 2017, Muangchoo-in et al. [4] introduced the notion of a monotone (α, β)-nonexpansive mapping T in an ordered Banach space E and proved some existence theorems of fixed points by using the assumption $\liminf_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$. and some weak and strong convergence theorems of Ishikawa type iteration as follows are obtained :

$$\begin{cases} y_n = (1 - s_n)x_n + s_n Tx_n, \\ x_{n+1} = (1 - s_n)x_n + s_n T(y_n) \end{cases} \tag{1.1}$$

for each $n \geq 1$, where $\{s_n\}$ is the sequences in $[0, 1]$. Under the control condition

$$\liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0 \textbf{ or } \limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0.$$

In 2013, Sahu, D.R. [5] introduced Normal S-iteration process defined as follows : For C a convex subset of normed space X and a non-linear mapping T of C into itself, for each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = (1 - s_n)x_n + s_n Tx_n, \\ x_{n+1} = T(y_n) \end{cases} \tag{1.2}$$

for each $n \geq 1$, where $\{s_n\}$ is the sequences in $(0, 1)$.

Motivated by the results mentioned above, in this paper, we show some existence of a fixed point of a monotone (α, β)-nonexpansive mapping in ordered Banach spaces by do not use the condition $\liminf_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0$. And we prove some weak and strong convergence theorems of Normal S-iteration for a monotone (α, β)-nonexpansive mapping under the condition

$$\limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0, \quad \liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0.$$

Finally, we give a numerical example to illustrate the main result in this paper.

2. Preliminaries

Let P be a closed and convex cone of a real Banach space E. A *partial order* "$\leq$" with respect to P in E is defined as follows:

$$x \leq y \ (x < y) \text{ if and only if } \ y - x \in P \ (y - x \in \mathring{P}),$$

for all $x, y \in E$, where $\mathring{P}$ is the interior of P.

In this paper, let E be a Banach space with the norm $\|\cdot\|$ and the partial order $\leq$. Let $F(T) = \{x \in E : Tx = x\}$ denote the set of all fixed points of a mapping T. Also, we assume that the order intervals are convex and closed. Recall that an order interval is any of the subsets

$$[x, \rightarrow) = \{p \in E; x \leq p\} \ \text{ or } \ (\leftarrow, x] = \{p \in E; p \leq x\}$$

for any $a \in C$. An *order interval* $[x, y]$ for all $x, y \in E$ is given by

$$[x, y] = [x, \rightarrow) \cap (\leftarrow, y] = \{z \in E : x \leq z \leq y\}. \tag{2.1}$$

Then the convexity of the order interval $[x, y]$ implies that

$$x \leq tx + (1-t)y \leq y, \tag{2.2}$$

for all $x, y \in E$ with $x \leq y$.

A Banach space E is said to be:

(1) *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;
(2) *uniformly convex* if, for all $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$.

The following inequality was shown by Xu [6] in a uniformly convex Banach space E, which is known as *Xu's inequality.*

Lemma 2.1. [6] *For any real numbers $q > 1$ and $r > 0$, a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex function $g : [0, +\infty) \longrightarrow [0, +\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^q \leq t\|x\|^q + (1-t)\|y\|^q - \omega(q, t)g(\|x - y\|), \tag{2.3}$$

for all $x, y \in B_r(0) = \{x \in E; \|x\| \leq r\}$ and $t \in [0, 1]$, where $\omega(q, t) = t^q(1-t) + t(1-t)^q$.
In particular, take $q = 2$ and $t = \frac{1}{2}$,

$$\Big\|\frac{x+y}{2}\Big\|^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{4}g(\|x - y\|). \tag{2.4}$$

Lemma 2.2. [7] *Let K be a nonempty closed convex subset of a reflexive Banach space E. Assume that $\rho : K \longrightarrow R$ is a proper convex lower semi-continuous and coercive function. Then the function ρ attains its minimum on K, that is, there exists $x \in K$ such that*

$$\rho(x) = \inf_{y \in K} \rho(y).$$

Lemma 2.3. [8] *A Banach space E is said to satisfy Opial's condition if, whenever any sequence $\{x_n\}$ in E converges weakly to a point x,*

$$\limsup_{n \longrightarrow \infty} ||x_n - x|| < \limsup_{n \longrightarrow \infty} ||x_n - y||$$

for any $y \in E$ such that $y \neq x$.

Definition 2.4. [4] Let K be a nonempty closed subset of an ordered Banach space $(E, \leq)$. A mapping $T : K \to K$ is said to be :

(1) *monotone (α, β)-nonexpansive* if T is monotone and, for some $\alpha, \beta < 1$,
$\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \beta\|Ty - x\|^2 + (1 - (\alpha + \beta))\|x - y\|^2,$
for all $x, y \in K$ with $x \leq y$, which is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \frac{\alpha + \beta}{1 - \beta}\|Tx - x\|^2 + \frac{2}{1 - \beta}\|Tx - x\| \left[\|x - y\| + \|Tx - Ty\|\right]. \tag{2.5}$$

(2) *monotone quasi-nonexpansive* if T is monotone, $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$
for all $p \in F(T)$ and $x \in K$ with $x \leq$ p or $p \leq x$.

Remark 2.5. If $\beta = \alpha$, then (α, β)-nonexpansive is α-nonexpansive mapping.

3. Main Results

3.1. **The existence of fixed points.** We denote

$$F_{\leq}(T) = \{p \in F(T) : p \leq x_1\}, \quad F_{\geq}(T) = \{p \in F(T) : x_1 \leq p\}.$$

Note that, since the partial order $\leq$ is defined by the closed convex cone P, it is obvious that both $F_{\leq}(T)$ and $F_{\geq}(T)$ are closed convex.

Now, we introduce the following lemma to find fixed points of a monotone (α, β)-nonexpansive mapping in Banach space E:

Lemma 3.1. *Let K be a nonempty closed and convex subset of a Banach space $(E, \leq)$. Let $T : K \to K$ be a monotone mapping and assume that the sequence $\{x_n\}$ defined by Normal S-iteration (1.2) and $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). Then we have*

(1) *$x_n \leq y_n \leq x_{n+1}$ (or $x_n \geq y_n \geq x_{n+1}$);*
(2) *$x_n \leq x$ (or$x \leq x_n$) for all $n \leq 1$ if $\{x_n\}$ weakly converges to a point $x \in K$.*

Proof. (1) Let $k_1, k_2 \in K$ such that $k_1 \leq k_2$. Then we have

$$k_1 \leq (1 - \alpha)k_1 + \alpha k_2 \leq k_2$$

for all $\alpha \in [0, 1]$ since order intervals are convex. By the assumption, we have $x_1 \leq Tx_1$ and so the inequality is true for $n = 1$. Assume that $x_k \leq Tx_k$ for $k \geq 2$. We will show that $x_{k+1} \leq Tx_{k+1}$ by convexity and monotonicity, we have

$$x_k \leq (1 - s_k)x_k + s_k Tx_k = y_k \leq Tx_k,$$

i.e., $x_k \leq y_k \leq Tx_k \leq Ty_k = x_{k=1}$. since $y_k \leq x_{k+1}$ by T is monotone then $Ty_k = x_{k+1} \leq Tx_{k+1}$. By induction, we can conclude that $x_n \leq Tx_n$ is true for all $n \geq 1$.
Now we have $x_n \leq Tx_n$ for all $n \geq 1$ by convexity

$$x_n \leq (1 - s_n)x_n + s_n Tx_n = y_n \leq Tx_n,$$

since T is monotonicity $x_n \leq y_n$ then $Tx_n \leq Ty_n$, that is $x_n \leq y_n \leq Tx_n \leq Ty_n = x_{n+1}$. Hence, we conclude that $x_n \leq y_n \leq x_{n+1}$
On the other hand, if we assume $Tx_1 \leq x_1$, then we can show that $x_n \geq y_n \geq x_{n+1}$

(2) From Dehaish and Khamsi [1, Lemma 3.1]), we have the conclusion. This completes the proof. ☐

Next, we show some existence theorems of fixed points of monotone (α, β)-nonexpansive mappings in a uniformly convex ordered Banach space $(E, \leq)$.

Theorem 3.1. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E,\leq)$ and a mapping $T: K \to K$ be a monotone (α,β)-nonexpansive mapping. Assume $x_1 \leq Tx_1$ and the sequence $\{x_n\}$ defined by Normal S-iteration (1.2) is bounded with $x_n \leq w$ for some $w \in K$. Then $F_{\geq}(T) \neq \emptyset$.*

Proof. From Lemma 3.1, we have $x_1 \leq \cdots \leq x_n \leq x_{n+1}$. Let $C_n = \{z \in K : x_n \leq z\}$ for all $n \geq 1$. Then C_n is closed convex and $w \in C_n$. So C_n is nonempty. Let $C^* = \bigcap_{n=1}^{\infty} C_n$. Then C^* is a nonempty and closed convex subset of K. Since $\{x_n\}$ is bounded, we can define a function $\rho : C^* \longrightarrow [0,+\infty)$ by

$$\rho(z) = \limsup_{n\longrightarrow\infty} \|x_n - z\|^2,$$

for all $z \in C^*$. it follows from Lemma 2.2 that, there exists $z^* \in C^*$ such that

$$\rho(z^*) = \inf_{z\in C^*} \rho(z). \tag{3.1}$$

By the definition of C^*, we have

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots \leq z^*.$$

Since T is monotone, it follows from Lemma 3.1 that

$$x_n \leq Tx_{n+1} \leq Tz^*,$$

for each $k \geq 1$, which means that $Tz^* \in C^*$ and hence $\frac{z^*+Tz^*}{2} \in C^*$. Thus, by (3.1), we have

$$\rho(z^*) \leq \rho\Big(\frac{z^*+Tz^*}{2}\Big), \quad \rho(z^*) \leq \rho(Tz^*). \tag{3.2}$$

On the other hand, it follows from Definition 2.4 that

$$\begin{aligned}\|Tx_n - Tz^*\|^2 &\leq \|x_n - z^*\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx_n - x_n\|^2 \\ &\quad + \frac{2}{1-\beta}\|Tx_n - x_n\|\big[|\alpha|\|x_n - z^*\| + |\beta|\|Tx_n - Tz^*\|\big].\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded and $\liminf_{k\longrightarrow\infty} \|x_n - Tx_n\| = 0$, we have

$$\|Tx_n - Tz^*\|^2 \leq \|x_n - z^*\|^2,$$

and then

$$\limsup_{k\longrightarrow\infty} \|Tx_n - Tz^*\|^2 \leq \limsup_{k\longrightarrow\infty} \|x_n - z^*\|^2. \tag{3.3}$$

Thus, using (3.3), we have

$$\begin{aligned}\rho(Tz^*) &= \limsup_{k\longrightarrow\infty} \|x_n - Tz^*\|^2 \\ &= \limsup_{k\longrightarrow\infty} \|Tx_n - Tz^*\|^2 \\ &\leq \limsup_{k\longrightarrow\infty} [\|x_n - z^*\|^2 \\ &= \rho(z^*).\end{aligned} \tag{3.4}$$

Now, we show that $z^* = Tz^*$. From Lemma 2.1 with $q = 2$ and $t = \frac{1}{2}$ and (3.4) that is,

$$\begin{aligned}\rho\Big(\frac{z^*+Tz^*}{2}\Big) &= \limsup_{k\longrightarrow\infty} \Big\|x_n - \frac{z^*+Tz^*}{2}\Big\|^2 \\ &= \limsup_{k\longrightarrow\infty} \Big\|\frac{x_n - z^*}{2} + \frac{x_n - Tz^*}{2}\Big\|^2\end{aligned}$$

$$\leq \limsup_{k\longrightarrow\infty}\Big(\frac{1}{2}\|x_n - z^*\|^2 + \frac{1}{2}\|x_n - Tz^*\|^2 - \frac{1}{4}g(\|z^* - Tz^*\|)\Big)$$
$$\leq \frac{1}{2}\rho(z^*) + \frac{1}{2}\rho(Tz^*) - \frac{1}{4}g(\|z^* - Tz^*\|)$$
$$= \rho(z^*) - \frac{1}{4}g(\|z^* - Tz^*\|).$$

By Lemma 2.1, we have

$$\frac{1}{4}g(\|z^* - Tz^*\|) \leq \rho(z^*) - \rho\Big(\frac{z^* + Tz^*}{2}\Big) \leq 0.$$

Thus we have $g(\|z^* - Tz^*\|) = 0$ and so $z^* = Tz^*$ by the property of g. □

Theorem 3.2. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume $Tx_1 \leq x_1$ and the sequence $\{x_n\}$ defined by Normal S-iteration (1.2) is bounded with $w \leq x_n$ for some $w \in K$ Then $F_{\leq}(T) \neq \emptyset$.*

Proof. the proof same Theorem 3.1, by let $x_{n+1} \leq x_n \leq \cdots \leq x_1$. □

3.2. **The convergence of Normal S-iteration.** In this section, we prove some convergence theorems of Normal S-iteration for a monotone (α, β)-nonexpansive mapping in an ordered Banach space E.

Theorem 3.3. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$) and $F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$). Then we have*

1. *the sequence $\{x_n\}$ is bounded;*
2. *$\|x_{n+1} - p\| \leq \|x_n - p\|$ and $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists for all $p \in F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$);*
3. *$\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$ provided $\limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0$;*
4. *$\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$ provided $\liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0$.*

Proof. Without loss of generality, we assume that $x_1 \leq p \in F_{\geq}(T) \neq \emptyset$. Now, we claim $x_n \leq p$ for all $n \geq 1$. In fact, a mapping T is monotone, we have $x_1 \leq Tx_1 \leq Tp = p$ and $x_1 \leq y_1 \leq Tx_1 \leq p$ then we have $y_1 \leq p$. Again from T is monotone, then $Ty_1 \leq Tp = p$ from $x_1 \leq Ty_1$. By convex we can get $x_2 \leq p$, and so $x_1 \leq x_2 \leq p$. Suppose that $x_k \leq p$ for some $k \geq 2$. Then $Tx_k \leq Tp = p$ by monotonicity, from the condition (1) of Lemma 3.1 we have $x_k \leq y_k \leq Tx_k \leq Ty_k$ and $x_k \leq y_k \leq Tx_k \leq p$. Since $y_k \leq p$ then $Ty_k \leq Tp = p$. And $x_k \leq Ty_k$ by convexity

$$x_k \leq (1 - s_k)x_k + s_k Ty_k = x_{k+1} \leq Ty_k.$$

That is, we get $x_{k+1} \leq p$. Hence we conclude $x_n \leq p$ for all $n \leq 1$.
It follows from Lemma 3.1 that $\|Tx_n - p\| \leq \|x_n - p\|$ for all $n \geq 1$ and so

$$\begin{aligned}\|y_n - p\| &= \|(1 - s_n)x_n + s_n Tx_n - p\|\\ &\leq (1 - s_n)\|x_n - p\| + s_n\|T(x_n) - p\|\\ &\leq (1 - s_n)\|x_n - p\| + s_n\|x_n - p\|\\ &= \|x_n - p\|.\end{aligned}$$

Consequently, we have

$$\|x_{n+1} - p\| = \|T(y_n) - p\|$$

$$\begin{aligned}
&\leq \|y_n - p\| \\
&\leq \|x_n - p\| \\
&\cdots \\
&\leq \|x_1 - p\|.
\end{aligned}$$

Then the sequence $\{\|x_n - p\|\}$ is non-increasing and bounded and hence the conclusions (1) and (2) hold.

Now, we show that the conclusion (3) and (4) hold. From Lemma 2.1 with $q = 2$, $t = s_n$ and Lemma 3.1 it follows that,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|Ty_n - p\|^2 \\
&= \|y_n - p\|^2 \\
&\leq \|(1 - s_n)(x_n - p) + s_n(Tx_n - p)\|^2 \\
&\leq (1 - s_n)\|x_n - p\|^2 + s_n\|x_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Tx_n\|) \\
&= \|x_n - p\|^2 - s_n(1 - s_n)g(\|x_n - Tx_n\|)
\end{aligned}$$

which implies that

$$s_n(1 - s_n)g(\|x_n - Tx_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Then it follows from the conclusion (2) that

$$\limsup_{n \longrightarrow \infty} s_n(1 - s_n)g(\|x_n - Tx_n\|) = 0.$$

From the conclusion (3), since $\limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0$,

$$\left(\limsup_{n \longrightarrow \infty} s_n(1 - s_n)\right)\left(\liminf_{n \longrightarrow \infty} g(\|x_n - Tx_n\|)\right) \leq \limsup_{n \longrightarrow \infty} s_n(1 - s_n)g(\|x_n - Tx_n\|),$$

we have

$$\liminf_{n \longrightarrow \infty} g(\|x_n - Tx_n\|) = 0.$$

Hence we have

$$\liminf_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0,$$

by the properties of g. From the conclusion (4), since $\liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0$,

$$\left(\liminf_{n \longrightarrow \infty} s_n(1 - s_n)\right)\left(\limsup_{n \longrightarrow \infty} g(\|x_n - Tx_n\|)\right) \leq \limsup_{n \longrightarrow \infty} s_n(1 - s_n)g(\|x_n - Tx_n\|),$$

we have

$$\lim_{n \longrightarrow \infty} g(\|x_n - Tx_n\|) = \limsup_{n \longrightarrow \infty} g(\|x_n - Tx_n\|) = 0.$$

Hence we have

$$\lim_{n \longrightarrow \infty} \|x_n - Tx_n\| = 0,$$

by the properties of g. □

Theorem 3.4. *Let K be a nonempty and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume that E satisfies Opial's condition and the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). If $F_{\geq}(T) \neq \emptyset$ (or $F_{\leq}(T) \neq \emptyset$) and $\liminf_{n\longrightarrow\infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges weakly to a fixed point z of T.*

Proof. It follows from Theorem 3.3 that $\{x_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a point $z \in K$. From Lemma 3.1, it follows that $x_1 \leq x_{n_k} \leq z$ (or $z \leq x_{n_k} \leq x_n$) for all $k \geq 1$.
From Definition 2.4 that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2 + \frac{2}{1-\beta}\|Tx - x\|\big[|\alpha|\|x - y\| + |\beta|\|Tx - Ty\|\big].$$

Since the sequence $\{x_n\}$ is bounded and $\lim_{k\longrightarrow\infty}\|x_{n_k} - Tx_{n_k}\| = 0$, we have

$$\limsup_{k\longrightarrow\infty}\|Tx_{n_k} - Tz\|^2 \leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - z\|^2$$

and hence

$$\limsup_{k\longrightarrow\infty}\|Tx_{n_k} - Tz\| \leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - z\|. \tag{3.5}$$

Now, we prove that $z = Tz$. In fact, suppose that $z \neq Tz$. Then, by (3.5) and Opial's condition, we have

$$\begin{aligned}\limsup_{k\longrightarrow\infty}\|x_{n_k} - z\| &\leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - Tz\|\\ &\leq \limsup_{k\longrightarrow\infty}(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|)\\ &\leq \limsup_{k\longrightarrow\infty}\|x_{n_k} - z\|,\end{aligned}$$

which is a contraction. This implies that $z \in F_{\geq}(T)$ (or $z \in F_{\leq}(T)$). Using the conclusion (2) of Theorem 3.3, $\lim_{n\longrightarrow\infty}\|x_n - z\|$ exists.

Now, we show that the sequence $\{x_n\}$ converge weakly to the point z. Suppose that this does not hold. Then there exists a subsequence $\{x_{n_j}\}$ to converge weakly to a point $x \in K$ and $z \neq x$. Similarly, we must have $x = Tx$ and $\lim_{n\longrightarrow\infty}\|x_n - x\|$ exists. It follows from Opial's condition that

$$\lim_{n\longrightarrow\infty}\|x_n - z\| < \lim_{n\longrightarrow\infty}\|x_n - x\| = \limsup_{j\longrightarrow\infty}\|x_{n_j} - x\| < \lim_{n\longrightarrow\infty}\|x_n - z\|,$$

which is a contradiction and hence we get $x = z$. □

Theorem 3.5. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1 \leq Tx_1$. If $\limsup_{n\longrightarrow\infty} s_n(1 - s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\geq}(T)$.*

Proof. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p \in K$. From Lemma 3.1, it follows that $x_1 \leq x_{n_k} \leq p$ for all $k \geq 1$. By Theorem 3.1, we have $F_{\geq}(T) \neq \emptyset$ and it follows from Theorem 3.3 that $\{x_n\}$ is bounded and

$$\liminf_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0.$$

Assume that

$$\liminf_{k\longrightarrow\infty}\|x_{n_k} - Tx_{n_k}\| = 0.$$

From Definition 2.4 that

$$\|Tx - Tp\|^2 \leq \|x - p\|^2 + \frac{\alpha+\beta}{1-\beta}\|Tx - x\|^2$$

$$+\frac{2}{1-\beta}\|Tx-x\|\big[|\alpha|\|x-p\|+|\beta|\|Tx-Tp\|\big].$$

Since the sequence $\{x_{n_k}\}$ is bounded and

$$\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|=0,\quad \lim_{k\longrightarrow\infty}\|x_{n_k}-Tx_{n_k}\|=0,$$

we have

$$\limsup_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|^2\le 0$$

and hence

$$\lim_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|=0. \tag{3.6}$$

Therefore, we have

$$\limsup_{k\longrightarrow\infty}\|x_{n_k}-Tp\|\le\limsup_{k\longrightarrow\infty}(\|x_{n_k}-Tx_{n_k}\|+\|Tx_{n_k}-Tp\|)=0$$

and so $\lim_{k\longrightarrow\infty}\|x_{n_k}-Tp\|=0$, which implies that $p\in F_{\ge}(T)$. Using the conclusion (2) of Theorem 3.3, $\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|$ exists and so $\lim_{k\longrightarrow\infty}\|x_n-p\|=0$. □

Theorem 3.6. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E,\le)$ and a mapping $T:K\to K$ be a monotone (α,β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $x_1\le Tx_1$. If $\liminf_{n\longrightarrow\infty}s_n(1-s_n)>0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p\in F_{\ge}(T)$.*

Proof. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to a point $p\in K$. From Lemma 3.1, it follows that $x_1\le x_{n_k}\le p$ for all $k\ge 1$. By Theorem 3.1, we have $F_{\ge}(T)\ne\emptyset$ and it follows from Theorem 3.3 that $\{x_n\}$ is bounded and

$$\liminf_{n\longrightarrow\infty}\|x_n-Tx_n\|=0.$$

Without loss of generality, we can assume that

$$\liminf_{k\longrightarrow\infty}\|x_{n_k}-Tx_{n_k}\|=0.$$

From Definition 2.4 that

$$\|Tx-Tp\|^2\le\|x-p\|^2+\frac{\alpha+\beta}{1-\beta}\|Tx-x\|^2$$
$$+\frac{2}{1-\beta}\|Tx-x\|\big[|\alpha|\|x-p\|+|\beta|\|Tx-Tp\|\big].$$

Since the sequence $\{x_{n_k}\}$ is bounded and

$$\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|=0,\quad \lim_{k\longrightarrow\infty}\|x_{n_k}-Tx_{n_k}\|=0,$$

we have

$$\liminf_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|^2\le 0$$

and hence

$$\lim_{k\longrightarrow\infty}\|Tx_{n_k}-Tp\|=0. \tag{3.7}$$

Therefore, we have

$$\liminf_{k\longrightarrow\infty}\|x_{n_k}-Tp\|\le\liminf_{k\longrightarrow\infty}(\|x_{n_k}-Tx_{n_k}\|+\|Tx_{n_k}-Tp\|)=0$$

and so $\lim_{k\longrightarrow\infty}\|x_{n_k}-Tp\|=0$, which implies that $p\in F_{\ge}(T)$. Using the conclusion (2) of Theorem 3.3, $\lim_{k\longrightarrow\infty}\|x_{n_k}-p\|$ exists and so $\lim_{k\longrightarrow\infty}\|x_n-p\|=0$. □

Similarly, the following theorem can be proved:

Theorem 3.7. *Let K be a nonempty compact and closed convex subset of a uniformly convex ordered Banach space $(E, \leq)$ and a mapping $T : K \to K$ be a monotone (α, β)-nonexpansive mapping. Assume the sequence $\{x_n\}$ is defined by Normal S-iteration(1.2) with $Tx_1 \leq x_1$. If either $\liminf_{n\longrightarrow\infty} s_n(1-s_n) > 0$ or $\limsup_{n\longrightarrow\infty} s_n(1-s_n) > 0$, then the sequence $\{x_n\}$ converges strongly to a fixed point $p \in F_{\leq}(T)$.*

From Theorem 3.5, we have the following:

3.3. **The numerical examples.** Now, we give two numerical examples to illustrate the following examples, by we add Normal-S iteration for compare with Mann's iteration and Ishiwaka's iteration of [4] in the first example. And the last, we show the example between Mann's iteration and Normal-S iteration.

Example 3.2. Let $T : [0, 1] \to [0, 1]$ be a mapping defined by

$$Tx = \begin{cases} 0.25 & if \quad x \neq 1, \\ 0.5 & if \quad x = 1. \end{cases}$$

for any $x \in [0, 1]$. Then T is a $(0.8, 0.2)$-nonexpansive mapping. Define the sequences $s_n = \frac{1}{4} + \frac{1}{n^2}$ for each $n \geq 1$, then $\limsup_{n\to\infty} s_n(1-s_n) > 0$. Then all the conditions of Theorem 3.5 are satisfied. Also, 0.25 is a fixed point of T.

Table 1. The convergent step of $\{x_n\}$ for Example with $s_n = \frac{1}{4} + \frac{1}{n^2}$

Number of iterations	Sequence of Mann	Sequence of Ishikawa	Sequence of Normal-S
1	0.5000000	0.5000000	0.5000000
2	0.1875000	0.3294046	0.2500000
4	0.2300347	0.2518192	0.2500000
6	0.2402544	0.2502132	0.2500000
8	0.2448648	0.2500322	0.2500000
10	0.2472181	0.2500053	0.2500000
12	0.2484731	0.2500009	0.2500000
14	0.2491557	0.2500001	0.2500000
16	0.2495311	0.2500000	0.2500000

Example 3.3. Let $T_1 : [-1.5, -1] \to [-1.5, -1]$ or $T_2 : [1, 1.5] \to [1, 1.5]$ be the mappings defined by

$$Tx = arctan(5x).$$

The fixed points of mappings T_1 and T_2 are -1.4320322 and 1.4320322 respectively.

It is easy to see that T is monotone. Next we will show that T is a $(0.9, 0.1)$-nonexpansive mapping. By using Matlab R2015b software, we get

$$\min_{x,y\in[1,1.5]} \big\{0.9\|arctan(5x) - y\|^2 + 0.1\|arctan(5y) - x\|^2 + (1 - 0.9 - 0.1)\|x - y\|^2 - \|arctan(5x) - arctan(5y)\|^2\big\} = 4.37 \cdot 10^{-0.6} > 0.$$

then implies that

$$\|arctan(5x)-arctan(5y)\|^2 \leq 0.9\|arctan(5x)-y\|^2+0.1\|arctan(5y)-x\|^2+(1-0.9-0.1)\|x-y\|^2$$

for all $x, y \in [1, 1.5]$. And it is true for all $x, y \in [-1.5, -1]$ too. Therefore T is a monotone $(0.9, 0.1)$-nonexpansive mapping.

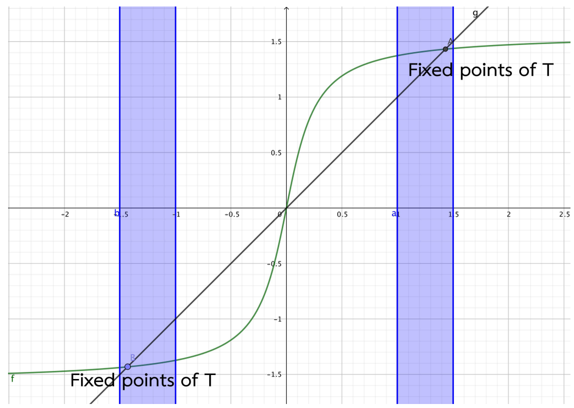

FIGURE 1. The fixed points of T are -1.4320322 and 1.4320322

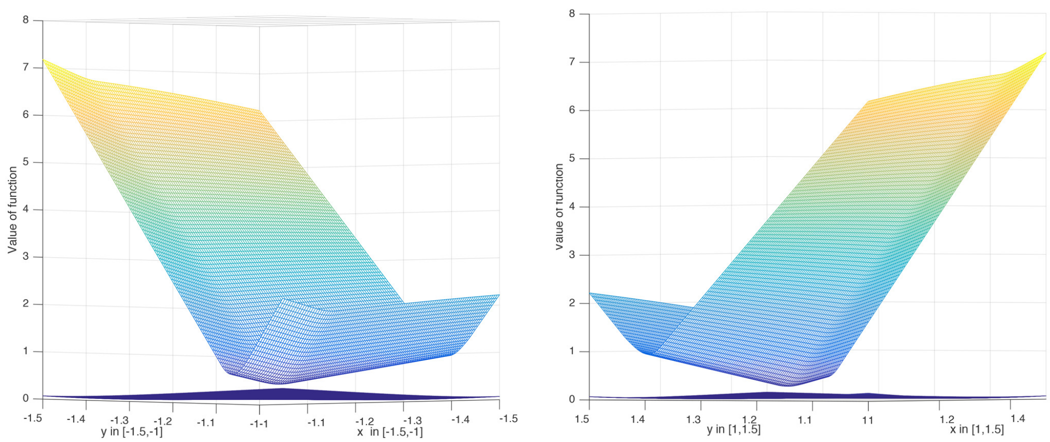

FIGURE 2. The value of mappings T_1 and T_2

Next we show the numerical solution of T, the numerical solution of this example is presented in Table 2.

Note that, if we set $x = 1.5$, $y = 1$ and $\alpha = \beta = 0.9$ then, the mapping T is not α-nonexpansive mapping.

From observing the numerical behavior, if we choose x_0 nearly is the solution then the sequence convergence is fast. Next we will show the convergent behavior of $\{s_n\}$ for iterative comparison between Mann's iteration, Ishikawa's iteration and normal-S iteration. by fixing $x_0 = 1.2$ and using three groups of sequences s_n for $n \geq 1$ are :

(i) $s_n = \frac{1}{4} + \frac{1}{n^k}, k \in \{0.01, 2, 5\}$;

(ii) $s_n = \frac{1}{4} + \frac{1}{\log^k(n+1)}, k \in \{0.01, 2, 5\}$;

(iii) $s_n = \frac{1}{4} + \frac{\log^k(n+1)}{n+2}, k \in \{0.01, 2, 5\}$:

All these sequences satisfy all condition of convergence theorems, Next figures describe the convergent behavior of three situations for value k.

TABLE 2. The convergent step of $\{x_n\}$ for Example 3.3 with $s_n = \frac{1}{4} + \frac{1}{n^2}$

Number of Iterations	Sequence value of Mann		Sequence value of Normal S	
	$x_0 = 1.2$	$x_0 = -1.3$	$x_0 = 1.2$	$x_0 = -1.3$
1	1.2000000	-1.3000000	1.2000000	-1.3000000
2	1.4570595	-1.4476837	1.4343860	-1.4335135
3	1.4457227	-1.4405986	1.4321554	-1.4321098
4	1.4412475	-1.4377994	1.4320401	-1.4320372
5	1.4386414	-1.4361689	1.4320327	-1.4320325
6	1.4369073	-1.4350837	1.4320322	-1.4320322
7	1.4356822	-1.4343169	1.4320322	-1.4320322
8	1.4347894	-1.4337581	1.4320322	-1.4320322
9	1.4341269	-1.4333435	1.4320322	-1.4320322
10	1.4336299	-1.4330323	1.4320322	-1.4320322

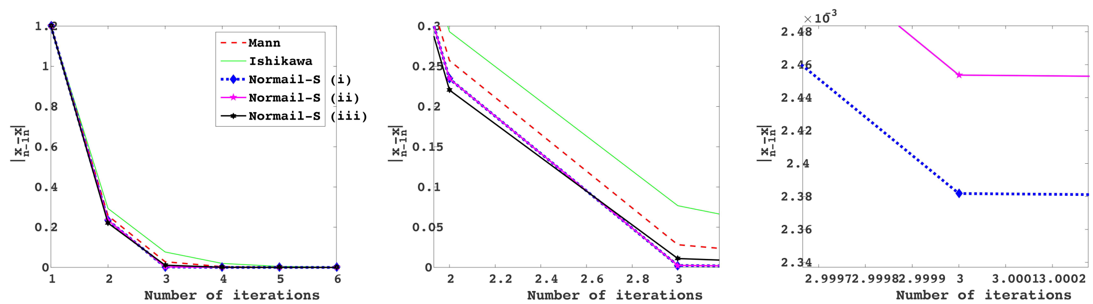

FIGURE 3. The behavior of sequence by fixing $k = 0.01$

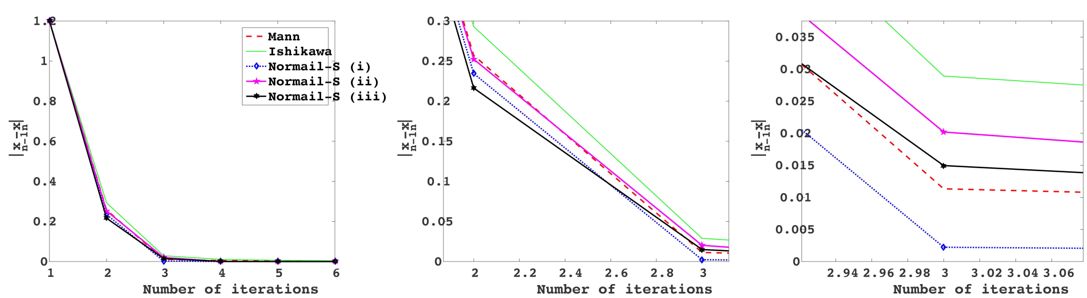

FIGURE 4. The behavior of sequence by fixing $k = 2$

The last figure describes the convergent behaviour for comparison k in three groups

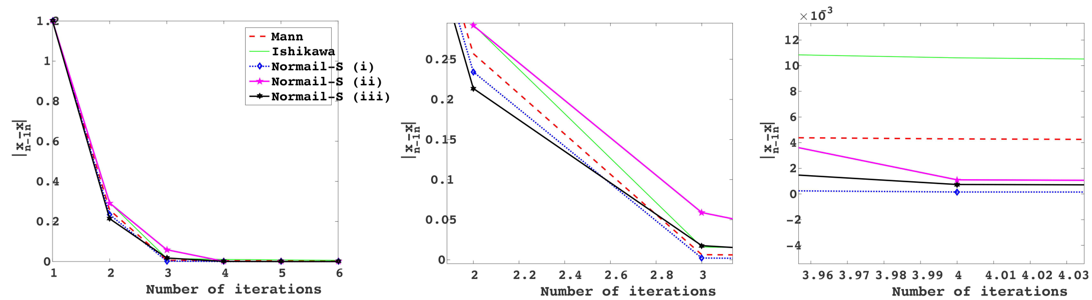

FIGURE 5. The behavior of sequence by fixing $k = 5$

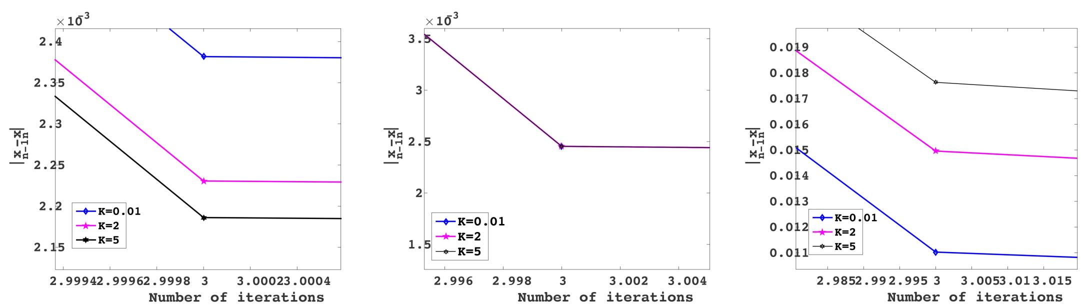

FIGURE 6. The convergent behaviour of each k for cases of group(i), group(ii) and group(iii)

4. CONCLUSION

We get the results about the convergence theorems of monotone (α, β)-nonexpansive mapping for the sequence $\{x_n\}$ is defined by normal-S iteration. In part of numerical, we give the examples for show the convergent behavior of sequence $\{s_n\}$ of normal-S iteration (in Figure 3 4 5 6)

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