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INEXACT PROXIMAL POINT ALGORITHM FOR MULTIOBJECTIVE OPTIMIZATION

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ABSTRACT. The main aim of this article is to present an inexact proximal point algorithm for constrained multiobjective optimization problems under the locally Lipschitz condition of the cost function. Convergence analysis of the considered method, Fritz-John necessary optimality condition of ϵ-quasi weakly Pareto solution in terms of Clarke subdifferential is derived. The suitable conditions to guarantee that the accumulation points of the generated sequences are Pareto-Clarke critical points are provided.

KEYWORDS:Multiobjective optimization; Quasi-convex functions; Lipschitz continuous function; Clarke subdifferential; Pareto-Clarke critical point.
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1. Introduction

With the development of optimization theory, multiobjective optimization problems have increasingly received much attentions, and have been greatly applied to management, decision-making disciplines, resource planning, engineering, the design of aircraft control systems and so on, see, for example, [30, 31]. In multiobjective optimization, one considers optimization problems with several conflicting objective functions. It is usually hard to find an optimal solution that satisfies all objectives from the mathematical point of view (i.e., there is no ideal minimizer), but we obtain a set of alternatives with different trade-offs, called efficient solutions.

The multiobjective optimization problem is considering the following context: For $I = \{1, 2, ..., m\}$, we put $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_j \geq 0, j \in I\}$, and $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m : x_j > 0, j \in I\}$. For $y, z \in \mathbb{R}^m$, ($z \succeq y$ or $y \preceq z$) means that $z - y \in \mathbb{R}^m_+$, and ($z \succ y$

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or $y \prec z$) means that $z - y \in \mathbb{R}^m_{++}$. By using these relations, we consider the efficient solution concepts of the (constrained) multiobjective minimization problem

$$\min_{x \in C} F(x), \tag{1.1}$$

where $C \subset \mathbb{R}^n$ is the constrained set and $F : \mathbb{R}^n \to \mathbb{R}^m$ is the objective mapping.

There are a number of works that pay attention to the methods for finding efficient solutions of multiobjective optimization problem (1.1). Such as, in 2007, Ceng and Yao [9] developed both an absolute and a relative version of approximate proximal point algorithm. They considered the approximate proximal method via the subproblems of finding weakly efficient points for suitable regularizations of the original mapping. Later, in 2015, Papa Quiroz et al. [29] proposed an inexact proximal point method of constrained multiobjective problems involving locally Lipschitz quasiconvex objective functions. They used proximal distances and assumed that the function is also bounded from below, lower semicontinuous for convergence analysis of the method. They proved that the sequence generated by the proposed method converges to a stationary point of the problem. After that, in 2018, João Carlos de O. Souza [33] studied the convergence of exact and inexact versions of the proximal point method with a generalized regularization function in Hadamard manifolds for solving scalar and vectorial optimization problems involving Lipschitz functions. In 2018, Bento et al. [5] considered the exact proximal point method of the constrained nonsmooth multiobjective optimization problem. They used non-scalarization approach for convergence analysis of the method, where the first order optimality condition of the problem is replaced by a necessary condition for weak Pareto points of a multiobjective problem. For more information on the related works in this direction, ones may see [1, 4, 5, 6, 7, 16, 17, 33]) and the references therein.

In this paper, our interest is to consider an inexact proximal point method for solving the multiobjective optimization problem (1.1).

Using the same technique as in Bento et al. [5], we propose an inexact proximal point algorithm for constrained nonsmooth multiobjective optimization problem. In terms of Clarke subdifferential, we introduce Fritz-John optimality condition of an ϵ-quasi weak Pareto solution, which we use for convergence analysis of our method. We also show that our proposed algorithm is well defined and the sequence achieved by the proposed algorithm converges to a Pareto-Clarke critical point. For a convex objective function F, we obtain the convergence to a weak Pareto solution of the problem.

2. Preliminaries

In this section, we present some basic concepts and results that are of fundamental importance for the development of our work.

The domain of f, denoted by $\operatorname{dom} f$, is the subset of $\mathbb{R}^n$ on which f has a finite valued. A function f is said to be proper when $\operatorname{dom} f \neq \emptyset$. We denote the closed unit ball in $\mathbb{R}^n$ by $\mathbb{B}_{\mathbb{R}^n}$. We say that a scalar valued function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exist a neighborhood U of x and a positive real number L such that

$$|f(z) - f(y)| \leq L\|z - y\|, \quad \forall z, y \in U.$$

A vector valued function $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz if all components of F are locally Lipschitz.

Next, we recall some concepts of Clarke directional derivative.

The Clarke directional derivative of a proper locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ at $x \in \mathbb{R}^n$ in the direction of $d \in \mathbb{R}^n$ is denoted by $f^{\diamond}(x, d)$, and is defined as

$$f^{\diamond}(x, d) = \lim_{t \to 0} \sup_{y \to x} \frac{f(y + td) - f(y)}{t}.$$

Now, we recall some concepts involving locally Lipschitz functions and nonconvex constrained sets.

Let $C \subset \mathbb{R}^n$ be a nonempty and closed set. We denote the distance function $d : \mathbb{R}^n \to \mathbb{R}$ of a point $x \in \mathbb{R}^n$ to a set $C \subset \mathbb{R}^n$ as

$$d_C(x) := \inf\{\|x - c\| : c \in C\}. \tag{2.1}$$

We say that a point $x \in C$ is a Pareto-Clarke critical point of F in C if, for any element $v \in T_C(x)$, there exists $i = 1, ..., m$ such that

$$f_i^{\diamond}(x, v) \geq 0, \tag{2.2}$$

where f_i is the ith component of F and $T_C(x) := \{v \in \mathbb{R}^n : d_C^{\diamond}(x, v) = 0\}$ denotes the set of all tangent vectors to C at x. As mentioned in [10], page 11, a vector v belongs to $T_C(x)$ if and only if it satisfies the following property: for every sequence $\{x^k\}$ in C converging to x and every sequence t_k in $(0, \infty)$ converging to 0, there is a sequence v^k converging to v such that $x^k + t_k v^k$ belongs to C for all k. The normal cone is the one obtained from tangent cone $T_C(x)$ by polarity. Therefore, the normal cone $N_C(x)$ to C at x is as follows:

$$N_C(x) := \{\varsigma \in \mathbb{R}^n : \langle \varsigma, v \rangle \leq 0, \forall v \in T_C(x)\},$$

see [5]. If C is convex, $N_C(x)$ coincides with normal cones in the sense of convex analysis; (see [10], Proposition 2.4.4).

Now, we remind some basic concepts and properties of multiobjective optimization, which can be found in [24].

A sequence $\{x^k\} \subset \mathbb{R}^m$ is called a decreasing sequence if $x^p \prec x^k$ for $k < p$. A point $\bar{x}$ is said to be an infimum of $\{x^k\}$, if there is no x such that $x \preceq \bar{x}$ and $x \preceq x^k$ satisfying $\bar{x} \preceq x^k$, for all $k \in \mathbb{N}$.

Next, we recall some definitions of optimal solutions and approximate optimal solutions of multiobjective function $F : \mathbb{R}^n \to \mathbb{R}^m$.

Consider a nonempty subset $C \subset \mathbb{R}^n$ and $\epsilon := (\epsilon_1, ..., \epsilon_m) \in \mathbb{R}^m_+$, a point $x^* \in C$ is called

(i) a weak Pareto solution of problem (1.1) if there exists no $x \in C$ such that $f_i(x) < f_i(x^*)$, for all $i \in \{1, ..., m\}$.

(ii) an ϵ-weak Pareto solution of problem (1.1) if there exists no $x \in C$ such that $f_i(x) + \epsilon_i < f_i(x^*)$, for all $i \in \{1, ..., m\}$.

(iii) an ϵ-quasi weak Pareto solution of problem (1.1) if there is no $x \in C$ such that $f_i(x) + \epsilon_i \|x - x^*\| < f_i(x^*)$, for all $i \in \{1, ..., m\}$.

We denote the set of weak Pareto, ϵ-weak Pareto and ϵ-quasi weak Pareto solutions of problem (1.1) by $\arg\min_w\{F(x)|x \in C\}$, $\arg\min_{\epsilon w}\{F(x)|x \in C\}$ and $\arg\min_{\epsilon q-w}\{F(x)|x \in C\}$, respectively. For the detail, see [13] and [25].

Remark 2.1. It is apparent that, if $\epsilon = 0$, then the notions of an ϵ-weakly Pareto solution and an ϵ-weakly quasi Pareto solution defined above coincide with the usual one of a weak Pareto solution. Also, for the case, $\epsilon \neq 0$, it is easy to see that, $\arg\min_w\{F(x)|x \in C\} \subset \arg\min_{\epsilon w}\{F(x)|x \in C\}$ and $\arg\min_w\{F(x)|x \in C\} \subset \arg\min_{\epsilon q-w}\{F(x)|x \in C\}$. While, the sets $\arg\min_{\epsilon w}\{F(x)|x \in C\}$ and $\arg\min_{\epsilon q-w}\{F(x)|x \in C\}$ might be two different sets. For detail, see [13].

Now, we remind Clarke subdifferential concept of scalar and vector functions. The Clarke subdifferential of scalar valued function $f:\mathbb{R}^n\to\mathbb{R}$ at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) := \{w\in\mathbb{R}^n : \langle w,d\rangle \le f^{\diamond}(x,d),\ \ \forall d\in\mathbb{R}^n\},$$

see Clarke [11].
The Clarke subdifferential of $F:\mathbb{R}^n\to\mathbb{R}^m$ at $x\in\mathbb{R}^n$, denoted by $\partial F(x)$, is defined as

$$\partial F(x) := \{U\in\mathbb{R}^{m\times n} : U^T d \preceq F^{\diamond}(x,d),\ \ \forall d\in\mathbb{R}^n\},$$

where $F^{\diamond}(x,d) := \{f_1^{\diamond}(x,d),...,f_m^{\diamond}(x,d)\}$.

Proposition 2.2. *([11], Proposition 1.4)* $f^{\diamond}(x;v)=\max\{\xi\cdot v : \xi\in\partial f(x)\}$.

Remark 2.3. It is noted in [5] that, combining (2.2) with Proposition 2.2, we have the following alternative definition: a point $x\in\mathbb{R}^n$ is a Pareto-Clarke critical point of F in C if, for any $v\in T_C(x)$, there exist $i\in\{1,...,m\}$ and $\xi\in\partial f_i(x)$ such that $\langle\xi,v\rangle\ge 0$. Thus, if x is not a Pareto-Clarke critical point of F in C, there exists $v\in T_C(x)$ such that $Uv\prec 0$, $\forall U\in\partial F(x)$.

The necessary condition for a point to be a Pareto-Clarke critical point of a vector-valued function can be found in Bento et al. ([5] Lemma 1), and is given below.

Proposition 2.4. [5] *Let $w\in\mathbb{R}^m_+\setminus\{0\}$ and assume that C is closed and nonempty set. If $-U^Tw\in N_C(x)$ for some $U\in\partial F(x)$, then x is a Pareto-Clarke critical point of F.*

For the nonconvex case, a formula for the Clarke sudifferential of the distance function (2.1) defined in Burke, Ferris and Qian [3] is as follows:

Proposition 2.5. [3] *Let $C\subset\mathbb{R}^m$ be a nonempty and closed set: If $x\in C$, then*

$$\partial d_C(x)\subset\mathbb{B}[0,1]\cap N_C(x), \tag{2.3}$$

where $\mathbb{B}[0,1]$ denotes the closed unit ball in $\mathbb{R}^m$.

Now, we recall some basic definitions of multiobjective functions.
Consider a vector function $F:\mathbb{R}^n\to\mathbb{R}^m$, we say that
i) F is called $\mathbb{R}^m_+$-convex if, for every $x,y\in\mathbb{R}^n$, the following condition holds:

$$F((1-t)x+ty)\preceq(1-t)F(x)+tF(y),\quad \forall t\in[0,1].$$

ii) F is called $\mathbb{R}^m_+$-quasiconvex if, for every $x,y\in\mathbb{R}^n$, the following condition holds:

$$F((1-t)x+ty)\preceq\max\{F(x),F(y)\},\quad \forall t\in[0,1],$$

where the maximum is considered coordinate by coordinate.

Remark 2.6. A vector function $F:\mathbb{R}^n\to\mathbb{R}^m$ is convex (resp. quasi-convex) iff F is componentwise convex (resp. quasi-convex), see Definition 6.2 and Corollary 6.6 of [24], pages 29, 31, respectively.

Next propositions will be useful in the following section.

Proposition 2.7. *([34], Theorem 3.2.1) Let $C\subset\mathbb{R}^n$ be a nonempty set and $f:\mathbb{R}^n\to\mathbb{R}$ be a Lipschitz function on $\mathbb{R}^n$ with constant L. If $\bar{x}$ is a minimizer for the constrained minimization problem,*

$$\min f(x),\quad x\in C, \tag{2.4}$$

and $\tau \geq L$, then $\bar{x}$ is also a minimizer for the unconstrained minimization problem

$$\min\{f(x) + \tau d_C(x)\}, \quad x \in \mathbb{R}^n. \tag{2.5}$$

If $\tau > L$ and C is a closed set, then the converse assertion is also true: Any minimizer $\bar{x}$ for the unconstrained problem (2.5) is also a minimizer for the constrained problem (2.4).

Proposition 2.8. *([1], Proposition 2.6.1) Let $C = \mathbb{R}^n$ and $\hat{x}$ be a Pareto-Clarke critical point of a locally Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}^m$. If F is $\mathbb{R}^m_+$-convex, then $\hat{x}$ is a weak Pareto solution of the problem (1.1).*

Proposition 2.9. *([28], Proposition 5.3(ii)) For a function $f : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz at $\bar{x} \in \mathbb{R}^n$ with modulus $l > 0$, it holds that*

$$\|x^*\| \leq l, \quad \forall x^* \in \partial f(\bar{x}). \tag{2.6}$$

Proposition 2.10. *([28], Theorem 5.10) Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions at $\bar{x} \in \mathbb{R}^n$, then*

$$\partial(f_1 + f_2)(\bar{x}) \subset \partial f_1(\bar{x}) + \partial f_2(\bar{x}). \tag{2.7}$$

Proposition 2.11. [10] *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i = 1, 2, ..., m$, be locally Lipschitz function at $x \in \mathbb{R}^n$ for all $i = \{1, ..., m\}$. Then, the function $f(x) := \max\{f_i(x) | i \in \{1, ..., m\}\}$ is also locally Lipschitz at x and*

$$\partial f(x) \subset \bigcup \left\{ \partial(\sum_{i=1}^m \lambda_i f_i)(x) | \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, \lambda_i [f_i(x) - f(x)] = 0 \right\}.$$

Proposition 2.12. *([2], Theorem 2.1) Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper quasiconvex locally Lipschitz function on $\mathbb{R}^n$. If $x^* \in \partial f(x)$ such that $\langle x^*, \hat{x} - x \rangle > 0$, then $f(x) \leq f(\hat{x})$.*

The next definition and result will be useful for the existence of the set of minimizers of a vector function which can be found in [24].

Definition 2.13. [24] A subset A of $\mathbb{R}^m$ is said to be $\mathbb{R}^m_+$-complete, if any decreasing sequence of A is bounded by an element of A, i.e., whenever $\{x^k\} \subset A$ is a decreasing sequence, then there exists $x \in A$ such that $x \preceq x^k$, for all $k \geq 0$.

Proposition 2.14. *([24], Lemma 3.5) If $A \subset \mathbb{R}^m$ is closed, has a lower bound (i.e., $\exists$ some $a \in A$ such that, for all $x \in A$, $a \preceq x$), then A is $\mathbb{R}^m_+$-complete.*

Proposition 2.15. *([24], Theorem 3.3) Consider the multiobjective problem (1.1). Then, $\arg\min\{F(x) | x \in C\}$ is nonempty iff $F(C)$ has a $\mathbb{R}^m_+$-complete section.*

3. Necessary optimality condition

In this section, we consider multiobjective optimization problem (1.1) of finding the quasi-weak Pareto point of a vector valued function F subject to the following constrained set

$$C := \{x \in D | g_j(x) \leq 0, \ j = 1, ..., p\},$$

where $D \subset \mathbb{R}^n$ is a nonempty and closed set, and $g_j : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function. We provide necessary conditions for a point $x^* \in C$ to be an ϵ-quasi weak Pareto solution associated to the problem (1.1).

Proposition 3.1. *Let* $x^* \in \arg\min_{\epsilon q-w}\{F(x)|x \in C\}$*. Then, there exist* $t_i \geq 0$ *and* $\mu_j \geq 0$ *for* $i \in \{1, ..., m\}$ *and* $j \in \{1, ..., p\}$ *with* $\sum_{i=1}^m t_i + \sum_{j=1}^p \mu_j = 1$ *and* $\tau > 0$ *such that*

$$0 \in \sum_{i=1}^m t_i \partial f_i(x^*) + \sum_{j=1}^p \mu_j \partial g_j(x^*) + \sum_{i=1}^m t_i \epsilon_i \mathbb{B}_{x^*} + \tau \partial d_D(x^*),$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$*,* $\epsilon_i \in \mathbb{R}^m_+$ *for* $i \in \{1, ..., m\}$ *and* $\mathbb{B}_{x^*}$ *denotes the closed unit ball of* x^**.*

Proof. For each $x \in C$, put $\Psi(x) = \max_{\substack{i\in\{1,...,m\}\\ j\in\{1,...,p\}}}\{f_i(x) - f_i(x^*) + \epsilon_i\|x - x^*\|, g_j(x)\}$.
Observe that $\Psi(x^*) = 0$.
Next, since x^* is an ϵ-quasi weak Pareto optimal point, then there is no $x \in C$ such that

$$f_i(x) + \epsilon_i\|x - x^*\| < f_i(x^*), \quad \forall i \in \{1, ..., m\}. \tag{3.1}$$

It can be easily verified that $0 \leq \Psi(x)$, which infers that for all $x \in C$, we have

$$\Psi(x^*) = \inf_{x\in C} \Psi(x).$$

It follows that x^* is also a minimizer to the constrained optimization problem

$$\min_{x\in C} \Psi(x).$$

Proposition 2.11 and locally Lipschitz properties of functions f_i and g_j imply that the function Ψ is also locally Lipschitz around x^*. Let L be a locally Lipschitz constant of Ψ at x^* and $\tau \geq L$, then applying the Proposition 2.7 to the last problem, we obtain

$$0 \in \partial(\Psi(x^*) + \tau d_D(x^*)). \tag{3.2}$$

Also, the sum rule (2.7) implies that

$$0 \in \partial\Psi(x^*) + \tau\partial d_D(x^*). \tag{3.3}$$

Now, by Proposition 2.11 and invoking the sum rule (2.7) applied to the Ψ, there exist non-negative real numbers $t_i \geq 0$ and $\mu_j \geq 0$ such that $\sum_{i=1}^m t_i + \sum_{j=1}^p \mu_j = 1$ and

$$\partial\Psi(x^*) \subset \left\{ \sum_{i=1}^m t_i \partial f_i(x^*) + \sum_{i=1}^m t_i \epsilon_i B_{x^*} + \sum_{j=1}^p \mu_j \partial g_j(x^*) \right\}. \tag{3.4}$$

and the desired result follows by combining (3.3) with (3.4). □

4. Inexact Proximal Point Algorithm

In this section, we consider $C \subset \mathbb{R}^n$ a nonempty and closed set and $F : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz function.

Next, we consider the inexact proximal point algorithm, for obtaining a Pareto-Clarke critical point of F in C. Take a bounded sequence of positive real numbers $\{\lambda_k\}$, and a sequence $\{e^k\} \subset \mathbb{R}^m_{++}$ such that $\|e^k\| = 1$, for all $k \in \mathbb{N}$. The method generates the sequence $\{x^k\} \in C$ as follows.

4.1. **Algorithm.**
INITIALIZATION: Choose an arbitrary initial point

$$x^1 \in C. \tag{4.1}$$

STOPPING CRITERION: Given x^k, if x^k is a Pareto-Clarke critical point, then stop. Otherwise go to the iterative step.
ITERATIVE STEP: Take the next iterate $x^{k+1} \in C$ as y such that there exists $\epsilon^k \in \mathbb{R}^m_+$ satisfying

$$y \in \arg\min{}_{\epsilon^k q-w}\{F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k | x \in \Omega_k\}, \tag{4.2}$$

$$\epsilon^k \preceq \sigma_k \frac{\lambda_k}{2}\|y - x^k\| e^k, \tag{4.3}$$

where $\Omega_k := \{x \in C | F(x) \preceq F(x^k)\}$ and $\{\sigma_k\} \subset [0, 1)$.
From now on, we will assume that $0 \prec F$.

4.2. **Existence of iterates.**

Proposition 4.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Then, the sequence $\{x^k\}$, generated by Algorithm 4.1, is well defined.*

Proof. We proceed by induction: It holds for $k = 1$, due to (4.1). Assume that x^k exists and define

$$F_k(x) := F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k.$$

Since $x^k \in \Omega_k$, we have, $F_k(\Omega_k) \neq \emptyset$. By assumption on F, that is $0 \prec F$, we get, $0 \prec F_k(x)$. Now, let $\{y^p\} \subset F_k(\Omega_k)$ such that $y^p \to y$. Since $y^p \in F_k(\Omega_k)$, there exists $z^p \in \Omega_k$ satisfying $y^p = F_k(z^p)$, for any p. We claim that $\{z^p\}$ is bounded, if not, then there is $\{p_j\} \subset \{p\}$ such that $z^{p_j} \to \infty$ as $j \to \infty$, then coercivity of F_k infers that $\|F_k(z^{p_j})\| \to +\infty$ as $j \to \infty$. On the other hand, $\|F_k(z^p)\| \to \|y\|$ because $y^p = F_k(z^p)$ and $y^p \to y$, which is a contradiction. Hence, we proved that $\{z^p\}$ is a bounded sequence. Subsequently, there are $\{z^{p_j}\} \subset \{z^p\}$ and $z \in \mathbb{R}^n$ such that $z^{p_j} \to z$ as $j \to \infty$. Moreover, by the continuity of F, we know that Ω_k is a closed set. Hence, $z \in \Omega_k$. Applying continuity of F_k and using uniqueness of limit, we can assert that $y \in F_k(\Omega_k)$. This proves $F_k(\Omega_k)$ is closed.
Subsequently, by Proposition 2.14 and property of $\mathbb{R}^m_+$ that all decreasing sequences having lower bound converges to its infimum, we know that $F_k(\Omega_k)$ is $\mathbb{R}^m_+$-complete. Thus, Proposition 2.15 infers that

$$\arg\min{}_w\{F_k(x) | x \in \Omega_k\}$$

is not empty. Therefore, by Remark 2.1, it follows that $\arg\min_{\epsilon^k q-w}\{F_k(x)|x \in \Omega_k\} \neq \emptyset$. □

Remark 4.2. Note that if Algorithm 4.1 terminates after finite number of iterations, then it terminates at a Pareto-Clarke critical point.

4.3. **Convergence Analysis.**
In this section, first we present some results which play an important role in our subsequent considerations. Then, we show that the sequence generated by our algorithm converges to a Pareto-Clarke critical point.

Proposition 4.3. *For all $k \in \mathbb{N}$, there exists $A_k \in \mathbb{R}^{m\times n}$, $\alpha^k, \beta^k \in \mathbb{R}^m_+$, $\tau_k > 0$ and $w^k \in \mathbb{R}^n$ such that*

$$A_k^T(\alpha^k + \beta^k) + \lambda_{k-1}\langle e^{k-1}, \alpha^k\rangle(x^k - x^{k-1}) + \langle \epsilon^{k-1}, \alpha^k\rangle v^k + \tau_k w^k = 0, \tag{4.4}$$

where $v^k \in \mathbb{B}_{x^k}$, $w^k \in \mathbb{B}[0,1] \cap N_C(x^k)$ *and* $\sum_{i=1}^m (\alpha_i^k + \beta_i^k) = 1$, $\forall k \in \mathbb{N}$.

Proof. For every k, consider the functions

$$W_k(x) := F(x) - F(x^k), \text{ and } F_k(x) := F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k.$$

As, F and $\|x - x^k\|^2$ are locally Lipschitz, the coordinate functions $(W_k)_i(.) := F(.) - F(x^k)$ and $(F_k)_i(.) := F(.) + \frac{\lambda_k}{2}\|. - x^k\|^2 e^k$, $i \in \{1, ..., m\}$, of $W_k(x)$ and $F_k(x)$, respectively, are also locally Lipschitz.

Since x^k is an ϵ-quasi weak Pareto solution for

$$\min F_{k-1}(x) \text{ such that } W_{k-1}(x) \preceq 0,$$

hence the desired result follows by applying Proposition 3.1, for each $k \in \mathbb{N}$ fixed with f_i and g_j by F_{k-1} and W_{k-1}, respectively, and taking into account that, from Proposition 2.5, we have

$$\partial d_C(x^k) \subset \mathbb{B}[0,1] \cap N_C(x^k), \ \ \forall k \in \mathbb{N}.$$

In this case, $A_k^T = [u_1^k....u_m^k]$, where $u_i^k \in \partial f_i(x^k)$ with $i \in \{1, ..., m\}$, $\alpha^k = (\alpha_1^k, ..., \alpha_m^k)^T$ and $\beta^k = (\beta_1^k, ..., \beta_m^k)^T$. □

Proposition 4.4. *If there exists $k \in \mathbb{N}$ such that $x^{k+1} = x^k$, then x^k is a Pareto-Clarke critical point of F.*

Proof. Suppose that for any $k \in \mathbb{N}$, $x^{k+1} = x^k$, which implies that $\epsilon^k = 0$. Then by Proposition 4.3, we obtain

$$A_{k+1}^T(\alpha^{k+1} + \beta^{k+1}) + \tau_k w^{k+1} = 0, \tag{4.5}$$

which infers that

$$-A_{k+1}^T(\alpha^{k+1} + \beta^{k+1}) \in N_C(x^{k+1}). \tag{4.6}$$

Since $\sum_{i=1}^m (\alpha_i^{k+1} + \beta_i^{k+1}) = 1$, we can say that $(\alpha^{k+1} + \beta^{k+1}) \in \mathbb{R}_+^m \setminus \{0\}$. Moreover, $A_{k+1} \in \partial F(x^{k+1})$, then using Proposition 2.4, we obtain the desired result. □

Proposition 4.5. *Let $k_0 \in \mathbb{N}$ be such that $\alpha^{k_0} = 0$. Then x^{k_0} is a Pareto-Clarke critical point of F.*

Proof. If there exists $k_0 \in \mathbb{N}$ such that $\alpha_{k_0} = 0$ then, from (4.4), we have

$$A_{k_0}^T \beta^{k_0} + \tau_{k_0} w^{k_0} = 0, \tag{4.7}$$

where $\tau_{k_0} > 0$, $w^{k_0} \in N_C(x^{k_0})$. Since $A_{k_0} \in \partial F(x^{k_0})$ and $\beta^{k_0} \in \mathbb{R}_+^m \setminus \{0\}$, the desired result follows by using Proposition 2.4. □

From now on, we will assume the sequences $\{\lambda_k\}$, $\{\epsilon^k\}$ and $\{x^k\}$ are infinite sequences generated by Algorithm 4.1, then $\alpha^k \neq 0$ and $x^{k+1} \neq x^k$, in view of Proposition 4.4 and 4.5, respectively.
Next we prove that every cluster point of x^k, if any, is Pareto-Clarke critical point.

Theorem 4.1. *Assume that there exist scalars $a, b, c, d \in \mathbb{R}_{++}$ such that $a \leq \lambda_k \leq b$, $c \leq e_i^k \leq d$, $\sigma_k \leq d < 1$, for all $k \in \mathbb{N}$ and $i \in \{1, ..., m\}$. Then, every cluster point of $\{x^k\}$, if any, is a Pareto-Clarke critical point of F.*

Proof. Since

$$x^{k+1} \in \arg\min_{\epsilon^k q-w}\{F(x) + \frac{\lambda_k}{2}\|x - x^k\|^2 e^k | x \in \Omega_k\},$$

we have

$$\max_{1\leq i\leq m}\{f_i(x^k)-f_i(x^{k+1})+\epsilon_i^k\|x^k-x^{k+1}\|-\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_i^k\}\geq 0.$$

Hence for any k, there exists some index $i_0 := i_0(k) \in \{1,...,m\}$, where the maximum in the last inequality is attained. Thus,

$$f_{i_0}(x^k)-f_{i_0}(x^{k+1})+\epsilon_{i_0}^k\|x^k-x^{k+1}\|-\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k\geq 0,$$

which provides us

$$\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k-\epsilon_{i_0}^k\|x^k-x^{k+1}\|\leq f_{i_0}(x^k)-f_{i_0}(x^{k+1}).$$

By (4.3) and boundedness assumption of $\{\lambda_k\}$ and $\{e^k\}$, we obtain

$$\begin{aligned}f_{i_0}(x^k)-f_{i_0}(x^{k+1})&\geq\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k-\epsilon_{i_0}^k\|x^{k+1}-x^k\|\\&\geq\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k-\sigma_k\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k\\&\geq(1-\sigma_k)\frac{\lambda_k}{2}\|x^{k+1}-x^k\|^2e_{i_0}^k.\end{aligned}$$

Then, from the boundedness of $\{\lambda_k\}$, $\{\epsilon^k\}$ and $\{\sigma_k\}$, we obtain

$$(1-d)\frac{ac}{2}\|x^{k+1}-x^k\|^2\leq f_{i_0}(x^k)-f_{i_0}(x^{k+1}).\tag{4.8}$$

Combining (4.2) with the definition of Ω_k, it follows that $\{F(x^k)\}$ is nonincreasing sequence, and by assumption on F, i.e, $0\prec F$, we have that $\{F(x^k)\}$ is a convergent sequence. Hence, by taking $k\to+\infty$ on (4.8), we get

$$\lim_{k\to+\infty}(x^{k+1}-x^k)=0.\tag{4.9}$$

Take $\bar{x}$ as a cluster point of $\{x^k\}$, then there exists subsequence $\{x^{k_j}\}$ of $\{x^k\}$ converging to $\bar{x}$. Therefore, by applying Proposition 4.3 for the sequence $\{x^{k_j}\}$, we have that there exist sequences $A_{k_j+1}\in\partial F(x^{k_j+1})$, $\alpha^{k_j+1},\beta^{k_j+1}\in\mathbb{R}^m_+$ and $v^{k_j+1}\in B_{x^{k_j+1}}$ such that

$$A_{k_j+1}^T(\alpha^{k_j+1}+\beta^{k_j+1})+\lambda_{k_j}\langle e^{k_j},\alpha^{k_j+1}\rangle(x^{k_j+1}-x^{k_j})+\langle\epsilon^{k_j},\alpha^{k_j+1}\rangle v^{k_j+1}+\tau_{k_j+1}w^{k_j+1}=0,\tag{4.10}$$

where $\sum_{i=1}^m(\alpha_i^{k_j+1}+\beta_i^{k_j+1})=1$ and $w^{k_j+1}\in N_C(x^{k_j+1})$.

From the convergence of $\{x^{k_j}\}$, we obtain that $\{x^{k_j}\}$ is bounded. By locally Lipschitz property of F, it follows by (2.6) that their subgradients are bounded. So from the above conditions, the sequences A_{k_j}, v^{k_j}, α^{k_j}, β^{k_j}, w^{k_j} are bounded. Thus, equality (4.10) implies that τ_{k_j} is also bounded. Now, without loss of generality, we may assume that the sequences A_{k_j}, v^{k_j}, α^{k_j}, β^{k_j}, w^{k_j} and τ_{k_j} converge to $\bar{A}$, $\bar{v}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{w}$ and $\bar{\tau}$ respectively. Also, since $\lambda_{k_j}\langle e^{k_j},\alpha^{k_j+1}\rangle$ is bounded, then by letting k_j goes to infinity in (4.10), we obtain

$$\bar{A}^T(\bar{\alpha}+\bar{\beta})+\bar{\tau}\bar{w}=0.\tag{4.11}$$

Since $\bar{w}\in N_C(\bar{x})$, $(\bar{\alpha}+\bar{\beta})\in\mathbb{R}^m_+\setminus\{0\}$, $\bar{A}\in\partial F(\bar{x})$, it follows from (4.11) that

$$-\bar{A}^T(\bar{\alpha}+\bar{\beta})\in N_C(\bar{x}),$$

and this together with Proposition 2.4, enables us to say that $\bar{x}$ is a Pareto-Clarke critical point of F. This completes the proof. □

Next, we will present full covergence theorem of proposed Algorithm 4.1. The following definition and lemma will be useful in our proof.

Definition 4.6. Let $\Omega \subset \mathbb{R}^n$ be a nonempty set. A sequence $\{z^k\} \subset \Omega$ is said to be Fejér convergent to a nonempty set Ω iff, for all $z \in \Omega$,

$$\|z^{k+1} - z\|^2 \leq \|z^k - z\|^2 + \vartheta_k, \quad k = 0, 1, ...$$

where $\{\vartheta_k\} \subset (0, \infty)$ satisfies $\sum_{k=1}^{\infty} \vartheta_k < \infty$.

The following result on Fejér convergence is well known.

Lemma 4.7. [15] *Let $\Omega \subset \mathbb{R}^n$ be a nonempty set and $\{z^k\} \subset \Omega$ be a Fejér convergent sequence to Ω, then:*

- *The sequence $\{z^k\}$ is bounded.*
- *If a cluster point $\bar{z}$ of $\{z^k\}$ belongs to Ω, the whole sequence $\{z^k\}$ converges to $\bar{z}$ as k goes to $+\infty$.*

Now, we will consider that $F : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-quasiconvex, C is convex set, and the following well-known assumption.
$\mathcal{H}\mathbf{1}$: The set $(F(x^0) - \mathbb{R}^m_+) \cap F(C)$ is $\mathbb{R}^m_+$-complete.

Theorem 4.2. *Assume that $\mathcal{H}\mathbf{1}$ holds true and $\sum_{k=0}^{+\infty} \sigma_k < +\infty$. Then, the sequence $\{x^k\}$ generated by the Algorithm 4.1, converges to a Pareto-Clarke critical point of F.*

Proof. Define

$$E := \bigcap_{k=0}^{+\infty} \Omega_k.$$

Assumption $\mathcal{H}\mathbf{1}$ implies that E is nonempty. Take $x^* \in E$, which infers that $x^* \in \Omega_k$ for $k \in \mathbb{N}$. It is easy to see that:

$$\|x^k - x^*\|^2 = \|x^{k+1} - x^*\|^2 + \|x^k - x^{k+1}\|^2 + 2\langle x^k - x^{k+1}, x^{k+1} - x^*\rangle, \ \forall k \in \mathbb{N}. \tag{4.12}$$

Following the steps of the proof of Theorem 4.1,

$$\lambda_k \langle e^k, \alpha^{k+1}\rangle (x^k - x^{k+1}) = A_{k+1}^T(\alpha^{k+1} + \beta^{k+1}) + \langle \epsilon^k, \alpha^{k+1}\rangle v^{k+1} + \tau_{k+1} w^{k+1}, \ \forall k \in \mathbb{N}. \tag{4.13}$$

Now, combining (4.12) with (4.13), we get

$$\begin{aligned} &\frac{\lambda_k b_k}{2}\left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^k - x^{k+1}\|^2\right) \\ &= \left\langle A_{k+1}^T(\alpha^{k+1} + \beta^{k+1}) + \langle \epsilon^k, \alpha^{k+1}\rangle v^{k+1} + \tau_{k+1} w^{k+1}, x^{k+1} - x^* \right\rangle \\ &= \sum_{i=1}^m (\alpha_i^{k+1} + \beta_i^{k+1})\langle u_i^{k+1}, x^{k+1} - x^*\rangle + \sum_{i=1}^m \alpha_i^{k+1}\epsilon_i^k \langle v^{k+1}, x^{k+1} - x^*\rangle \\ &+ \tau_{k+1}\langle w^{k+1}, x^{k+1} - x^*\rangle, \end{aligned} \tag{4.14}$$

where $b_k = \langle e^k, \alpha^{k+1}\rangle$, $u_i^{k+1} \in \partial f_i(x^{k+1}), \forall k \in \mathbb{N}$ and $i \in \{1, ..., m\}$. Since F is $\mathbb{R}^m_+$-quasiconvex function, in particular, f_i is quasiconvex for each $i \in \{1, ..., m\}$. As $x^* \in \Omega_k$ and $u_i^{k+1} \in \partial f_i(x^{k+1})$, it follows by Proposition 2.12 that

$$\sum_{i=1}^m (\alpha_i^{k+1} + \beta_i^{k+1})\langle u_i^{k+1}, x^{k+1} - x^*\rangle \geq 0, \ \forall k \in \mathbb{N}. \tag{4.15}$$

As C is a convex set, $w^{k+1} \in N_C(x^{k+1})$ together with $\tau_{k+1} > 0$ and characterization of convex normal cone imply that

$$\tau_{k+1}\langle w^{k+1}, x^{k+1} - x^*\rangle \geq 0, \ \forall k \in \mathbb{N}. \tag{4.16}$$

By combining the inequalities (4.15), (4.16) with (4.14), we obtain

$$\begin{aligned} \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^k - x^{k+1}\|^2 &\geq -\frac{2}{\lambda_k b_k}\sum_{i=1}^{m} \alpha_i^{k+1}\epsilon_i^k \langle v^{k+1}, x^{k+1} - x^*\rangle \\ &\geq -\sigma_k \|x^{k+1} - x^k\|\|x^* - x^{k+1}\|, \ \forall k \in \mathbb{N}. \end{aligned} \tag{4.17}$$

As, $r + s \geq 2\sqrt{rs}$ holds for $r, s \geq 0$, taking $s := \|x^{k+1} - x^k\|$ and $r := \|x^* - x^{k+1}\|$, we obtain

$$\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 - \|x^k - x^{k+1}\|^2 \geq -\frac{\sigma_k}{2}\left[\|x^{k+1} - x^k\|^2 + \|x^* - x^{k+1}\|^2\right], \ \forall k \in \mathbb{N}.$$

Thus, we get

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \left(\frac{1}{1-\sigma_k}\right)\|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 \\ &\leq \left(1 + \frac{\sigma_k}{1-\sigma_k}\right)\|x^k - x^*\|^2, \ \forall k \in \mathbb{N}. \end{aligned} \tag{4.18}$$

Since $\sum_{k=0}^{\infty} \sigma_k^2 < +\infty$, it follows that

$$K_0 := \sum_{k=k_0}^{+\infty} \frac{2\sigma_k^2}{1 - 2\sigma_k^2} < +\infty \ \text{ and } \ K_1 := \prod_{j=k_0}^{+\infty}\left(1 + \frac{2\sigma_j^2}{1-2\sigma_j^2}\right) < +\infty.$$

By (4.18), observe that for all $k \geq k_0$

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \left(1 + \frac{2\sigma_k^2}{1-2\sigma_k^2}\right)\|x^k - x^*\|^2 \\ &\leq \left(1 + \frac{2\sigma_{k-1}^2}{1-2\sigma_{k-1}^2}\right)\left(1 + \frac{2\sigma_k^2}{1-2\sigma_k^2}\right)\|x^{k-1} - x^*\|^2 \\ &\ \ \vdots \\ &\leq \prod_{j=k_0}^{k}\left(1 + \frac{2\sigma_j^2}{1-2\sigma_j^2}\right)\|x^{k_0} - x^*\|^2 \\ &\leq \prod_{j=k_0}^{\infty}\left(1 + \frac{2\sigma_j^2}{1-2\sigma_j^2}\right)\|x^{k_0} - x^*\|^2 \\ &= K_1\|x^{k_0} - x^*\|^2. \end{aligned}$$

This shows that $\{x^k\}$ is bounded. Then (4.18) becomes

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \frac{2\sigma_k^2}{1-2\sigma_k^2}K^2, \ \forall k \in \mathbb{N}. \tag{4.19}$$

where $K = \sup_k \|x^k - x^*\|$. Take $\eta_k = \frac{2\sigma_k^2}{1-2\sigma_k^2}K^2$. Since $\eta_k > 0$ and $\sum_{k=1}^{\infty} \eta_k < +\infty$, we obtain that $\{x^k\}$ is quasi-Fejér convergent to E and boundedness of $\{x^k\}$ implies that the sequence $\{x^k\}$ has a cluster point $\bar{x}$. Since Theorem 4.1 implies that $\bar{x} \in E$. Therefore using Lemma 4.7 with $U = E$, we conclude that the whole sequence $\{x^k\}$ converges to $\bar{x}$ as k goes to $+\infty$, where $\bar{x}$ is a Pareto-Clarke critical point of F. □

Corollary 4.8. *If $C = \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex and locally Lipschitz function, then the sequence $\{x^k\}$ converges to a weak Pareto optimal point of F.*

Proof. It is immediate from Proposition 2.8. □

5. Conclusion

Bento et al. [5] proposed an exact proximal point method for nonconvex and non-differentiable constrained multiobjective optimization problems. Later, Bento et al. [6] extended the above work in the Riemannian context. Furthermore, for full convergence analysis, they assumed that the objective function is convex. After that Lucas Vidal de [23] proposed and analyzed an inexact version of proximal point method presented by Bento et al. [6]. They also derived the Fritz John necessary optimality condition in terms of Mordukovich subdifferential for convergence analysis of the algorithm.

In this article, we developed an inexact version of proximal point method of Bento et al. [5]. In terms of Clarke subdifferential, we introduced Fritz-John necessary optimality condition of ϵ-quasi weakly Pareto solution, which we apply for convergence analysis of our proposed method. We also presented that the proposed method is well defined and under some suitable conditions the sequence attained by our proposed method converges to a Pareto-Clarke critical point. The newly proposed inexact proximal point algorithm is important because of its practical point of view. Notice that, the proximal point method is a conceptual algorithm, and its computational performance strongly depends on the method used to solve the subproblems. Hence, in practice computations introduce numerical errors in order to solve the auxiliary minimization problems and these methods usually provide only approximate solutions of the subproblems. Clearly, it is very important, from the view of practice, to study the asymptotic behavior of iterations of the algorithm in the presence of computational errors.

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References

1. Apolinario, H.C., Papa Quiroz, E.A., Oliveria, P.R.: A scalarization proximal point method for quasiconvex multiobjective minimization. J. Glob. Optim. 64, 79–96 (2016)
2. Aussel, D.: Subdifferential properties of quasiconvex and pseudoconvex functions: unified approach. J. Optim. Theory Appl. 97(1), 29–45 (1998)
3. Burke, J. V., Ferris, M. C., Qian M.: On the Clarke subdifferential of the distance function of a closed set, J. Math. Anal. Appl., 166, 199–213 (1992)
4. Branke, J., Deb, K., Miettinen, K., Slowinski, R. eds.: Practical approaches to multiobjective optimization, Dagstuhl Seminar, Dagstuhl, Wadern, Germany, 06501, (2007)
5. Bento, G.D.C., Cruz Neto, J.X., López, G., Soubeyran, A., Souza, J.C.O.,: The proximal point method for locally Lipschitz functions in multiobjective optimization with application to the compromise problem. SIAM Journal on Optimization, 28(2), pp.1104-1120 (2018)
6. Bento, G.D.C., da Cruz Neto, J.X. and de Meireles, L.V.: Proximal Point Method for Locally Lipschitz Functions in Multiobjective Optimization of Hadamard Manifolds. Journal of Optimization Theory and Applications, 179(1), pp.37-52 (2018)
7. Burachik, R. S., Kaya, C. Y., Rizvi, M.M.: A new scalarization technique to approximate Pareto fronts of problems with disconnected feasible sets, J. Optim. Theory Appl., 162, 428-446 (2014)
8. Bolte, J., Daniilidis, A., Lewis, A., Shiota, M.: Clarke subgradients of stratifiable functions. SIAM J. Optim. 18, 556–572 (2007)
9. Ceng, Lu-Chuan, and Jen-Chih Yao. "Approximate proximal methods in vector optimization." European Journal of Operational Research 183(1), 1-19 (2007)
10. Clarke, H.F.: Optimization and Nonsmooth Analysis, Classics in Applied Mathematics. SIAM, New York (1983)

11. Clarke, F. H.: Generalized gradients and applications, Trans. Amer. Math. Soc., 205, 247–262 (1975)
12. Custodio, A.L., Madeira, J.F.A., Vaz, A.I.F., Vicente, L.N.: Direct multisearch for multiobjective optimization. SIAM J. Optim. 21, 1109–1140 (2011)
13. Chuong, T.D., Kim, D.S.: Approximate solutions of multiobjective optimization problems. Positivity 20, 187–207 (2016)
14. Chuasuk, P., A. Farajzadeh, and A. Kaewcharoen. An iterative algorithm for solving split feasibility problems and fixed point problems in p-uniformly convex and smooth Banach spaces, Journal of Computational Analysis and Applications, 49 (2020)
15. Combettes, P. L.: Quasi-Fejerian analysis of some optimization algorithms, in Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, vol. 8 of Studies in Computational Mathematics, pp. 115–152, North-Holland, Amsterdam, (2001)
16. Durea M., Strugariu, R.: Some remarks on proximal point algorithm in scalar and vectorial cases, Nonlinear Funct. Anal. Appl., 15, 307-319 (2010)
17. Eichfelder, G.: An adaptive scalarization method in multiobjective optimization, SIAM J. Optim., 19, 1694-1718 (2009)
18. Gal, T., Hanne, T.: On the development and future aspects of vector optimization and MCDM. A tutorial. In: Multicriteria Anal., (Coimbra, 1994), 130-145. Springer, Berlin (1997)
19. Gromicho, J.: Quasiconvex Optimization and Location Theory. Kluwer Academic Publishers, Dordrecht, The Netherlands (1998)
20. Gutiérrez, C., Jiménez, B., Novo, V.: ϵ-Pareto optimality conditions for convex multiobjective programming via max function. Numer. Funct. Anal. Optim. 27, 57–70 (2006)
21. Ioffe, A.D., Tihomirov, V.M.: Theory of extremal problems. North-Holland Publishing Co., Amsterdam-New York (1979)
22. Khonchaliew, M.; Farajzadeh, A.; Petrot, N. Shrinking Extragradient Method for Pseudomonotone Equilibrium Problems and Quasi-Nonexpansive Mappings, 11, 480, Symmetry (2019)
23. Meireles, Lucas Vidal de.: Proximal point methods for multiobjective optimization in riemannian manifolds (PhD Thesis), (2019).
24. Luc, D.T.: Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems. Springer, Berlin (1989)
25. Miettinen, K.M.: Nonlinear Multiobjective Optimization. Kluwer Academic Publishers, Boston (1999)
26. Mordukhovich, B.S., Shao, Y.H.: Nonsmooth sequential analysis in Asplund spaces. Trans. Amer. Math. Soc. 348(4), 1235–1280 (1996)
27. Mas-Colell, A., Whinston, M.D., Green, J.R.: Microeconomic Theory. Oxford University Press, New York, NY, USA (1995)
28. Penot J. P. Circa-Subdifferentials, Clarke Subdifferentials. In: Calculus Without Derivatives. Graduate Texts in Mathematics, vol 266. Springer, New York, NY (2013)
29. Quiroz, E.P., Ramirez, L.M. and Oliveira, P.R.: An inexact proximal method for quasiconvex minimization. European Journal of Operational Research, 246(3), pp.721-729 (2015)
30. Schy, A., Giesy, D.P.: Multicriteria optimization techniques for design of aircraft control systems. In: Stadler, W. (ed.) Multieriteria Optimization in Engineering and in the Sciences. Plenum Press, New York (1988)
31. Stadler, W.: Multicriteria Optimization in Engineering and in the Sciences. Plenum Press, New York (1988)
32. Son, T.Q., Van Tuyen, N. and Wen, C.F.: Optimality conditions for approximate Pareto solutions of a nonsmooth vector optimization problem with an infinite number of constraints arXiv preprint arXiv:1808.10100 (2018)
33. Souza, J.C.D.O.,: Proximal Point Methods for Lipschitz Functions on Hadamard Manifolds: Scalar and Vectorial Cases. Journal of Optimization Theory and Applications, 179(3), pp.745-760 (2018)
34. Vinter, R.B.: Optimal Control. Birkhauser, Basel (2000)