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ITERATIVE SCHEME FOR FIXED POINT PROBLEM OF ASYMPTOTICALLY NONEXPANSIVE SEMIGROUPS AND SPLIT EQUILIBRIUM PROBLEM IN HILBERT SPACES

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ABSTRACT. The main objective of this work is to modify the sequence $\{x_n\}$ of the explicit projection algorithm of asymptotically nonexpansive semigroups. We prove the strong convergence theorem of a sequence $\{x_n\}$ to the common fixed point of asymptotically nonexpansive semigroups and the solutions of split equilibrium problems. Our main results extended and improved the results of Pei Zhou and Gou-Jie Zhao [17] and many authors.

KEYWORDS: asymptotically nonexpansive mappings; asymptotically nonexpansive semigroup; fixed point; split equilibrium problem.
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1. Introduction

$\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C\times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by EP.

The split equilibrium problem was introduced by Moudafi [12], he considers the following pair of equilibrium problems in different spaces. Let H_1 and H_2 be two real Hilbert spaces, let $F_1 : C\times C \to \mathbb{R}$ and $F_2 : Q\times Q \to \mathbb{R}$ be nonlinear bifunctions and let $A : H_1 \to H_2$ be a bounded linear operator which C and Q are closed convex subsets of H_1 and H_2, respectively. Then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C. \tag{1.2}$$

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and such that

$$y^* \in Ax^* \in Q, \quad F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.3}$$

The solution set of SEP (1.2)-(1.3) is denote by $\Omega = \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\}$.

Recall that, a mapping $T : C \to C$ and a self mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0,1)$ and T is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. Recall also that a one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see, e. g., [15]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$

(ii) $T(s + t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(iii) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(iv) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

In 2010, Tian [16] introduced the following general iterative scheme for finding an element of set of solutions to the fixed point of nonexpansive mapping in a Hilbert space. Define the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n, \tag{1.4}$$

where B is $k-$Lipscitzian and $\eta-$strongly monotone operator. Then he prove that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, the sequence $\{x_n\}$ gererate by (1.4) converges strongly to the unique solution $x^* \in Fix(T)$ of variational inequality

$$\langle(\gamma f - \mu B)x^*, x - x^*\rangle \leq 0, \forall x \in F_{ix}(T). \tag{1.5}$$

In 2011, Ceng et al. [4] added the metric project to the method of Tian (1.4) and studied the following explicit iterative scheme to find fixed points:

$$x_{n+1} = P_C\left[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)Tx_n\right]. \tag{1.6}$$

They prove the strong converge of $\{x_n\}$ to a fixed point $x^* \in Fix(T)$ of the same variational in equality (1.5).

In 2008, Plubtieng and Punpaeng [13] introduced the following implicit iterative algorithm to prove a strong convergence theorem for fixed point problem with nonexpansive semigroup:

$$x_n = \alpha_n f(x_n) + (I - \alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \tag{1.7}$$

where x_n is a continuous net and s_n is a positive real divergent net.

In 2014, Kazmi and Rizvi [8] studied the following implcit iterative algorithm. Under some asummptions, they obtain some strong convergence theorem for EP(1.1) and the fixed point problem:

$$u_n = T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n),$$

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \tag{1.8}$$

where s_n and r_n are the continuous nets in $(0,1)$.

In the same year, Zhou and Zhao [17] introduce an explicit iterative scheme for finding a common element of the set of solutions SEP and fixed point for a nonexpansive semigroup in real Hilbert spaces. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} &= P_C\left[\alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right]. \end{aligned} \tag{1.9}$$

Under suitable conditions, some strong convergence theorems for approximating to these common elements are proved.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.1. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x-y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 1.2. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x-y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \not< 1$, then $\mathcal{T}$ is not necessary nonexpansive semigroup.

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

Inspired and motivate by above and [17], the purpose of this paper to introduce an explicit iterative scheme for finding a common element of the set of solutions SEP and fixed point for an asymptotically nonexpansive semigroup in real Hilbert spaces.

2. Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space, then the following hold:*
(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H$;
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, t \in [0,1], \forall x, y \in H$.
(iii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H$.

Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$

foe all $y \in C$, such P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \forall x \in H, z \in C. \tag{2.1}$$

It is easy to see that (2.1) is equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|P_C x - z\|^2, \quad \forall x \in H, z \in C. \tag{2.2}$$

Let $B : C \to H$ be a nonlinear mapping. Recall the following definitions.

Definition 2.2. B is said to be

(i) monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C, \tag{2.3}$$

(ii) strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C, \tag{2.4}$$

for such a case, B is said to be $\alpha-$strongly monotone,

(iii) $\alpha-$inverse strongly monotone($\alpha-$ism) if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C, \tag{2.5}$$

(iv) $k-$Lipschitz continuous if exists a constant $k \geq 0$ such that

$$\|Bx - By\| \leq k\|x - y\|, \quad \forall x, y \in C. \tag{2.6}$$

Remark 2.3. Let $\mathcal{F} = \mu B - \gamma f$, where B is a $k-$Lipschitz and $\eta-$strongly monotone operator on H with $k > 0$ and f is a Lipschitz mapping on H with coefficient $L > 0$, $0 < \gamma \leq \mu\eta/L$. It is a simple matter to see that the operator $\mathcal{F}$ is $(\mu\eta - \gamma L)-$strongly monotone over H; that is

$$\langle \mathcal{F}x - \mathcal{F}y, x - y\rangle \geq (\mu\eta - \gamma L)\|x - y\|^2, \quad \forall x, y \in H, \tag{2.7}$$

Lemma 2.4. [6] *Let T be a nonexpansive mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever the sequence of x_n is weakly convergent to x and $(I - T)x_n$ is strongly convergent to y, then $(I - T)x = y$.*

Lemma 2.5. [10] *Assume that A is a strongly positive linear bounded operator on Hilbert space H with coefficient $\tau > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\tau$.*

Lemma 2.6. [7] *Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, If $\{x_n\}$ is a sequence in C satisfying the properties:*
(i) $x_n \rightharpoonup z$; and
(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$,
then $z \in Fix(\mathcal{T})$.

Lemma 2.7. [7] *Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,*

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(u)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.8. [9] *Let T be an asymptotically nonexpansive mapping defined on a bounded convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $x \in F(T)$.*

Lemma 2.9. [11] *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|$$

for all $n \geq 1$, then $x_n \to q$.

Definition 2.10. [12] A mapping $T : H \to H$ is said to be averaged if it can be written as the average of the identity mapping and a nonexpansive mapping; that is,

$$T = (1-\epsilon)I + \epsilon S, \tag{2.8}$$

where $\epsilon \in (0,1), S : H \to H$ is nonexpansive, and I is the identity operator on H.

Proposition 2.11. [12]

(i) *If $T = (1-\epsilon)S + \epsilon V$, where $S : H \to H$ is averaged, $V : H \to H$ is nonexpansive, and $\epsilon \in (0,1)$, then T is averaged.*
(ii) *The composite of finite many averaged mappings is averaged.*
(iii) *If T is $\nu - ism$, then for $\gamma > 0, \gamma T$ is $(\nu/\gamma) - ism$.*
(iv) *T is averaged if and only if its complement $I - T$ is $\nu - ism$ for some $\nu > \frac{1}{2}$.*

Assumption 2.12. [1] *For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:*
($A1$) *$F(x,x) = 0$ for all $x \in C$;*
($A2$) *F is monotone, that is $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;*
($A3$) *for each $x, y \in C$,*

$$\lim_{t \to 0} F(tz + (1-t)x, y) \leq F(x,y); \tag{2.9}$$

($A4$) *for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.*

Lemma 2.13. [2] *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying ($A1$)($A4$). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(x,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r^F(x) = \left\{ z \in C : F(x,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}, \tag{2.11}$$

for all $x \in H$. Then the following hold:

(i) *T_r^F is single valued;*
(ii) *T_r^F is firmly nonexpansive; that is, for any $x, y \in H$*

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.12}$$

(iii) *$F(T_r^F) = \mathrm{EP}(F)$;*
(iv) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 2.14. [3] *Let C be a nonempty closed convex subset of a Hilbert space H, and let $F : C \times C \to \mathbb{R}$ be a bifunction. Let $x \in C$ and $r_1, r_2 \in (0, \infty)$. Then*

$$\left\|T_{r_1}^F x - T_{r_2}^F x\right\| \leq \left|1 - \frac{r_2}{r_1}\right| \left(\left\|T_{r_1}^F x\right\| + \|x\|\right). \tag{2.13}$$

Lemma 2.15. [14] *Assume that* $\{a_n\}, \{b_n\}, \{c_n\}$ *are sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n, n \geq 0$$

where $\{a_n\}$ *is asequence in* $(0, 1)$ *and* $\{b_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\Sigma_{n=0}^{\infty} c_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{b_n}{c_n} \leq 0$ *or* $\Sigma_{n=0}^{\infty}|b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Let $f : H_1 \to H_1$ be a contractive mapping with constant $\beta \in (0, 1)$ and let $A : H_1 \to H_2, B : H_1 \to H_1$ be a $\eta-$strongly monotone and $\theta-$Lipschitzian with $\theta > 0, \eta > 0$. In this work, we may assume that $0 < \mu < \frac{2\eta}{\theta^2}$ and $0 < \gamma < \mu(\eta - \frac{\mu\theta^2}{2})/\beta = \frac{\tau}{\beta}$. Let $\Im = \{T(s) : 0 \leq s < \infty\}$ be an aymptotically nonexpansive semigroup on C such that $\Gamma = F(\Im) \cap \Omega \neq \emptyset$. Assume $\{r_n\}$ and $\{s_n\}$ are the continuous nets of positive real numbers such that $\lim_{n\to 0} r_n = r > 0$ and $\lim_{n\to 0} s_n = +\infty$.

In this section, we introduce the following explicit iterative scheme that the nets $\{u_n\}$ and $\{x_n\}$ are generated by

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} &= P_C\left[\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right], \end{aligned} \tag{3.1}$$

where $P_C : H_1 \to C, \delta \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A.

We prove the strong convergence of $\{u_n\}$ and $\{x_n\}$ to a fixed point $x^* \in F(\Im)$ which solve the following variational inequality:

$$\langle(\mu F - \gamma g)x^*, x^* - \overline{x}\rangle \leq 0, \forall\overline{x} \in \Gamma = F(\Im) \cap \Omega. \tag{3.2}$$

In the sequel, we denote by $\{y_n\}$ the sequence defined by

$$y_n = \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds. \tag{3.3}$$

Theorem 3.1. *Let* H_1 *and* H_2 *be two real Hilbert spaces and let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed subsets. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *are the bifunctions satisfying Assumption 2.12 and* F_2 *is upper semicontinuous in the first argument. Let the sequence* $\{u_n\}$ *and* $\{x_n\}$ *be generated by (3.1), and suppose that the sequence* $\{\alpha_n\}$ *satisfies the following conditions:*

(i) $\alpha_n \in (0, 1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = 0$;
(iii) *either* $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

where $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_s^T ds \to 1$ *as* $n \to \infty$. *Then the sequence* $\{u_n\}$ *and* $\{x_n\}$ *converge strongly to* $x^* \in \Gamma = F(\Im) \cap \Omega$, *where* $x^* = P_\Gamma(I - \mu B + \gamma f)x^*$, *which is the unique solution of the variational inequality (3.2).*

Proof. For $\alpha_n \in (0, 1)$ and $\forall x \in H_1$, define a mapping $G : H_1 \to H_2$ by

$$Gx = P_C\left[\alpha_n\gamma f(x) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(x + \delta A^*(T_{r_n}^{F_2} - I)Ax)ds\right]. \tag{3.4}$$

From Lemma 2.13 we easily know that $T_{r_n}^{F_1}$ and $T_{r_n}^{F_2}$ both are firmly nonexpansive mappings and are averaged operators. From Proposition 2.11, we can obtain that the operator $(I+\delta A^*(T_{r_n}^{F_2}-I)A)$ is averaged and hence nonexpansive. Following Lemma 2.14 and $\forall x, y \in H_1$, we get

$$\begin{aligned}
\|Gx-Gy\| &= \|P_C\left[\alpha_n\gamma f(x)+(I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(x+\delta A^*(T_{r_n}^{F_2}-I)Ax)ds\right] \\
&- P_C\left[\alpha_n\gamma f(y)+(I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(y+\delta A^*(T_{r_n}^{F_2}-I)Ay)ds\right]\| \\
&\leq \|\left[\alpha_n\gamma f(x)+(I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(x+\delta A^*(T_{r_n}^{F_2}-I)Ax)ds\right] \\
&- \left[\alpha_n\gamma f(y)+(I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)T_{r_n}^{F_1}(y+\delta A^*(T_{r_n}^{F_2}-I)Ay)ds\right]\| \\
&\leq \alpha_n\gamma\|f(x)-f(y)\| \\
&\quad +(1-\alpha_n\tau)\|\frac{1}{s_n}\int_0^{s_n}[T(s)(T_{r_n}^{F_1}(x+\delta A^*(T_{r_n}^{F_2}-I)Ax)) \\
&\quad -T(s)(T_{r_n}^{F_1}(y+\delta A^*(T_{r_n}^{F_2}-I)Ay))ds]\| \\
&\leq \alpha_n\gamma\|f(x)-f(y)\| \\
&\quad +(1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n}\|T(s)(T_{r_n}^{F_1}(x+\delta A^*(T_{r_n}^{F_2}-I)Ax) \\
&\quad -T(s)(T_{r_n}^{F_1}(y+\delta A^*(T_{r_n}^{F_2}-I)Ay))\|ds \\
&\leq \alpha_n\gamma\|f(x)-f(y)\| \\
&\quad +(1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T\|T_{r_n}^{F_1}(x+\delta A^*(T_{r_n}^{F_2}-I)Ax) \\
&\quad -T_{r_n}^{F_1}(y+\delta A^*(T_{r_n}^{F_2}-I)Ay))\|ds \\
&\leq \alpha_n\gamma\|f(x)-f(y)\|+(1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T\,\|x-y\|\,ds \\
&\leq \alpha_n\gamma\|f(x)-f(y)\|+(1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T ds\,\|x-y\| \\
&\leq \alpha_n\gamma\beta\|x-y\|+(1-\alpha_n\tau)\widetilde{s_n}\|x-y\| \\
&= (1-\alpha_n(\tau\widetilde{s_n}-\gamma\beta))\|x-y\|. \qquad (3.5)
\end{aligned}$$

Since $\gamma<\frac{\tau}{\beta}$ and $\alpha_n\in(0,1)$ then $(1-\alpha_n(\tau\widetilde{s_n}-\gamma\beta))<1$, it follows that G is contraction, by Banach contraction principle, there exists a unique a fixed point x^*. Next, we proved that $\{u_n\},\{x_n\}$ are bounded. Let $p\in\Gamma=F(S)\cap\Omega$, we obtain that $p=T_{r_n}^{F_1}p$ and $p=T_{r_n}^{F_2}Ap$ and $p=T(s)p$. From (3.1), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n - p\|^2 \\
&= \|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n - T_{r_n}^{F_1}p\|^2 \\
&\leq \|x_n + \delta A^*(T_{r_n}^{F_2}-I)Ax_n - p\|^2 \\
&\leq \|x_n - p\|^2 + \|\delta A^*(T_{r_n}^{F_2}-I)Ax_n\|^2 + 2\delta\langle x_n - p, A^*(T_{r_n}^{F_2}-I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + \delta^2\langle (T_{r_n}^{F_2}-I)Ax_n, AA^*(T_{r_n}^{F_2}-I)Ax_n\rangle \\
&\quad +2\delta\langle A(x_n-p), (T_{r_n}^{F_2}-I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + L\delta^2\langle (T_{r_n}^{F_2}-I)Ax_n, (T_{r_n}^{F_2}-I)Ax_n\rangle \\
&\quad +2\delta\langle A(x_n-p) + (T_{r_n}^{F_2}-I)Ax_n - (T_{r_n}^{F_2}-I)Ax_n, A^*(T_{r_n}^{F_2}-I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + L\delta^2\|(T_{r_n}^{F_2}-I)Ax_n\|^2 \\
&\quad +2\delta\left\{\langle T_{r_n}^{F_2}Ax_n - Ap, (T_{r_n}^{F_2}-I)Ax_n\rangle - \|(T_{r_n}^{F_2}-I)Ax_n\|^2\right\} \\
&\leq \|x_n - p\|^2 + L\delta^2\|(T_{r_n}^{F_2}-I)Ax_n\|^2 \\
&\quad +2\delta\left\{\frac{1}{2}\|(T_{r_n}^{F_2}-I)Ax_n\|^2 - \|(T_{r_n}^{F_2}-I)Ax_n\|^2\right\} \\
&\leq \|x_n - p\|^2 + L\delta^2\|(T_{r_n}^{F_2}-I)Ax_n\|^2 - \delta\|(T_{r_n}^{F_2}-I)Ax_n\|^2 \\
&= \|x_n - p\|^2 + \delta(L\delta-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2. \qquad (3.6)
\end{aligned}$$

Since $\delta \in (0, 1/L)$, we have

$$\|u_n - p\| \leq \|x_n - p\|. \qquad (3.7)$$

Put $y_n = \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds$, it follows that

$$\begin{aligned}
\|y_n - p\| &= \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&\leq \frac{1}{s_n}\left\|\int_0^{s_n}(T(s)u_n - T(s)p)ds\right\| \\
&\leq \|u_n - p\| \leq \|x_n - p\|. \qquad (3.8)
\end{aligned}$$

And we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|P_C\left[\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right] - p\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&= \left\|\alpha_n(\gamma f(x_n) - \mu Bp) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - (I-\mu\alpha_n B)p\right\| \qquad (3.9) \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1-\alpha_n\tau)\left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n}\|T(s)u_n - T(s)p\|\,ds \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T\,\|u_n - p\|\,ds \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1-\alpha_n\tau)\frac{1}{s_n}\int_0^{s_n} L_s^T ds\,\|u_n - p\| \\
&\leq \alpha_n\|\gamma f(x_n) - \mu Bp\| + (1-\alpha_n\tau)\widetilde{s_n}\,\|u_n - p\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-\mu Bp\|+(1-\alpha_n\tau)\widetilde{s_n}\,\|u_n-p\| \\
&\leq \alpha_n\gamma\beta\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Bp\|+(1-\alpha_n\tau)\widetilde{s_n}\,\|x_n-p\| \\
&\leq [\widetilde{s_n}-\alpha_n(\tau\widetilde{s_n}-\gamma\beta)]\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Bp\| \qquad (3.10)
\end{aligned}
$$

Since $\{\widetilde{s_n}-\alpha_n(\tau\widetilde{s_n}-\gamma\beta)\}$ is convergence sequence of real number then it is a bounded dequence, we have $K\in\mathbb{R}$ such that

$$\|x_{n+1}-p\|\leq K\|x_n-p\|+\alpha_n\|\gamma f(p)-\mu Bp\|, \qquad (3.11)$$

we have $\{x_n\}$ is bounded and therefore $\{u_n\},\{y_n\}$ and $\{f(x_n)\}$ are bounded. From (3.10), $\{\|x_n-p\|\}$ is bounded and decreasing sequence, hence $\lim_{n\to\infty}\|x_n-p\|$ exists.

Next, we claim that $\lim_{n\to\infty}\|x_n-u_n\|=0$. From (3.9), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \left\|\alpha_n(\gamma f(x_n)-\mu Bp)+(I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds-(I-\mu\alpha_n B)p\right\|^2 \\
&\leq (1-\alpha_n\tau)^2\left\|\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds-p\right\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-\gamma f(p)+\gamma f(p)-\mu Bp, x_n-p\rangle \\
&\leq (1-\alpha_n\tau)^2\|u_n-p\|^2+2\alpha_n\gamma\beta\|x_n-p\| \\
&\quad +2\alpha_n\langle\gamma f(p)-\mu Bp, x_n-p\rangle \\
&\leq \|u_n-p\|^2+\alpha_n\tau^2\|x_n-p\|^2+2\alpha_n\gamma\beta\|x_n-p\| \\
&\quad +2\alpha_n\|\gamma f(p)-\mu Bp\|\|x_n-p\| \\
&\leq \|x_n-p\|^2+\delta(L\delta-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2+\alpha_n\tau^2\|x_n-p\|^2 \\
&\quad +2\alpha_n\gamma\beta\|x_n-p\|+2\alpha_n\|\gamma f(p)-\mu Bp\|\|x_n-p\|. \qquad (3.12)
\end{aligned}
$$

From (3.12), we obtain

$$
\begin{aligned}
\delta(1-L\delta)\|(T_{r_n}^{F_2}-I)Ax_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&\quad +\alpha_n(\tau^2\|x_n-p\|^2+2\gamma\beta\|x_n-p\| \\
&\quad\quad +2\|\gamma f(p)-\mu Bp\|\|x_n-p\|). \qquad (3.13)
\end{aligned}
$$

Since $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n-p\|$ exists, $\lim_{n\to\infty}\alpha_n=0$ and $\delta(1-L\delta)>0$, we obtain that

$$\lim_{n\to\infty}\|(T_{r_n}^{F_2}-I)Ax_n\|=0. \qquad (3.14)$$

From (3.1), we have

$$
\begin{aligned}
\|u_n-p\|^2 &= \left\|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n-p\right\|^2 \\
&= \left\|T_{r_n}^{F_1}(I+\delta A^*(T_{r_n}^{F_2}-I)A)x_n-T_{r_n}^{F_1}p\right\|^2 \\
&\leq \langle u_n-p, x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n-p\rangle \\
&= \frac{1}{2}\{\|u_n-p\|^2+\|x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n-p\|^2 \\
&\quad -\|u_n-p-[x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n-p]\|^2\} \qquad (3.15) \\
&\leq \frac{1}{2}\left\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n-\delta A^*(T_{r_n}^{F_2}-I)Ax_n\|^2\right\} \\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2-\delta\|A^*(T_{r_n}^{F_2}-I)Ax_n\|^2 \\
&\quad +2\delta\|A(u_n-x_n)\|\|(T_{r_n}^{F_2}-I)Ax_n\|\}.
\end{aligned}
$$

Hence, we obtain

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n - p\|^2 - \|u_n - x_n\|^2 - \delta\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 \\ &\quad +2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\| \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\|. \end{aligned} \tag{3.16}$$

It follows from (3.12) and (3.16) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|u_n - p\|^2 + \alpha_n\tau^2\|x_n - p\|^2 + 2\alpha_n\gamma\beta\|x_n - p\| \\ &\quad +2\alpha_n\|\gamma f(p) - \mu Bp\|\|x_n - p\| \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\| \\ &\quad +\alpha_n\tau^2\|x_n - p\|^2 \\ &\quad +2\alpha_n\gamma\beta\|x_n - p\| + 2\alpha_n\|\gamma f(p) - \mu Bp\|\|x_n - p\| \qquad (3.17) \\ &= \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\| \\ &\quad +\alpha_n\tau^2 M_1, \end{aligned}$$

where $M_1 = \tau^2\|x_n - p\|^2 + 2\gamma\beta\|x_n - p\| + 2\|\gamma f(p) - \mu Bp\|\|x_n - p\|$. From (3.17), we obtain

$$\|u_n - x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + +2\delta\|A(u_n - x_n)\|\|(T_{r_n}^{F_2} - I)Ax_n\| + \alpha_n\tau^2 M_1 \tag{3.18}$$

From (3.18), (3.14), $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - p\|$ exists $\lim_{n\to\infty}\alpha_n = 0$ and $\delta > 0$, we obtain

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.19}$$

Next, we prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (1.4) and Lemma 2.14, we have

$$\begin{aligned} \|u_n - u_{n+1}\| &= \|T_{r_n}^{F_1}(I + \delta A^*(T_{r_n}^{F_2} - I)A)x_n - T_{r_n}^{F_1}(I + \delta A^*(T_{r_{n-1}}^{F_2} - I)A)x_{n-1}\| \\ &\leq \|(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - (x_{n-1} + \delta A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1})\| \\ &\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\ &\qquad - (x_{n-1} + \delta A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1})\| \\ &\leq \|x_n - x_{n-1} - \delta A^*A(x_n - x_{n-1})\| \\ &\quad +\delta\|A\|\|T_{r_{n-1}}^{F_2}Ax_n - T_{r_{n-1}}^{F_2}Ax_{n-1}\| \\ &\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\ &\qquad - (x_{n-1} + \delta A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1})\| \\ &\leq \left(\|x_n - x_{n-1}\|^2 - 2\delta\|A(x_n - x_{n-1})\|^2 + \delta^2\|A\|^4\|x_n - x_{n-1}\|^2\right)^{\frac{1}{2}} \\ &\quad +\delta\|A\|\left(\|Ax_n(x_n - x_{n-1})\| + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\|\right) \\ &\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\ &\qquad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\ &\leq \left(1 - 2\delta\|A\|^2 + \delta^2\|A\|^4\right)^{\frac{1}{2}}\|x_n - x_{n-1}\| \\ &\quad +\delta\|A\|^2\left(\|x_n - x_{n-1}\| + \|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\|\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\
&\qquad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
&\le (1 - \delta\|A\|^2)\|x_n - x_{n-1}\| + \delta\|A\|^2(\|x_n - x_{n-1}\| \\
&\qquad + |1 - \delta\|A\|\frac{r_{n-1}}{r_n}|\|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\|) \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\
&\qquad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
&= \|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|\|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\| \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) \\
&\qquad - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\| \\
&= \|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n), \qquad (3.20)
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_n &= \|T_{r_n}^{F_2}Ax_n - Ax_{n-1}\| \\
\xi_n &= \|T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n) - (x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n)\|. \qquad (3.21)
\end{aligned}
$$

From (3.3), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\| \\
&\le \|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_{n-1}ds\| \\
&\quad + \|\frac{1}{s_n}\int_0^{s_n} T(s)u_{n-1}ds - \frac{1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\| \\
&\le \frac{1}{s_n}\int_0^{s_n} \|T(s)(u_n - u_{n-1})\|ds \\
&\quad + \|\frac{1}{s_n}\int_0^{s_n} T(s)u_{n-1}ds - \frac{s_1}{s_{n-1}}\int_0^{s_{n-1}} T(s)u_{n-1}ds\| \\
&\le \widetilde{s_n}\|u_n - u_{n-1}\| + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| \\
&\quad + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|. \qquad (3.22)
\end{aligned}
$$

From (3.20) and (3.22), we obtain

$$
\begin{aligned}
\|y_n - y_{n-1}\| &\le \|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n) \\
&+ \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|. \qquad (3.23)
\end{aligned}
$$

From (3.1) again, we obtain

$$
\|x_{n+1} - x_n\| = \|P_C[\alpha_n\gamma f(x_n) + (I - \mu\alpha_n B)y_n]
$$

$$
\begin{aligned}
&\quad - P_C\left[\alpha_{n-1}\gamma f(x_{n-1}) + (I-\mu\alpha_{n-1}B)y_{n-1}\right]\| \\
\leq\ & \|(\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)y_n) - (\alpha_{n-1}\gamma f(x_{n-1}) \\
&\qquad + (I-\mu\alpha_{n-1}B)y_{n-1})\| \\
=\ & \|(\alpha_n\gamma(f(x_n) - f(x_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1}) \\
&\quad + (I-\mu\alpha_n B)(y_n - y_{n-1}) + \mu(\alpha_n - \alpha_{n-1})y_{n-1})\| \\
\leq\ & \alpha_n\gamma\beta\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + (I-\alpha_n\tau)\|y_n - y_{n-1}\| + \mu|\alpha_n - \alpha_{n-1}|\|y_{n-1})\| \\
\leq\ & \alpha_n\gamma\beta\|x_n - x_{n-1}\| + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
& +(I-\alpha_n\tau)(\|x_n - x_{n-1}\| + \delta\|A\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n) \\
& + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|) \\
& +\mu|\alpha_n - \alpha_{n-1}|\|y_{n-1}\| \\
=\ & (1-\alpha_n(\tau-\gamma\beta))(\|x_n - x_{n-1}\| \\
&\quad + \gamma|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\||1 - \frac{r_{n-1}}{r_n}|(\delta\|A\|\varepsilon_n + \xi_n)) \\
& + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| \left\|\int_0^{s_{n-1}} T(s)u_{n-1}ds\right\| + \frac{1}{s_n}\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\| \\
& +\mu|\alpha_n - \alpha_{n-1}|\|y_{n-1}\| \\
\leq\ & (1-\alpha_n(\tau-\gamma\beta))\|x_n - x_{n-1}\| \\
& +M_2(\gamma|\alpha_n - \alpha_{n-1}| + \left|1 - \frac{r_{n-1}}{r_n}\right| + \left|\frac{1}{s_n} - \frac{1}{s_{n-1}}\right| + \left|\frac{1}{s_{n-1}}\right| \\
&\quad + \mu|\alpha_n - \alpha_{n-1}|),
\end{aligned}
\tag{3.24}
$$

where

$$
M_2 = \max\left\{\sup_{n\leq 1}(\delta\|A\|\varepsilon_n + \xi_n), \sup_{n\leq 1}\left(\left\|\int_{s_{n-1}}^{s_n} T(s)u_{n-1}ds\right\|\right), \sup_{n\leq 1}\|y_{n-1}\|\right\}. \tag{3.25}
$$

Since $\{x_n\}, \{u_n\}$ and $\{y_n\}$ are bounded, we have $\{Ax_n\}$ and $\{T(s)u_{n-1}\}$ are bounded. Then $M_2 < \infty$.

It follows from condition $(1)-(3)$ we have $\lim_{n\to\infty} r_n = r > 0$, $\lim_{n\to\infty} s_n = +\infty$ and Lemma 2.15, we obtain that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.26}
$$

Next, we claim that $\lim_{n\to\infty}\|T(s)x_n - x_n\| = 0$. From (3.1) and (3.3), we obtain

$$
\begin{aligned}
\|x_{n+1} - y_n\| &\leq \left\|P_C\left[\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right] - P_C y_n\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I-\mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - y_n\right\| \\
&\leq \alpha_n\left\|\gamma f(x_n) - \mu B\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|.
\end{aligned}
\tag{3.27}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$ and $\{x_n\}, \{u_n\}$ are bounded, we have

$$
\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.28}
$$

From (3.26) and (3.28), we get

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, \tag{3.29}$$

it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.30}$$

On the other hand, from (3.1), we have

$$\begin{aligned}
\|T(s)x_n - x_n\| &= \left\|T(s)x_n - T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&\quad + \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n\right\| \\
&\leq \left\|x_n - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&\quad + \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| \\
&\quad + \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n\right\| \\
&\leq 2\|x_n - y_n\| + \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|,
\end{aligned} \tag{3.31}$$

So without loss of generality, we assume that $\Im = \{T(s) : 0 \leq s < +\infty\}$ is asymptotically nonexpansive semigroup on C, and from Lemma 2, we have

$$\lim_{n\to\infty} \left\|T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\| = 0. \tag{3.32}$$

It follows from (3.30), (3.31) and (3.32), we have

$$\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0. \tag{3.33}$$

Next, we claim that there exists a common fixed point of $EP(F_1) \cap EP(F_2)$.

Since $\{x_n\}$ is bounded on Hilbert space, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some $z \in X$. From (3.19), $y_{n_i} \rightharpoonup z$. Now, we show that $z \in EP(F_1)$. From (3.1) and $(A2)$, for any $y \in H$, we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n) \tag{3.34}$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i}}{-} x_{n_i} r_{n_i} \right\rangle \geq F_1(y, u_{n_i}). \tag{3.35}$$

Since $\frac{u_{n_i}}{-} x_{n_i} r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup z$, from $(A1)$, it follows that $0 \geq F_1(y, z)$ for all $y \in H$. For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1-t)z$, then we get $0 \geq F_1(y_t, z)$. From $(A1)$ and $(A2)$, we have

$$0 = F_1(y_t, y_t) \leq tF_1(y_t, y) + (1-t)F_1(y_t, z) \leq tF_1(y_t, y) \tag{3.36}$$

and hence $0 \leq F_1(y_t, y)$. From $(A3)$, we have $0 \leq F_1(z, y)$ for all $y \in H$. Therefor, $z \in EP(F_1)$.

Since $x_{n_i} \rightharpoonup z$ and A is a bounded linear operator, we obtain $Ax_{n_i} \rightharpoonup Az$. Let $v_{n_j} = Ax_{n_j} - T_{r_{n_j}}^{F_2} x_{n_j}$. It follows from (3.14), we have $\lim_{n\to\infty} v_{n_j} = 0$ and $Ax_{n_j} - v_{n_j} = T_{r_{n_j}}^{F_2} x_{n_j}$. Then from Lemma 2.13, we get

$$F_2(Ax_{n_j} - v_{n_j}, y) + \frac{1}{r_{n_j}} \left\langle y - (Ax_{n_j} - v_{n_j}), (Ax_{n_j} - v_{n_j}) - Ax_{n_j} \right\rangle \geq 0, \forall y \in Q. \tag{3.37}$$

Since F_2 is upper semicontinuous in the first argument, and $\limsup_{n\to\infty} r_n = r > 0$, we taking $j \to \infty$, we have

$$F_2(Az - v_{n_j}, y) \geq 0, \forall y \in Q \tag{3.38}$$

that is $Az \in EP(F_2)$ and hence $z \in \Omega$.

Next, we claim that $\langle (\mu F - \gamma f)x^*, x^* - \overline{x} \rangle \leq 0, \forall \overline{x} \in \Gamma = F(S) \cap \Omega$. From (3.1), putting

$$z_n = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \tag{3.39}$$

we can observe that

$$x_{n+1} = P_C z_n \quad = \quad P_C z_n - z_n + \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \tag{3.40}$$

it follows that

$$(\mu B - \gamma f)x_n \quad = \quad \frac{1}{\alpha_n}(P_C z_n - z_n) + \frac{1}{\alpha_n}(x_n - x_{n+1}) + \frac{1}{\alpha_n}(I - \mu\aleph_n B)(y_n - x_n). \tag{3.41}$$

Hence, for each $p \in \Gamma = F(S) \cap \Omega$, we obtain that

$$\begin{aligned} \langle (\mu B - \gamma f)x_n, x_n - p\rangle &= \frac{1}{\alpha_n}\langle P_C z_n - z_n, x_n - p\rangle + \frac{1}{\alpha_n}\langle x_n - x_{n+1}, x_n - p\rangle \\ &\quad + \frac{1}{\alpha_n}\langle (I - \mu\aleph_n B)(y_n - x_n), x_n - p\rangle \\ &= \frac{1}{\alpha_n}\langle P_C z_n - z_n, x_n - p\rangle + \frac{1}{\alpha_n}\langle x_n - x_{n+1}, x_n - p\rangle \\ &\quad + \frac{1}{\alpha_n}\langle y_n - x_n, x_n - p\rangle + \frac{1}{\alpha_n}\langle By_n - Bx_n, x_n - p\rangle. \end{aligned} \tag{3.42}$$

From (3.42) taking limit $n \to \infty$, we have $By_n - Bx_n \to Bx^* - Bx^* = 0, y_n - x_n \to 0$ and $P_C z_n - z_n \to P_C x^* - x^* = 0$, we have

$$\langle (\mu B - \gamma f)x_n, x_n - p\rangle \leq 0, \tag{3.43}$$

which implies that $z = P_\Gamma(I - \mu B + \gamma f)$.

Next, we claim that $z \in \Gamma = F(S) \cap \Omega$. From (3.1), we have $x_{n+1} = P_C z_n$, and for $x^* \in \Gamma$, we have

$$\begin{aligned} x_{n+1} - x^* &= P_C z_n - z_n + z_n - x^* \\ &= P_C z_n - z_n + \alpha_n(\gamma f(x_n) - \mu Bx^*) + (I - \mu\alpha_n B)y_n - (I - \mu\alpha_n B)x^*. \end{aligned} \tag{3.44}$$

Since P_C is the metric projection from H_1 onto C, we obtain

$$\langle P_C z_n - z_n, P_C z_n - x^* \rangle \leq 0. \tag{3.45}$$

It follows from (3.44) and (3.45), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle P_C z_n - z_n, x_{n+1}-x^*\rangle + \alpha_n\langle(\gamma f(x_n)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +\langle(I-\mu\alpha_n B)(y_n-x^*), x_{n+1}-x^*\rangle \\
&\leq \alpha_n\langle(\gamma f(x_n)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +\langle(I-\mu\alpha_n B)(y_n-x^*), x_{n+1}-x^*\rangle \\
&\leq \alpha_n\gamma\langle f(x_n)-f(x^*), x_{n+1}-x^*\rangle \\
&\quad +\alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +\langle(I-\mu\alpha_n B)(y_n-x^*), x_{n+1}-x^*\rangle \\
&\leq \alpha_n\gamma\beta\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +(1-\alpha_n\tau)\|y_n-x^*\|\|x_{n+1}-x^*\| \\
&\leq \alpha_n\gamma\beta\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\quad +(1-\alpha_n\tau)\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\leq (1-\alpha_n(\tau-\gamma\beta))\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\quad +\alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\leq \frac{(1-\alpha_n(\tau-\gamma\beta))}{2}\left(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\right) \\
&\quad +\alpha_n\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle, \qquad (3.46)
\end{aligned}$$

it implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\leq \frac{(1-\alpha_n(\tau-\gamma\beta))}{(1+\alpha_n(\tau-\gamma\beta))}\|x_n-x^*\|^2 \\
&\quad +\frac{2\alpha_n}{(1+\alpha_n(\tau-\gamma\beta))}\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\leq (1-\alpha_n(\tau-\gamma\beta))\|x_n-x^*\|^2 \\
&\quad +\frac{2\alpha_n}{(1+\alpha_n(\tau-\gamma\beta))}\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle \\
&\leq (1-a_n)\|x_n-x^*\|^2+\alpha_n b_n, \qquad (3.47)
\end{aligned}$$

where

$$\begin{aligned}
a_n &= \alpha_n(\tau-\gamma\beta), \\
b_n &= \frac{2}{(1+\alpha_n(\tau-\gamma\beta))}\langle\gamma f(x^*)-\mu Bx^*), x_{n+1}-x^*\rangle. \qquad (3.48)
\end{aligned}$$

We see that $\Sigma_{n=0}^{\infty} = +\infty$ and $\limsup_{n\to\infty} b_n \leq 0$. From Lemma 2.15, we have $x_n \to x^*$. This completes the proof. □

Corollary 3.2. [17] *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.12 and F_2 is upper semicontinuous in the first argument. Let $\Im = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $\Gamma = F(\Im) \cap \Omega \neq \emptyset$. Let the sequence $\{u_n\}$ and $\{x_n\}$ be generated by (3.1), and suppose that the sequence $\{\alpha_n\}$ satisfies the following conditions:*

(i) *$\alpha_n \in (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$;*
(ii) *$\Sigma_{n=0}^{\infty}\alpha_n = 0$;*
(iii) *either $\Sigma_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.*

where $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_s^T ds \to 1$ *as* $n \to \infty$. *Then the sequence* $\{u_n\}$ *and* $\{x_n\}$ *generated by (3.1) converge strongly to* $x^* \in \Gamma = F(\Im) \cap \Omega$, *where* $x^* = P_\Gamma(I - \mu B + \gamma f)x^*$, *which is the unique solution of the variational inequality (3.2).*

Proof. From example 1.1 and example 1.2, we see that a nonexpansive semigroup is $\mathcal{T}$ is asymptotically nonexpansive semigroup. Then this theorem cover by theorem 3.1. □

Corollary 3.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces and let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty closed subsets. Let* $A : H_1 \to H_2$ *be a bounded linear operator. Assume that* $F_1 : C \times C \to \mathbb{R}$ *and* $F_2 : Q \times Q \to \mathbb{R}$ *are the bifunctions satisfying Assumption 2.12 and* F_2 *is upper semicontinuous in the first argument. Let the sequence* $\{u_n\}$ *and* $\{x_n\}$ *be generated by are generated by*

$$\begin{aligned} u_n &= T_{r_n}^{F_1}(x_n + \delta A^*(T_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \mu\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{aligned} \tag{3.49}$$

the sequence $\{\alpha_n\}$ *satisfies the following conditions:*

(i) $\alpha_n \in (0,1)$ *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\Sigma_{n=0}^{\infty}\alpha_n = 0$;
(iii) *either* $\Sigma_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

where $\widetilde{s_n} = \frac{1}{s_n}\int_0^{s_n} L_s^T ds \to 1$ *as* $n \to \infty$. *Then the sequence* $\{u_n\}$ *and* $\{x_n\}$ *converge strongly to* $x^* \in \Gamma = F(\Im) \cap \Omega$, *where* $x^* = P_\Gamma(I - \mu B + \gamma f)x^*$, *which is the unique solution of the variational inequality (3.2).*

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