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COUPLED COINCIDENCE POINT THEOREMS OF MAPPINGS IN PARTIALLY ORDERED METRIC SPACES

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ABSTRACT. In this paper, we introduce a new generalized weakly contractive condition involving expressions of Kannan type contraction and establish coupled coincidence point and coupled common fixed point theorems of a pair of mappings satisfying the new contractive condition.

KEYWORDS: Coupled coincidence point; Coupled common fixed point; Mixed g- monotone property; Partially ordered set.
AMS Subject Classification: 47H10, 54F05

1. Introduction

Nowadays, fixed point techniques are widely applied in many branches of mathematics, especially in nonlinear analysis. One of the most important theorems in this regard is the fundamental theorem in metric fixed point theory, known as Banach contraction principle, which guarantees the existence and uniqueness of fixed point of contraction mappings (a mapping $T : X \to X$ is called a contraction if there exists a constant $c \in [0, 1)$ such that $d(T(x), T(y)) \leq c \cdot d(x, y),\ \forall\ x, y \in X$) defined on a complete metric space. There are many generalizations and extensions of this important result in literature (see, for example [8, 9, 11, 12, 18]). One of the notable extensions of this into partially ordered metric space is done by Ran and Reurings [16]. Further, a lot of research work is done in this line, including the results of Nieto and Lopez [14, 15]. By weakening the condition on contraction, Alber et al. [1] introduced weakly contractive maps and generalized the Banach contraction principle in Hilbert spaces. Afterwards Rhodes [18] obtained a fixed point theorem for weakly contractive maps in complete metric spaces. Followed by this, fixed points of weakly contractive maps and generalized weakly contractive maps are studied.
It is very clear that contraction maps are continuous, so the Banach contraction

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principle is applicable only for continuous functions. But Kannan [12] established a fixed point theorem for functions satisfying contraction condition called Kannan contraction, which need not be continuous.
In 2006, Gnana Bhaskar and Lakshmikantham followed the method of Nieto and Lopez, to weaken the contraction condition by considering a partial order on the metric space, and established coupled fixed point theorems of mixed monotone mappings on partially ordered complete metric space. Thereafter several research work dealing with coupled fixed point theorems are carried out. In 2009, Lakshmikantham and Ciric introduced a new concept called mixed g- monotone mapping and established coupled coincidence point and coupled common fixed point theorems for a mapping had a g and mixed g- monotone property. Also in 2011, Berinde [2] extended the result of Gnana Bhaskar and Lakshmikantham for mixed monotone mappings by weakening the contractive condition as follows:

$$d(F(x,y),F(u,v))+d(F(y,x),F(v,u)) \leq k\cdot[d(x,u)+d(y,v)]\ \forall\ x \geqslant u,\ y \leqslant v$$

Followed by this, several authors have done research in coupled, coupled coincidence and coupled common fixed points of mappings satisfying various contractive type conditions [3, 4, 5, 6, 7]. In 2011, Choudhary et al. [6] established the existence of coupled coincidence points for pairs of mappings g and mixed g- monotone mappings, which are compatible and satisfying the following contractive type condition:

$$\psi(d(F(x,y),F(u,v))) \leq \psi(\max\{d(gx,gu),d(gy,gv)\})-\phi(\max\{d(gx,gu),d(gy,gv)\}) \tag{1.1}$$

for all $x,y,u,v \in X$ for which $gx \leqslant gu$ and $gy \geqslant gv$, where ψ,ϕ are two control functions satisfying different conditions.
Inspired by the contractive type conditions defined by Berinde [2] and Choudhary et al. [6] and by incorporating the expressions of Kannan type contraction, we have introduced a new contractive type condition. In this paper, we have proved coupled coincidence point and coupled common fixed point theorems for pairs of mappings satisfying the newly introduced contractive condition under the settings of complete metric spaces.

2. Preliminaries

Some useful definitions are given in this section.

Definition 2.1. [13] Let $(X,\leqslant)$ be a partially ordered set and $F: X\times X \longrightarrow X$ and $g: X\longrightarrow X$. We say F has the mixed g- monotone property if F is monotone g- non-decreasing in its first argument and is monotone g- non-increasing in its second argument, that is, for any $x,y\in X$
$x_1,x_2 \in X,\ g(x_1)\leqslant g(x_2) \Longrightarrow F(x_1,y)\leqslant F(x_2,y)$ and
$y_1,y_2 \in X,\ g(y_1)\leqslant g(y_2) \Longrightarrow F(x,y_1)\geqslant F(x,y_2)$.

Definition 2.2. [5] Let (X,d) be a metric space, $F: X\times X \longrightarrow X$ and $g: X\longrightarrow X$ be two mappings. Mappings F and g are said to be compatible if

$$\begin{aligned}\lim_{n\longrightarrow\infty} d(g(F(x_n,y_n)),F(g(x_n),g(y_n))) &= 0,\ \textit{and}\\ \lim_{n\longrightarrow\infty} d(g(F(y_n,x_n)),F(g(y_n),g(x_n))) &= 0\end{aligned}$$

hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that
$\lim_{n\longrightarrow\infty} F(x_n,y_n)=\lim_{n\longrightarrow\infty} g(x_n)=x$ and $\lim_{n\longrightarrow\infty} F(y_n,x_n)=\lim_{n\longrightarrow\infty} g(y_n)=y$ for some $x,y\in X$ are satisfied.

Definition 2.3. [10] An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.4. [13] An element $(x, y) \in X \times X$ is said to be a coupled coincidence point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 2.5. [13] An element $(x, y) \in X \times X$ is said to be a coupled common fixed point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x, y) = g(x) = x$ and $F(y, x) = g(y) = y$.

Definition 2.6. [17] A function $f : X \to [0, \infty)$, where X is a metric space, is called lower semi continuous, if for all $x \in X$ and $\{x_n\} \subseteq X$ with $\lim_{n\to\infty} x_n = x$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

3. Main Results

In this section, we prove five coupled coincidence point theorems for pairs of mapping g and mixed g- monotone mappings. The first three theorems discuss the existence of coupled coincidence points. One of the results assures the uniqueness of coupled common fixed point and in the last theorem we give an additional condition by which the components of coupled coincidence points are proved to be the same. Throughout this paper let

$$\Psi = \{\psi : [0, \infty) \to [0, \infty) |\ \psi \text{ is continuous, monotone increasing and } \psi(t) = 0 \Leftrightarrow t = 0\} \tag{3.1}$$

and

$$\Phi = \{\phi : [0, \infty) \to [0, \infty) |\ \phi \text{ is lower semi continuous and } \phi(t) = 0 \ \Leftrightarrow \ t = 0\} \tag{3.2}$$

Let $(X, \leqslant)$ be a partial ordered set. Define a partial order $\preceq$ on $X \times X$ as:
$(x, y) \preceq (u, v) \ \Leftrightarrow \ x \leqslant u \ \text{ and } \ y \geqslant v, \ \forall\, x, y, u, v \in X.$

Theorem 3.1. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous, compatible functions with $F(X \times X) \subseteq g(X)$, F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u),\ g(y) \geqslant g(v)$:*

$$\psi[d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))] \leq \psi[M(x, y, u, v)] - \phi[M(x, y, u, v)] \tag{3.3}$$

where $\psi \in \Psi$, $\phi \in \Phi$ and
$M(x, y, u, v) = \max\{d(g(x), F(x, y)) + d(g(y), F(y, x)), d(g(u), F(u, v)) + d(g(v), F(v, u))\}$.
If there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. Given $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$.
Since $F(X \times X) \subseteq g(X)$ and F satisfies the mixed g- monotone property, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ such that $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ with $g(x_n) \leqslant g(x_{n+1})$ and $g(y_{n+1}) \leqslant g(y_n)$, for $n = 0, 1, 2, \cdots$
If for some $n \in \mathbb{N}$, $g(x_n) = g(x_{n+1})$ and $g(y_{n+1}) = g(y_n)$ then the proof is complete.
Otherwise we will proceed as follows:
Since $g(x_n) \leqslant g(x_{n+1})$ and $g(y_{n+1}) \leqslant g(y_n)$, for $n = 0, 1, 2, \cdots$, consider for all $n \in \mathbb{N}$,

$$\psi[d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1}))]$$

$$
\begin{aligned}
&= \psi[d(F(x_{n-1},y_{n-1}),F(x_n,y_n)) + d(F(y_{n-1},x_{n-1}),F(y_n,x_n))] \\
&\leq \psi[\max\{d(g(x_{n-1}),F(x_{n-1},y_{n-1})) + d(g(y_{n-1}),F(y_{n-1},x_{n-1})), \\
&\qquad\qquad d(g(x_n),F(x_n,y_n)) + d(g(y_n),F(y_n,x_n))\}] \\
&\quad -\phi[\max\{d(g(x_{n-1}),F(x_{n-1},y_{n-1})) + d(g(y_{n-1}),F(y_{n-1},x_{n-1})), \\
&\qquad\qquad d(g(x_n),F(x_n,y_n)) + d(g(y_n),F(y_n,x_n))\}] \\
&= \psi[\max\{d(g(x_{n-1}),g(x_n)) + d(g(y_{n-1}),g(y_n)), d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1}))\}] \\
&- \phi[\max\{d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n)), d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\}] \quad (3.4)
\end{aligned}
$$

Suppose that for some $m \in \mathbb{N}$
$d(g(x_{m-1}),g(x_m)) + d(g(y_{m-1}),g(y_m)) \leq d(g(x_m),g(x_{m+1})) + d(g(y_m),g(y_{m+1}))$.
Now by (3.4) we have

$$
\begin{aligned}
\psi[d(g(x_m),g(x_{m+1})) + d(g(y_m),g(y_{m+1}))] &\leq \psi[d(g(x_m),g(x_{m+1})) + d(g(y_m),g(y_{m+1}))] \\
&\quad -\phi[d(g(x_m),g(x_{m+1})) + d(g(y_m),g(y_{m+1}))] \\
&< \psi[d(g(x_m),g(x_{m+1})) + d(g(y_m),g(y_{m+1}))]
\end{aligned}
$$

which is a contradiction.
Therefore for all $n \in \mathbb{N}$,
$d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1})) < d(g(x_{n-1}),g(x_n)) + d(g(y_{n-1}),g(y_n))$.
Thus $\{d(g(x_n),g(x_{n+1}))+d(g(y_n),g(y_{n+1}))\}$ is a decreasing sequence of nonnegative reals, so there exists a $\delta \geq 0$ such that

$$\lim_{n\to\infty} d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1})) = \delta$$

Assume that $\delta > 0$.
By taking the upper limit on both sides of (3.4) we get

$$
\begin{aligned}
\psi[\delta] &\leq \psi[\delta] - \phi[\delta] \\
&< \psi[\delta]
\end{aligned}
$$

which is a contradiction. Therefore $\delta = 0$.
Next, we prove that both $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X.
We have $g(x_n) \leqslant g(x_{n+1})$ and $g(y_{n+1}) \leqslant g(y_n)$, for $n = 0, 1, 2, \cdots$.
Now consider for $n > m$,

$$
\begin{aligned}
&\psi[d(g(x_m),g(x_n)) + d(g(y_m),g(y_n))] \\
&= \psi[d(F(x_{m-1},y_{m-1}),F(x_{n-1},y_{n-1})) + d(F(y_{m-1},x_{m-1}),F(y_{n-1},x_{n-1}))] \\
&\leq \psi[\max\{d(g(x_{m-1}),F(x_{m-1},y_{m-1})) + d(g(y_{m-1}),F(y_{m-1},x_{m-1})), \\
&\qquad\qquad d(g(x_{n-1}),F(x_{n-1},y_{n-1})) + d(g(y_{n-1}),F(y_{n-1},x_{n-1}))\}] \\
&\quad -\phi[\max\{d(g(x_{m-1}),F(x_{m-1},y_{m-1})) + d(g(y_{m-1}),F(y_{m-1},x_{m-1})), \\
&\qquad\qquad d(g(x_{n-1}),F(x_{n-1},y_{n-1})) + d(g(y_{n-1}),F(y_{n-1},x_{n-1}))] \\
&= \psi[\max\{d(g(x_{m-1}),g(x_m))+d(g(y_{m-1}),g(y_m)), d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))\}] \\
&\quad -\phi[\max\{d(g(x_{m-1}),g(x_m))+d(g(y_{m-1}),g(y_m)), d(g(x_{n-1}),g(x_n))+d(g(y_{n-1}),g(y_n))\}]
\end{aligned}
$$

By taking the upper limit as $n, m \to \infty$ on both sides we get,

$$\lim_{n,m\to\infty} \psi[d(g(x_m),g(x_n)) + d(g(y_m),g(y_n))] = 0$$

Thus both $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X.
Since X is a complete metric space there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = y \tag{3.5}$$

Since $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ we have,

$$\lim_{n\to\infty} F(x_n, y_n) = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = y \tag{3.6}$$

Since F and g are compatible we have,

$$\begin{aligned} \lim_{n\longrightarrow\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) &= 0, \text{ and} \\ \lim_{n\longrightarrow\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) &= 0 \end{aligned}$$

Now, by the continuity of F and g we have, $F(x, y) = g(x)$ and $F(y, x) = g(y)$. Thus the proof. □

Corollary 3.2. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous, compatible functions with $F(X \times X) \subseteq g(X)$, F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u)$, $g(y) \geqslant g(v)$:*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq M(x, y, u, v) - \phi[M(x, y, u, v)]$$

where $\phi \in \Phi$ and
$M(x,y,u,v) = \max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(u), F(u,v)) + d(g(v), F(v,u))\}$.
If there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. By taking ψ as the identity function on $[0, \infty)$ in Theorem 3.1, we get the result. □

Corollary 3.3. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ and $g : X \to X$ be two continuous, compatible functions with $F(X \times X) \subseteq g(X)$, F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u)$, $g(y) \geqslant g(v)$:*

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\quad \leq k \cdot \max\{d(g(x), F(x, y)) + d(g(y), F(y, x)), d(g(u), F(u, v)) + d(g(v), F(v, u))\} \end{aligned}$$

If there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0, y_0)$ and $g(y_0) \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof. By taking $\phi(p) = (1 - k)p$, for $p \in [0, \infty)$ in Corollary 3.2, we get the result. □

Corollary 3.4. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose $F : X \times X \to X$ be continuous function with F satisfying the mixed monotone property and for all $x, y, u, v \in X$ with $x \leqslant u$, $y \geqslant v$:*

$$\begin{aligned} &d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\ &\qquad \leq k \cdot \max\{d(x, F(x, y)) + d(y, F(y, x)), d(u, F(u, v)) + d(v, F(v, u))\} \end{aligned}$$

If there exist $x_0, y_0 \in X$ with $x_0 \leqslant F(x_0, y_0)$ and $y_0 \geqslant F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof. By considering g as the identity function on X in Corollary 3.3, we get the result. □

The following theorem guarantees the existence of coupled coincidence points of F and g in which F need not be continuous.

Theorem 3.5. *Let $(X, d, \leqslant)$ be a partially ordered complete metric space and suppose that X has the following properties:*

(i) *if an increasing sequence* $\{x_n\}$ *converges to* x *then* $x_n \leqslant x, \ \forall n$
(ii) *if a decreasing sequence* $\{y_n\}$ *converges to* y *then* $y \leqslant y_n, \ \forall n.$

Let $F : X \times X \to X$ *and* $g : X \to X$ *be compatible functions with* $F(X \times X) \subseteq g(X)$*, g an order preserving, continuous function and F satisfying the mixed g- monotone property and F and g satisfy the following:*
For all $x, y, u, v \in X$ *with* $g(x) \leqslant g(u), \ g(y) \geqslant g(v)$

$$\psi[d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))] \leq \psi[M(x,y,u,v)] - \phi[M(x,y,u,v)]$$

where $\psi \in \Psi$*,* $\phi \in \Phi$ *and*
$M(x,y,u,v) = \max\{d(g(x),F(x,y)) + d(g(y),F(y,x)), d(g(u),F(u,v)) + d(g(v),F(v,u))\}$.
If there exist $x_0, y_0 \in X$ *with* $g(x_0) \leqslant F(x_0,y_0)$ *and* $g(y_0) \geqslant F(y_0,x_0)$*, then there exist* $x, y \in X$ *such that* $g(x) = F(x,y)$ *and* $g(y) = F(y,x)$*.*

Proof. Following as in Theorem 3.1 we can have, an increasing sequence $\{g(x_n)\}$ and a decreasing sequence $\{g(y_n)\}$ defined as $g(x_{n+1}) = F(x_n, y_n)$, $g(y_{n+1}) = F(y_n, x_n)$ such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y$$

By the hypothesis we have, $g(x_n) \leqslant x$ and $y \leqslant g(y_n), \ \forall n \in \mathbb{N}$
Since g is order preserving, we get $g(g(x_n)) \leqslant g(x)$ and $g(y) \leqslant g(g(y_n)), \ \forall n \in \mathbb{N}$.
Since g is continuous and F and g are compatible we have

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(F(x_n,y_n)) = \lim_{n\to\infty} F(g(x_n),g(y_n)) \\ \text{and } g(y) &= \lim_{n\to\infty} g(F(y_n,x_n)) = \lim_{n\to\infty} F(g(y_n),g(x_n)) \end{aligned}$$

Suppose $F(x,y) \neq g(x)$ or $F(y,x) \neq g(y)$.
Since $g(g(x_n)) \leqslant g(x)$ and $g(y) \leqslant g(g(y_n)), \ \forall n \in \mathbb{N}$, we have

$$\begin{aligned} &\psi[d(F(g(x_n),g(y_n)),F(x,y)) + d(F(g(y_n),g(x_n)),F(y,x))] \\ &\quad \leq \ \psi[\max\{d(g(g(x_n)),F(g(x_n),g(y_n))) + d(g(g(y_n)),F(g(y_n),g(x_n))), \\ &\qquad\qquad d(g(x),F(x,y)) + d(g(y),F(y,x))\}] \\ &\qquad -\phi[\max\{d(g(g(x_n)),F(g(x_n),g(y_n))) + d(g(g(y_n)),F(g(y_n),g(x_n))), \\ &\qquad\qquad d(g(x),F(x,y)) + d(g(y),F(y,x))\}] \end{aligned}$$

By taking the upper limit on both sides we get

$$\begin{aligned} \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] &\leq \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \\ &\quad -\phi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \\ &< \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \end{aligned}$$

which is a contradiction. Thus $g(x) = F(x,y)$ and $g(y) = F(y,x)$.
Hence the proof. □

In the following theorem we omit the completeness of the underlying space X and the compatibility and continuity conditions of the functions F and g assumed in Theorem 3.1. The following theorem guarantees the existence of coupled coincidence points of F and g.

Theorem 3.6. *Let* $(X, d, \leqslant)$ *be a partially ordered metric space and X has the following property:*

(i) *if an increasing sequence* $\{x_n\}$ *converges to* x *then* $x_n \leqslant x, \ \forall n$
(ii) *if a decreasing sequence* $\{y_n\}$ *converges to* y *then* $y \leqslant y_n, \ \forall n.$

Let $F : X \times X \to X$ and $g : X \to X$ be two functions with $F(X \times X) \subseteq g(X)$ and F satisfying the mixed g- monotone property and for all $x, y, u, v \in X$ with $g(x) \leqslant g(u),\ g(y) \geqslant g(v)$:

$$\psi[d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))] \leq \psi[M(x,y,u,v)] - \phi[M(x,y,u,v)]$$

where $\psi \in \Psi$, $\phi \in \Phi$ and
$M(x,y,u,v) = \max\{d(g(x),F(x,y)) + d(g(y),F(y,x)), d(g(u),F(u,v)) + d(g(v),F(v,u))\}$.
Suppose $g(X)$ is a complete subspace of X and if there exist $x_0, y_0 \in X$ with $g(x_0) \leqslant F(x_0,y_0)$ and $g(y_0) \geqslant F(y_0,x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Proof. Following as in Theorem 3.1, we get an increasing Cauchy sequence $\{g(x_n)\}$ and a decreasing Cauchy sequence $\{g(y_n)\}$ in X defined as $g(x_{n+1}) = F(x_n,y_n)$ and $g(y_{n+1}) = F(y_n,x_n)$.
Since $g(X)$ is a complete subspace of X, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = g(x) \text{ and } \lim_{n\to\infty} g(y_n) = g(y)$$

By the hypothesis we have, $g(x_n) \leqslant g(x)$ and $g(y) \leqslant g(y_n),\ \forall n \in \mathbb{N}$.
Suppose $F(x,y) \neq g(x)$ or $F(y,x) \neq g(y)$.
Now consider,

$$\begin{aligned}
&\psi[d(F(x_n,y_n),F(x,y)) + d(F(y_n,x_n),F(y,x))] \\
&\quad\leq \psi[\max\{d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1})), d(g(x),F(x,y)) + d(g(y),F(y,x))\}] \\
&\qquad -\phi[\max\{d(g(x_n),g(x_{n+1})) + d(g(y_n),g(y_{n+1})), d(g(x),F(x,y)) + d(g(y),F(y,x))\}]
\end{aligned}$$

Taking the upper limit on both sides we get

$$\begin{aligned}
\psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] &\leq \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \\
&\quad -\phi[d(g(x),F(x,y)) + d(g(y),F(y,x))] \\
&< \psi[d(g(x),F(x,y)) + d(g(y),F(y,x))]
\end{aligned}$$

which is a contradiction.
Thus $g(x) = F(x,y)$ and $g(y) = F(y,x)$. □

Theorem 3.7. *In addition to the hypothesis of Theorem 3.1 suppose that for any $(x,y),(u,v) \in X \times X$ there exist $(\alpha,\beta) \in X \times X$ such that $(g(\alpha),g(\beta))$ is comparable to $(F(\alpha,\beta),F(\beta,\alpha))$ and $(F(\alpha,\beta),F(\beta,\alpha))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$, then F and g have a unique coupled common fixed point.*

Proof. Theorem 3.1 ensures that the set of all coupled coincidence points of F and g is nonempty.
Let $(x,y),(u,v) \in X \times X$ be any two coupled coincidence points of F and g.
That is, $g(x) = F(x,y),\ g(y) = F(y,x)$ and $g(u) = F(u,v),\ g(v) = F(v,u)$.
First we shall prove that

$$g(x) = g(u),\ g(y) = g(v) \tag{3.7}$$

By the hypothesis there exist $(\alpha,\beta) \in X \times X$ such that $(g(\alpha),g(\beta))$ is comparable to $(F(\alpha,\beta),F(\beta,\alpha))$.
Following as in Theorem 3.1 we can construct an increasing, converging sequence $\{g(\alpha_n)\}$ and a decreasing, converging sequence $\{g(\beta_n)\}$ where $g(\alpha_{n+1}) = F(\alpha_n,\beta_n)$ and $g(\beta_{n+1}) = F(\beta_n,\alpha_n)$, $n \in \mathbb{N} \cup \{0\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$.
By the hypothesis $(F(\alpha,\beta),F(\beta,\alpha))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$.
Since (x,y) and (u,v) are coupled coincidence points of F and g and using the mixed

g- monotone property of F we get $(g(\alpha_n), g(\beta_n))$ is comparable to both $(g(x), g(y))$ and $(g(u), g(v))$.
Consider,

$$\begin{aligned}
&\psi[d(g(x), g(\alpha_{n+1})) + d(g(y), g(\beta_{n+1}))]\\
&= \psi[d(F(x,y), F(\alpha_n, \beta_n)) + d(F(y,x), F(\beta_n, \alpha_n))]\\
&\leq \psi[\max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))\}]\\
&\quad -\phi[\max\{d(g(x), F(x,y)) + d(g(y), F(y,x)), d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))\}]\\
&= \psi[d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))] - \phi[d(g(\alpha_n), F(\alpha_n, \beta_n)) + d(g(\beta_n), F(\beta_n, \alpha_n))]
\end{aligned}$$

Taking the upper limit on both sides as $n \to \infty$ we get
$\lim_{n\to\infty} \{\psi[d(g(x), g(\alpha_{n+1})) + d(g(y), g(\beta_{n+1}))]\} = 0$
Similarly, it can be proved that $\lim_{n\to\infty} \{\psi[d(g(u), g(\alpha_{n+1})) + d(g(v), g(\beta_{n+1}))]\} = 0$
Thus $g(x) = g(u)$ and $g(y) = g(v)$.
That is, for any two coupled coincidence points (x, y) and (u, v) of F and g, $g(x) = g(u)$ and $g(y) = g(v)$.
Let $\gamma = g(x)$ and $\delta = g(y)$.
Since (x, y) is a coupled coincidence point of F and g we have
$F(x, y) = \gamma$ and $F(y, x) = \delta$
Since F and g are compatible we have
$g(\gamma) = F(\gamma, \delta)$ and $g(\delta) = F(\delta, \gamma)$.
That is, (γ, δ) is a coupled coincidence point of F and g.
Therefore $g(\gamma) = g(x) = \gamma$ and $g(\delta) = g(y) = \delta$
Therefore (γ, δ) is a coupled common fixed point of F and g.
The uniqueness of coupled common fixed point of F and g follows from (3.7). □

Theorem 3.8. *In addition to the hypothesis of Theorem 3.1, suppose that $g(x_0)$ and $g(y_0)$ are comparable, then $x = y$.*

Proof. Without loss of generality assume that $g(x_0) \leqslant g(y_0)$.
By following Theorem 3.1 we get $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$
where $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$.
By the mixed g- monotone property of F, it can be easily verified that $g(x_n) \leqslant g(y_n),\ \forall n \in \mathbb{N}$. Now consider,

$$\begin{aligned}
&\psi[d(F(x_n, y_n), F(y_n, x_n)) + d(F(y_n, x_n), F(x_n, y_n))]\\
&\leq \psi[\max\{d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})), d(g(y_n), g(y_{n+1})) + d(g(x_n), g(x_{n+1}))\}]\\
&\quad -\phi[\max\{d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})), d(g(y_n), g(y_{n+1})) + d(g(x_n), g(x_{n+1}))\}]
\end{aligned}$$

By taking the upper limit as $n \to \infty$ on both sides we get, $\psi[d(x, y) + d(y, x)] = 0$
Thus $x = y$. □

The following example illustrates Theorem 3.5.

Example 3.9. Let $X = [0, 1]$ with the usual order $\leq$ and the usual metric $d(x, y) = |x - y|,\ \forall\ x, y \in X$.
Clearly X is a partially ordered complete metric space satisfying the two properties assumed in Theorem 3.5.
Let $g : X \to X$ and $F : X \times X \to X$ be defined as

$$g(x) = \frac{4}{5}x \quad \text{and} \quad F(x,y) = \begin{cases} 0 & \text{if } x \in [0, \frac{6}{7}) \\ \frac{1}{35} & \text{if } x \in [\frac{6}{7}, 1] \end{cases}$$

It can be seen that F and g are compatible mappings and F is a mixed g- monotone mapping.
Here $F(X \times X) \subseteq g(X)$, g is continuous and order preserving mapping on X.
Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be defined as $\psi(x) = x^2$ and $\phi(x) = \frac{400}{529}x^2$, then F and g satisfy the contractive type condition (3.3).
If all $x, y, u, v \in X$ satisfying $g(x) \le g(u)$, and $g(y) \ge g(v)$, belong to either $[0, \frac{6}{7})$ or $[\frac{6}{7}, 1]$ then the contractive type condition (3.3) is obvious. In the remaining possible cases for the values of $x, y, u, v \in X$ satisfying $g(x) \le g(u)$ and $g(y) \ge g(v)$, we consider three different cases and verify the validity of the contractive type condition (3.3), which will cover the remaining cases.
Case 1: When $x, v \in [0, \frac{6}{7})$ and $u, y \in [\frac{6}{7}, 1]$

$$\begin{aligned}
d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &= \left|0 - \frac{1}{35}\right| + \left|\frac{1}{35} - 0\right| \\
&= \frac{2}{35} \qquad (3.8)\\
d(g(x),F(x,y)) + d(g(y),F(y,x)) &= \left|\frac{4}{5}x - 0\right| + \left|\frac{4}{5}y - \frac{1}{35}\right| \\
&= \frac{4}{5}x + \left|\frac{4}{5}y - \frac{1}{35}\right| \\
&\ge \left|\frac{4}{5}\cdot\frac{6}{7} - \frac{1}{35}\right| \\
&= \frac{23}{35} \qquad (3.9)\\
d(g(u),F(u,v)) + d(g(v),F(v,u)) &= \left|\frac{4}{5}u - \frac{1}{35}\right| + \left|\frac{4}{5}v - 0\right| \\
&= \left|\frac{4}{5}u - \frac{1}{35}\right| + \frac{4}{5}v \\
&\ge \left|\frac{4}{5}\cdot\frac{6}{7} - \frac{1}{35}\right| \\
&= \frac{23}{35} \qquad (3.10)
\end{aligned}$$

By (3.9) and (3.10) we have,

$$\begin{aligned}
M(x,y,u,v) &= \max\{d(g(x),F(x,y)) + d(g(y),F(y,x)), d(g(u),F(u,v)) + d(g(v),F(v,u))\} \\
&\ge \frac{23}{35} \qquad (3.11)
\end{aligned}$$

Now, by (3.8) and (3.11) we have,

$$\begin{aligned}
\psi(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))) &= \frac{4}{1225} \\
\psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) &= \frac{129}{529}\cdot M(x,y,u,v)^2 \\
&\ge \frac{129}{529}\cdot\frac{529}{1225} \\
&= \frac{129}{1225}
\end{aligned}$$

Therefore,

$$\psi(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))) \le \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$$

Case 2: $x, u, v \in [0, \frac{6}{7})$ and $y \in [\frac{6}{7}, 1]$

$$\begin{aligned} d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &= |0-0| + \left|\frac{1}{35} - 0\right| \\ &= \frac{1}{35} \qquad (3.12) \\ d(g(x),F(x,y)) + d(g(y),F(y,x)) &= \left|\frac{4}{5}x - 0\right| + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &= \frac{4}{5}x + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &\geq \left|\frac{4}{5}\cdot\frac{6}{7} - \frac{1}{35}\right| \\ &= \frac{23}{35} \qquad (3.13) \\ d(g(u),F(u,v)) + d(g(v),F(v,u)) &= \left|\frac{4}{5}u - 0\right| + \left|\frac{4}{5}v - 0\right| \\ &= \frac{4}{5}(u+v) \qquad (3.14) \end{aligned}$$

By (3.13) and (3.14) we have,

$$\begin{aligned} M(x,y,u,v) &= \max\{d(g(x),F(x,y)) + d(g(y),F(y,x)), d(g(u),F(u,v)) + d(g(v),F(v,u))\} \\ &\geq \frac{23}{35} \qquad (3.15) \end{aligned}$$

Now by (3.12) and (3.15) we have,

$$\begin{aligned} \psi(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))) &= \frac{1}{1225} \\ \psi(M(x,y,u,v)) - \phi(M(x,y,u,v)) &= \frac{129}{529}\cdot M(x,y,u,v)^2 \\ &\geq \frac{129}{529}\cdot\frac{529}{1225} \\ &= \frac{129}{1225} \end{aligned}$$

Therefore

$$\psi(d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))) \leq \psi(M(x,y,u,v)) - \phi(M(x,y,u,v))$$

Case 3: $x \in [0, \frac{6}{7})$ and $y, u, v \in [\frac{6}{7}, 1]$

$$\begin{aligned} d(F(x,y),F(u,v)) + d(F(y,x),F(v,u)) &= \left|0 - \frac{1}{35}\right| + \left|\frac{1}{35} - \frac{1}{35}\right| \\ &= \frac{1}{35} \qquad (3.16) \\ d(g(x),F(x,y)) + d(g(y),F(y,x)) &= \left|\frac{4}{5}x - 0\right| + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &= \frac{4}{5}x + \left|\frac{4}{5}y - \frac{1}{35}\right| \\ &\geq \left|\frac{4}{5}\cdot\frac{6}{7} - \frac{1}{35}\right| \\ &= \frac{23}{35} \qquad (3.17) \\ d(g(u),F(u,v)) + d(g(v),F(v,u)) &= \left|\frac{4}{5}u - \frac{1}{35}\right| + \left|\frac{4}{5}v - \frac{1}{35}\right| \end{aligned}$$

$$\geq 2\left|\frac{4}{5}\cdot\frac{6}{7}-\frac{1}{35}\right| \tag{3.18}$$
$$= \frac{2\cdot 23}{35} \tag{3.19}$$

By (3.17) and (3.19) we have,

$$\begin{aligned} M(x,y,u,v) &= \max\{d(g(x),F(x,y))+d(g(y),F(y,x)),d(g(u),F(u,v))+d(g(v),F(v,u))\} \\ &\geq \frac{2\cdot 23}{35} \end{aligned} \tag{3.20}$$

Now by (3.16) and (3.20) we have,

$$\begin{aligned} \psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) &= \frac{1}{1225} \\ \psi(M(x,y,u,v))-\phi(M(x,y,u,v)) &= \frac{129}{529}\cdot M(x,y,u,v)^2 \\ &\geq \frac{129}{529}\cdot\frac{4\cdot 529}{1225} \\ &= \frac{516}{1225} \end{aligned}$$

Therefore

$$\psi(d(F(x,y),F(u,v))+d(F(y,x),F(v,u))) \leq \psi(M(x,y,u,v))-\phi(M(x,y,u,v))$$

Here $(0,0)$ is the only coupled common fixed point of F and g.

Remark 3.10. The above example also illustrates that the contractive type conditions (1.1) and (3.3) are independent.

For, take $x=y=v=\frac{6}{7}-\epsilon$ and $u=\frac{6}{7}$ where $0<\epsilon\leq\frac{6}{7}$.
Now

$$\psi(d(F(x,y),F(u,v))) = \psi\Big(\Big|0-\frac{1}{35}\Big|\Big) = \psi\Big(\frac{1}{35}\Big)$$

$$\begin{aligned} &\psi(\max(d(gx,gu),d(gy,gv)))-\phi(\max(d(gx,gu),d(gy,gv))) \\ &= \psi\Big(\Big|\frac{4}{5}\Big(\frac{6}{7}-\epsilon\Big)-\frac{4}{5}\cdot\frac{6}{7}\Big|\Big)-\phi\Big(\Big|\frac{4}{5}\Big(\frac{6}{7}-\epsilon\Big)-\frac{4}{5}\cdot\frac{6}{7}\Big|\Big) \\ &= \psi\Big(\frac{4}{5}\cdot\epsilon\Big)-\phi\Big(\frac{4}{5}\cdot\epsilon\Big) \end{aligned}$$

Since ψ and ϕ in (1.1) are continuous and $\psi^{-1}\{0\}=\{0\}$ and $\phi^{-1}\{0\}=\{0\}$
we have as $\epsilon\to 0$,
$\psi(\max(d(gx,gu),d(gy,gv)))-\phi(\max(d(gx,gu),d(gy,gv)))\to 0$
but $\psi(d(F(x,y),F(u,v)))=\psi(\frac{1}{35})>0$ for all $\epsilon>0$.
Thus F and g does not satisfy the contractive type condition (1.1).

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