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APPROXIMATION OF SOLUTIONS OF SPLIT INVERSE PROBLEM FOR MULTI-VALUED DEMI-CONTRACTIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. Let H_1 and H_2 be real Hilbert spaces and $A_j : H_1 \longrightarrow H_2$, $1 \leq j \leq r$ be bounded linear linear operators, $U_i : H_1 \longrightarrow 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \longrightarrow 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive operators. An iterative scheme is constructed and shown to converge weakly to a solution of generalized split common fixed points problem (GSCFPP). Under additional mild condition, the scheme is shown to converge strongly to a solution of GSCFPP. Moreover, our scheme is of special interest.

KEYWORDS: Fixed Point; Multivalued Demi-Contractive Mappings; Split Inverse Problem.
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1. Introduction

Let X and Y be two real Banach spaces. A split inverse problem is to find a point $x^* \in X$ that solves IP_1 such that $y^* = Ax^* \in Y$ solves IP_2, where IP_1 and IP_2 are two inverse problems. A simple generalization of inverse problem is split convex feasibility problem (SCFP) which was introduced in 1994 by Censor and Elfving [18] in finite dimensional Hilbert spaces for modelling inverse problems arising from signal detection, computer temography, image recovery and radiation therapy treatment planning (see, e.g.,[5], [16], [19] and [18]). The (SCFP) is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where H_1, H_2 are real Hilbert spaces, $A : H_1 \to H_2$ bounded linear operator, and $C \subseteq H_1$, $Q \subseteq H_2$ are non-empty, closed and convex sets.

In what follows we denote the solution set of the (SCFP) by

$$\Gamma \equiv \Gamma(U, A) := \{y \in C : Ay \in Q\}. \tag{1.2}$$

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In 2002, Byrne in [5] proved that x^* is a solution to (1.2) if and only if it is a fixed point of

$$P_C(I - rA^*(I - P_Q)A),$$

where A^* is the adjoint operator of A, P_C and P_D are the metric projections from H_1 onto C and from H_2 onto Q, respectively, and $r > 0$ is a positive constant. Indeed, this can be easily shown using characterization of projection mapping. Censor and Segal proposed in [21], the following algorithm to solve (1.2)
Algorithm: see [[21], Algorithm 2].
let $x^* \in H_1 := \mathbb{R}^n$ be arbitrary and for $k \in \mathbb{N}$ let

$$x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k), \tag{1.3}$$

where $\gamma \in (0, \frac{2}{L})$, L being the spectral radius of the operator A^*A and I is the identity operator.

In 2010, Moudafi [32] proved the following result for approximation of solution of SCFP involving demicontractive mappings. Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0. If $\Gamma \neq \emptyset$, then any sequence $\{x_k\}$ generated by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k),\ k \geq 0, \tag{1.4}$$

where $u_k = x_k + \gamma A^*(T - I)Ax_k$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0, 1)$,
converges weakly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu}{L}\right)$ and $\alpha_k \in \left(\delta, 1 - \beta - \delta\right)$ for small enough $\delta > 0$.

Recently, inspired and motivated by the result of Moudafi [32], Tang *et al.* [42] proposed a cyclic algorthm (Algorithm 2 below) to solve the SCFP for demi-contractive operators $\{U_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^r$. Then they proved that the sequence generated by the proposed algorithm converges weakly to the solution of SCFP. Their work extends those of Moudafi [32], Censor and Segal [21] and others.
Algorithm 2: [42]
Let $x_0 \in H_1$ be arbitrary and let the sequence $\{x_k\}$ be defined by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U_{i(k)}(u_k),\ k \geq 0, \tag{1.5}$$

where $u_k = x_k + \gamma A^*(T_{j(k)} - I)Ax_k$, $i(k) = k(mod\ p) + 1$ and $j(k) = k(mod\ r) + 1$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0, 1)$.

Very recently, in [25], Gibali proved the following strong convergence result for demicontractive operators; Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at 0 and that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle U(q) - q, \sigma\rangle \geq 0\ \forall\ q \in H_1, \\ \langle A^*(T - I)Ay, \sigma\rangle \geq 0\ \forall\ y \in H_1. \end{cases} \tag{1.6}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k),\ k \geq 0, \tag{1.7}$$

where $u_k = x_k + \gamma A^*(T - I)Ax_k$, $\gamma \in (0, \frac{1-\mu}{\lambda})$, λ being the spectral radius of the operator A^*A and $\alpha_k \in (0, 1)$,
converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu}{L}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta - \delta\big)$ for small enough $\delta > 0$.

Motivated by the works of Moudafi [32], A. Gibali [25], Censor and Segal [21], it is our purpose in this paper to solve a general split common fixed points problem formulated as follows:

$$\text{Find a point } x^* \in C := \cap_{i=1}^n C_i \text{ such that } A_j x^* \in Q_j, \tag{1.8}$$

where $A_j : H_1 \to H_2$ are bounded linear operators, $C_i = Fix(U_i)$, $1 \le i \le n$ and $Q_j = Fix(T_j)$, $1 \le j \le r$ with $U_i : H_1 \to H_1$ and $T_j : H_2 \to H_2$ multi-valued demi-contractive operators (with constants β_i, $1 \le i \le n$ and μ_j, $1 \le j \le r$, respectively).

2. Preliminaries

We begin with the following definitions and lemmas.

Definition 2.1. Let $T : H \to H$ be an operator and $D \subseteq H$ and $F(T) = \{x \in K : x = Tx\}$.

- The operator T is called nonexpansive, if $\forall x, y \in D$
$$\|Tx - Ty\| \le \|x - y\| \tag{2.1}$$
- T is called quasi-nonexpansive, if $\forall (x, q) \in D \times F(T)$
$$\|Tx - q\| \le \|x - q\| \tag{2.2}$$
- T is called k-strictly pseudo-contractive (see e.g., [28]), if there exists $k \in [0, 1)$ such that $\forall (x, y) \in D$
$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \tag{2.3}$$
- T is called demi-contractive (see e.g., [3, 20, 27]), if there exists $\beta \in [0, 1)$ such that $\forall (x, q) \in D \times Fix(T)$
$$\|Tx - q\|^2 \le \|x - q\|^2 + \beta\|x - Tx\|^2 \tag{2.4}$$

Definition 2.2. Let H be a real Hilbert space, an operator T is called demiclosed at $q \in H$ (see e.g., [2]), if
for any sequence $\{x_k\}_{k=1}^\infty$ such that $x_k \rightharpoonup x^*$ and $Tx_k \to q$, we have $Tx^* = q$.

Definition 2.3. Let H be a real Hilbert space. The map $D : 2^H \times 2^H \longrightarrow \mathbb{R}^+$ defined by

$$\begin{aligned} D(A, B) &= max\{\sup_{y\in A} d(y, B), \sup_{x\in B} d(x, A)\} \text{ for all } A, B \in 2^H, \\ where \quad & d(y, B) := \inf_{x\in A} d(x, y), \end{aligned}$$

is called Pompeiu-Hausedorff distance.

Remark 1. In general, the map D is not a metric. However, it becomes a metric if it is defined on a set of closed and bounded subsets of H.

Let $T : H \to 2^H$ be a multi-valued mapping. An element $x^* \in H$ is said to be a fixed point of T if $x^* \in Tx^*$. We denote by $F(T)$ the fixed points set of T i.e.,

$$F(T) := \{x \in H : x \in Tx\}. \tag{2.5}$$

Definition 2.4. Let H be a real Hilbert space and $CB(H)$ be a set of closed and bounded subsets of H. $T : H \to 2^{CB(H)}$ be a multi-valued mapping. Then, T is said to be demi-closed at zero if for any sequence $\{x_k\} \subset H$ with $x_k \rightharpoonup x^*$, and $d(x_k, Tx_k) \longrightarrow 0$, we have $x^* \in Tx^*$.

Definition 2.5. Let H be a real Hilbert space.

- A multi-valued mapping $T : \mathcal{D}(T) \subseteq H \to 2^{CB(H)}$ is said to be nonexpansive (see e.g., [22]), if
$$D(Tx, Ty) \leq \|x - y\| \ \forall \ x, y \in \mathcal{D}(T) \tag{2.6}$$
- The mapping $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and
$$D(Tx, Tx^*) \leq \|x - x^*\| \ \forall \ x \in \mathcal{D}(T), \ x^* \in F(T). \tag{2.7}$$
- The mapping $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be k-strictly pseudo-contractive if there exists there exists a constant $k \in [0, 1]$ such that for all $u \in Tx$, $v \in Ty$
$$(D(Tx, Ty))^2 \leq \|x - y\|^2 + k\|x - y - (u - v)\|^2; \text{ and} \tag{2.8}$$
- $T : \mathcal{D}(T) \subseteq H \to 2^H$ is said to be demi-contractive if $F(T) \neq \emptyset$ and there exists a constant $k \in [0, 1]$ such that for all $x \in \mathcal{D}(T)$, $u \in Tx$
$$(D(Tx, \{y\}))^2 \leq \|x - y\|^2 + k\|x - u\|^2. \tag{2.9}$$

The class of demi-contractive operators is a very important generalization of non-expansive operators Also some operators that arise in optimization problems are of demi-contractive type. See for example, Chidume and Maruster [11].

It is obvious that, the class of multi-valued quasi-nonexpansive is properly contained in the class of multi-valued demi-contractive operators. Indeed, consider the following example:

Example 1. (see e.g., [8]) Let $H = \mathbb{R}$ with the usual metric. Define $T : \mathbb{R} \to \mathbb{R}$ by
$$Tx = \begin{cases} [-3x, -\frac{5x}{2}], \ x \in [0, \infty), \\ [-\frac{5x}{2}, -3x], \ x \in (-\infty, 0]. \end{cases} \tag{2.10}$$

We have that $F(T) = \{0\}$ and T is a multi-valued demi-contractive mapping which is not quasi-nonexpansive. In fact, for each $x \in (-\infty, 0) \cup (0, \infty)$, we have
$$\begin{aligned} \big(D(Tx, T0)\big)^2 &= |-3x - 0|^2 \\ &= 9|x - 0|^2, \end{aligned}$$
which implies that T is not quasi-nonexpansive.
Also, we have that
$$\begin{aligned} \big(d(x, Tx)\big)^2 &= |x - (-\frac{5x}{2})|^2 \\ &= \frac{49}{4}|x|^2. \end{aligned}$$

Thus,
$$\begin{aligned} \big(D(Tx, T0)\big)^2 &= |x - 0|^2 + 8|x - 0|^2 \\ &= |x - 0|^2 + \frac{32}{49}\big(d(x, Tx)\big)^2. \end{aligned}$$

Therefore, T is a demi-contractive mapping with constant $k = \frac{32}{49} \in (0, 1)$.

Lemma 2.6. *Let $(X, \langle\cdot,\cdot\rangle)$ be an IPS. Then for any $x, y \in X$, and $\alpha \in [0,1]$ the following inequality holds:*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 - \alpha(1-\alpha)\|x-y\|^2 + (1-\alpha)\|y\|^2 \tag{2.11}$$

Lemma 2.7. *(see,e.g., [10]) Let $A, B \in CB(X)$ and $a \in A$. For every $\gamma > 0$, there exixts $b \in B$ such that*

$$d(a,b) \leq D(A,B) + \gamma. \tag{2.12}$$

Lemma 2.8. *(see,e.g., [10]) Let X be a reflexive real Banach space and $A, B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a-b\| \leq D(A,B). \tag{2.13}$$

Lemma 2.9. *(see,e.g., [12]) Let E be a normed linear space, $B_1 \in CB(E)$ and $x_0 \in E$ arbitrary. Then the following hold;*

$$D(\{x_0\}, B_1) = \sup_{b_1 \in B_1} \|x_0 - b_1\|$$

Lemma 2.10. ***(Opial's lemma)*** *Let H be a real Hilbert space and $\{x_k\}$ a sequence in H such that there exists a nonempty set $\Gamma \subset H$ satisfying the following;*

i) *For every $y \in \Gamma$, $\lim \|x_k - y\|$ exists.*
ii) *Any weak-cluster point of the sequence x_k belong to Γ.*
Then, there exists $\bar{x} \in \Gamma$ such that $\{x_k\}$ converges weakly to $\bar{x}$.

Lemma 2.11. *Let $T : \mathcal{D}(T) \subseteq H \to 2^H$ be a demi-contractive, then*

$$\langle x-u, x-p\rangle \geq \frac{1-\beta}{2}\|x-u\|^2 \ \forall u \in Tx. \tag{2.14}$$

Proof. Definition of T gives

$$\begin{aligned}(D(Tx,p))^2 &\leq \|x-p\|^2 + \beta\|x-u\|^2 \ \forall u \in Tx\\ D(Tx,p) &\leq \sqrt{\|x-p\|^2 + \beta\|x-u\|^2} \ \forall u \in Tx\end{aligned}$$

We have by lemma (2.9) that $D(Tx,p) = \sup_{u\in Tx}\|u-p\|$.
Using this result we get

$$-\beta\|x-u\|^2 \leq \|x-p\|^2 - \|u-p\|^2 \ \forall u \in Tx \ . \ . \ . \ (i)$$

We observe that $2\langle x-u, x-p\rangle = \|x-u\|^2 + \|x-p\|^2 - \|u-p\|^2$,
this implies $\|x-p\|^2 - \|u-p\|^2 = 2\langle x-u, x-p\rangle - \|x-u\|^2$.
Using this in (i) we have

$$-\beta\|x-u\|^2 \leq 2\langle x-u, x-p\rangle - \|x-u\|^2,$$

hence,

$$\frac{1-\beta}{2}\|x-u\|^2 \leq \langle x-u, x-p\rangle \ \forall u \in Tx,$$

which completes the proof. □

3. Main Result

We now prove weak and strong convergence for our proposed iterative scheme. However, we begin with the following lemma.

3.1. **Weak Convergence Result.**

$$\begin{cases} q_k = x_k + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_j x_k), \text{ where } b_{j,k} \in T_j(A_j x_k) \ \forall 1 \leq j \leq r \\ x_{k+1} = (1-\alpha_k)q_k + \frac{\alpha_k}{n} \sum_{i=1}^{n} u_{i,k}, \text{ where } u_{i,k} \in U_i(q_k) \ \forall \ 1 \leq i \leq n, \end{cases} \tag{3.1}$$

where U_i and T_j are multi-valued demi-contractive for each $1 \leq i \leq n$, $1 \leq j \leq r$, respectively, $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ with μ_{max} the maximum of demi-contractive constants of U_i and L being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.

Lemma 3.1. *Let $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i)$ and nonempty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i$ and j and $\forall x \in H_1$, $y \in H_2$. Then for arbitrary $x_0 \in H_1$, the sequence $\{x_k\}$ generated by algorithm (3.1) is Féjer monotone with respect to Γ, that is for every $x \in \Gamma$,*

$$\|x_{k+1} - x\| \leq \|x_k - x\| \ \forall \ k \in \mathbb{N},$$

provided that $\gamma \in \left(0, \frac{1-\mu_{max}}{rL}\right)$ and $\alpha_k \in (0,1)$.

Proof. Set $L := sup_{\substack{1\leq i\leq n \\ 1\leq j\leq r}} A_i^* A_j$, $\mu_{max} := sup_{1\leq i\leq n} U_i$, $\beta_{max} := sup_{1\leq j\leq r} T_j$; where U_i and T_j are demi-contractive constants of U_i and T_j, respectively.
Let $p \in \Gamma$ then from (3.1), we have

$$\begin{aligned} \|x_{k+1} - p\|^2 &= \|(1-\alpha_k)q_k + \frac{\alpha_k}{n}\sum_{i=1}^{n} u_{i,k} - p\|^2 \\ &= \|q_k - p + \frac{\alpha_k}{n}\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \\ &= \|q_k - p\|^2 + 2\frac{\alpha_k}{n}\langle q_k - p, \sum_{i=1}^{n}(u_{i,k} - q_k)\rangle \\ &+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \\ &= \|q_k - p\|^2 + 2\frac{\alpha_k}{n}\sum_{i=1}^{n}\langle u_{i,k} - q_k, q_k - p\rangle \\ &+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \\ &= \|q_k - p\|^2 - 2\frac{\alpha_k}{n}\sum_{i=1}^{n}\langle q_k - u_{i,k}, q_k - p\rangle \\ &+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \end{aligned}$$

Using lemma (2.11), we have

$$\begin{aligned} &\leq \|q_k - p\|^2 - \frac{\alpha_k}{n}\sum_{i=1}^{n}(1-\beta_i)\|q_k - u_{i,k}\|^2 \\ &+ \frac{\alpha_k^2}{n^2}\|\sum_{i=1}^{n}(u_{i,k} - q_k)\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \|q_k - p\|^2 - \frac{\alpha_k}{n}(1-\beta_{max_i})\sum_{i=1}^{n}\|q_k - u_{i,k}\|^2 \\ &+ \frac{\alpha_k^2}{n}\sum_{i=1}^{n}\|(u_{i,k} - q_k)\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_{k+1} - p\|^2 &\leq \|q_k - p\|^2 - \frac{\alpha_k}{n}(1-\beta_{max})\sum_{i=1}^{n}\|q_k - u_{i,k}\|^2 \\ &+ \frac{\alpha_k^2}{n}\sum_{i=1}^{n}\|(u_{i,k} - q_k)\|^2 \\ &= \|q_k - p\|^2 \\ &- \frac{\alpha_k}{n}((1-\beta_{max}) - \alpha_k)\sum_{i=1}^{n}\|u_{i,k} - q_k\|^2 \; \ldots \; (3.0.1) \end{aligned}$$

Also from (3.1), we obtain

$$\begin{aligned} \|q_k - p\|^2 &= \|x_k - p + \gamma\sum_{j=1}^{r}A_j^*(b_{j,k} - A_jx_k)\|^2 \\ &= \|x_k - p\|^2 + 2\gamma\sum_{j=1}^{r}\langle x_k - p, A_j^*(b_{j,k} - A_jx_k)\rangle \\ &+ \gamma^2\|\sum_{j=1}^{r}(b_{j,k} - A_jx_k)\|^2 \\ &= \|x_k - p\|^2 - 2\gamma\sum_{j=1}^{r}\langle A_jx_k - A_jp, A_jx_k - b_{j,k})\rangle \\ &+ \gamma^2\|\sum_{j=1}^{r}(b_{j,k} - A_jx_k)\|^2 \end{aligned}$$

Using lemma (2.11), we get

$$\begin{aligned} &\leq \|x_k - p\|^2 - \gamma\sum_{j=1}^{r}(1-\mu_j)\|b_{j,k} - A_jx_k\|^2 \\ &+ \gamma^2 rL\|b_{j,k} - A_jx_k\|^2 \end{aligned}$$

Hence,

$$\begin{aligned} \|q_k - p\|^2 &\leq \|x_k - p\|^2 - \gamma\sum_{j=1}^{r}(1-\mu_{max})\|b_{j,k} - A_jx_k\|^2 \\ &+ \gamma^2 rL\|b_{j,k} - A_jx_k\|^2 \\ &\leq \|x_k - p\|^2 - \gamma(1-\mu_{max})\sum_{j=1}^{r}\|b_{j,k} - A_jx_k\|^2 \\ &+ \gamma^2 rL\|b_{j,k} - A_jx_k\|^2 \end{aligned}$$

$$\leq \|x_k - p\|^2 - \gamma((1-\mu_{max}) - \gamma rL)\textstyle\sum_{j=1}^{r}\|b_{j,k} - A_jx_k\|^2.$$

Substituting this in (3.0.1) we have,

$$\begin{aligned}\|x_{k+1}-p\|^2 &\leq \|x_k-p\|^2-\gamma((1-\mu_{max})-\gamma rL)\sum_{j=1}^{r}\|b_{j,k}-A_jx_k\|^2\\ &- \frac{\alpha_k}{n}((1-\beta_{max})-\alpha_k)\sum_{i=1}^{n}\|u_{i,k}-q_k\|^2 \ . \ . \ . \ (3.0.2)\\ &\leq \|x_k-p\|^2\end{aligned}$$

provided $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(0, 1-\beta_{max}\big)$.
Hence, $\{x_k\}$ is Féjer monotone. □

Let $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i)$ and nonempty $Fix(U_i) = C_i$ and $Fix(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i \ and \ j$ and $\forall x \in H_1, \ y \in H_2$.
If $\Gamma \neq \emptyset$, then any sequence $\{x_k\}$ generated by algorithm (3.1) converges weakly to a split common fixed point $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(\delta, 1-\beta_{max}-\delta\big)$ for small enough $\delta > 0$.

Proof. From (3.0.2), we obtained that $\{\|x_k - p\|\}$ is monotone decreasing thus, $\{x_k\}$ is bounded and $\lim \|x_k - p\|$ exists say, y^*.
Since $\{x_k\}$ is bounded, we have that there exists $\{x_{k_v}\}$ such that

$$\begin{aligned}&x_{k_v} \rightharpoonup x^* \ as \ v \to \infty, \text{ which implies that}\\ &A_jx_{k_v} \longrightarrow A_jx^* \ as \ v \to \infty, \text{ and thus}\\ &A_jx_{k_v} \rightharpoonup A_jx^* \ . \ . \ . \ (3.0.3)\end{aligned}$$

From (3.0.2) also, we have

$$\begin{aligned}&\lim \|b_{j,k} - A_jx_k\| = 0 \ as \ k \to \infty,\\ &\text{which implies that } d(T_j(A_jx_k), A_jx_k) \leq \|b_{j,k} - A_jx_k\| \longrightarrow 0 \ \forall 1 \leq j \leq r,\\ &\text{then, } d(T_j(A_jx_k), A_jx_k) \longrightarrow 0,\\ &\text{thus, } d(T_j(A_jx_{k_v}), A_jx_{k_v}) \longrightarrow 0 \ \forall 1 \leq j \leq r \ . \ . \ . \ (3.0.4)\end{aligned}$$

Since $(T_j - I)$ is demi-closed at 0, we have from (3.0.3) and (3.0.4) that

$$\begin{aligned}&A_jx^* \in T_j(A_jx^*)\\ \Rightarrow \ &A_jx^* \in F(T_j) \ \forall 1 \leq j \leq r\end{aligned}$$

We also have that

$$q_{k_v} = x_{k_v} + \gamma \sum_{j=1}^{r} A_j^*(b_{j,k} - A_jx_{k_v})$$

Therefore,

$$q_{k_v} \longrightarrow x^* \ . \ . \ . \ (3.0.5)$$

From (3.0.2), we have $\|u_{i,k} - q_k\| \longrightarrow 0$ as $k \longrightarrow 0$,

$$\begin{aligned}&\text{this implies that } d(U_i(q_k), q_k) \leq \|u_{i,k} - q_k\| \ \forall 1 \leq i \leq n,\\ &\text{then, } d(U_i(q_k), q_k) \longrightarrow 0 \ \forall 1 \leq i \leq n,\\ &\text{hence, } d(U_i(q_{k_v}), q_{k_v}) \longrightarrow 0 \ \forall 1 \leq i \leq n.\end{aligned}$$

This together with (3.0.5) imply that $x^* \in U_i(x^*)$ with implies that $x^* \in F(U_i)$ for each $i = 1, 2, ..., n$,
hence, $x^* \in \cap_{i=1}^n F(U_i)$ and $A_j x^* \in F(T_j)$ for each $j = 1, 2, ..., r$. Hence, $x^* \in \Gamma$.
We have shown for any subsequence $\{x_{k_v}\}$ of $\{x_k\}$ such that $x_{k_v} \rightharpoonup x^*$ that $x^* \in \Gamma$.
Thus, applying Opial's lemma we have that there exists $x^{**} \in \Gamma$ such that the sequence $x_k \rightharpoonup x^{**}$.
Hence, weak convergence for $\{x_k\}$ is established. ☐

We now prove strong convergence for our iterative scheme.

3.2. **Strong Convergence Result.** Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \le j \le r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \le i \le n$ and $T_j : H_2 \to 2^{H_2}$, $1 \le j \le r$ be multi-valued demi-contractive (with constants β_i, μ_j, respectively) such that $U_i(p) = \{p\}$ for all $p \in F(U_i) = C_i$ and $T_j(p) = \{p\}$ for all $p \in F(T_j) = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i = 1, 2, ..., n$ *and* $j = 1, 2, ..., r$ and $\forall x \in H_1,\ y \in H_2$.
Suppose that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma \rangle \ge 0 \ \forall\ 1 \le i \le n,\ u_i \in U_i(q) \textit{ and } q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma \rangle \ge 0 \ \forall\ 1 \le j \le r,\ b_j \in T_j(A_j y) \textit{ and } y \in H_1. \end{cases} \tag{3.2}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(\delta, 1 - \beta_{max} - \delta\big)$ for small enough $\delta > 0$.

Proof. Let $x^* \in \Gamma$ and choose $x_0 \in H_1$ such that

$$\langle x_0 - x^*, \sigma \rangle > 0,$$

then there exists $\epsilon > 0$ such that

$$\langle x_0 - x^*, \sigma \rangle \ge \epsilon \|x_0 - x^*\|^2.$$

We now proof by induction that

$$\langle x_{k+1} - x^*, \sigma \rangle \ge \epsilon \|x_{k+1} - x^*\|^2 \ \forall\ k \ge 0. \tag{3.3}$$

Indeed, assume it holds up to some $k \ge 0$, then

$$\begin{aligned} \langle x_{k+1} - x^*, \sigma \rangle &= \langle x_{k+1} - x_k + x_k - x^*, \sigma \rangle \\ &= \langle x_{k+1} - x_k, \sigma \rangle + \langle x_k - x^*, \sigma \rangle \\ &= \langle \gamma \sum_{j=1}^r A_j^*(b_{j,k} - A_j x_k) + \frac{\alpha_k}{n} \sum_{i=1}^n (u_{i,k} - q_k), \sigma \rangle \\ &+ \langle x_k - x^*, \sigma \rangle \\ &= \gamma \sum_{j=1}^r \langle A_j^*(b_{j,k} - A_j x_k), \sigma \rangle + \frac{\alpha_k}{n} \sum_{i=1}^n \langle (u_{i,k} - q_k), \sigma \rangle \\ &+ \langle x_k - x^*, \sigma \rangle. \end{aligned}$$

Since $\gamma > 0$, $\alpha_k > 0$ and by (3.1) we get

$$\langle x_{k+1} - x^*, \sigma \rangle \ge \langle x_k - x^*, \sigma \rangle$$

by the induction assumption we have that

$$\langle x_{k+1} - x^*, \sigma \rangle \ge \epsilon \|x_k - x^*\|^2,$$

by lemma (3.1) the sequence $\{x_k\}$ generated by algorithm (3.1) is Féjer monotone with respect to Γ, so that

$$\langle x_{k+1} - x^*, \sigma \rangle \geq \epsilon \|x_{k+1} - x^*\|^2.$$

Therefore, (3.2) holds for all $k \geq 0$.
By theorem (3.3) we have

$$\begin{aligned} &x_k \rightharpoonup x^*, \text{ so that} \\ &\langle g, x_k \rangle \longrightarrow \langle g, x^* \rangle \ \forall\ g \in H_1. \end{aligned}$$

In particular, for $g = \sigma \in H_1$ we get

$$\langle \sigma, x_k \rangle \longrightarrow \langle \sigma, x^* \rangle which implies \langle \sigma, x_k - x^* \rangle \longrightarrow 0 \ as\ k \longrightarrow +\infty.$$

From (3.2) we have

$$\|x_k - x^*\|^2 \leq \frac{1}{\epsilon} \langle x_k - x^*, \sigma \rangle \longrightarrow 0 \ as\ k \longrightarrow +\infty.$$

$$\text{Thus} \quad \|x_k - x^*\|^2 \longrightarrow 0 \ as\ k \longrightarrow +\infty.$$

Consequently, $\|x_k - x^*\| \longrightarrow 0$ *as* $k \longrightarrow +\infty$; and hence $x_k \longrightarrow x^* \in \Gamma$ This completes the proof. □

The following corollary is a special case of theorem (3.3) when $i = j = 1$

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be multi-valued demi-contractive (with constants β, μ, respectively) such that $U(p) = \{p\}$ for all $p \in F(U) = C$ and $T(p) = \{p\}$ for all $p \in F(T) = Q$ with $U(x)$ and $T(y)$ closed and bounded $\forall x \in H_1,\ y \in H_2$.*
Assume that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u - q, \sigma \rangle \geq 0 \ \forall\ u \in U(q) \ and\ q \in H_1, \\ \langle A^*(b - Ay), \sigma \rangle \geq 0 \ \forall\ b \in T(Ay) \ and\ y \in H_1. \end{cases} \tag{3.4}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) *converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu}{L}\right)$ and $\alpha_k \in \left(\delta, 1 - \beta - \delta\right)$ for small enough $\delta > 0$.*

The following result generalizes theorem of Moudafi [32] which is a special case of theorem (3.3) where $n = r = 1$, and U and T are single-valued demi-contractive.

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator. Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be demi-contractive (with constants β, μ, respectively) with nonempty $Fix(U) = C$ and $Fix(T) = Q$. Assume that $U - I$ and $T - I$ are demi-closed at* 0 *and that there exists $\sigma \neq 0 \in H_1$, such that*

$$\begin{cases} \langle U(q) - q, \sigma \rangle \geq 0 \ \forall\ q \in H_1, \\ \langle A^*(T - I)Ay, \sigma \rangle \geq 0 \ \forall\ y \in H_1. \end{cases} \tag{3.5}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) *converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \left(0, \frac{1-\mu}{L}\right)$ and $\alpha_k \in \left(\delta, 1 - \beta - \delta\right)$ for small enough $\delta > 0$.*

Corollary 3.4. *Let H_1 and H_2 be two real Hilbert spaces and $A_j : H_1 \to H_2$, $1 \leq j \leq r$ be bounded linear operators, $U_i : H_1 \to 2^{H_1}$, $1 \leq i \leq n$ and $T_j : H_2 \to 2^{H_2}$, $1 \leq j \leq r$ be multi-valued quasi-nonexpansive such that $U_i(p) = \{p\}$ for all*

$p \in F(U_i) = C_i$ and $T_j(p) = \{p\}$ for all $p \in F(T_j = Q_j$ with $U_i(x)$ and $T_j(y)$ closed and bounded $\forall i = 1, 2, ..., n$ and $j = 1, 2, ..., r$ and $\forall x \in H_1,\ y \in H_2$.
Suppose that there exists $\sigma \neq 0 \in H_1$, such that

$$\begin{cases} \langle u_i - q, \sigma \rangle \geq 0 \ \forall\ 1 \leq i \leq n,\ u_i \in U_i(q) \text{ and } q \in H_1, \\ \langle A_j^*(b_j - A_j y), \sigma \rangle \geq 0 \ \forall\ 1 \leq j \leq r,\ b_j \in T_j(A_j y) \text{ and } y \in H_1. \end{cases} \tag{3.6}$$

If $\Gamma \neq \emptyset$, then for a suitable $x_0 \in H_1$ any sequence $\{x_k\}$ generated by algorithm (3.1) converges strongly to $x^* \in \Gamma$, provided that $\gamma \in \big(0, \frac{1-\mu_{max}}{rL}\big)$ and $\alpha_k \in \big(\delta, 1-\beta_{max}-\delta\big)$ for small enough $\delta > 0$.

3.3. **Numerical Examples.** In order to illustrate numerical application, we consider a special case of our scheme for $i = j = 1$ and $H_1 = H_2 = \mathbb{R}^3$.

All computations in this section were performed using python 3.5.2 terminal based on linux running 64-bit. The first 100 iterations of our scheme are presented in Table 1, and relationship between $\|x - x^*\|$ - values and number of iterations are given in Figure 1, where $x^* = 0 \in \Gamma$.
Now, for $x_0 = (1, 1, 1) \in \mathbb{R}^3$, $\gamma = 0.2$, $\alpha_k = 1 - \alpha_k = 0.5$, $\forall k \geq 1$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} \sqrt{\frac{3}{20}} & \sqrt{\frac{1}{20}} & 0 \\ \sqrt{\frac{1}{20}} & \sqrt{\frac{3}{20}} & \sqrt{\frac{3}{10}} \\ 0 & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} \end{bmatrix}, \text{ and } U = \begin{bmatrix} \sqrt{\frac{1}{10}} & 0 & \sqrt{\frac{3}{10}} \\ 0 & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{10}} \\ \sqrt{\frac{3}{20}} & 0 & \sqrt{\frac{3}{20}} \end{bmatrix}$$

we have the following iterations for $k = 100$.

Iterations	$\|x - x^*\|$
10	$1.09e^{-01}$
20	$7.00e^{-03}$
30	$4.00e^{-04}$
40	$3.37e^{-05}$
50	$2.30e^{-06}$
60	$1.54e^{-07}$
70	$1.04e^{-08}$
80	$6.10e^{-10}$
90	$4.72e^{-11}$
100	$3.20e^{-12}$

Table 1. The first 100 iterations generated by (3.1.6).

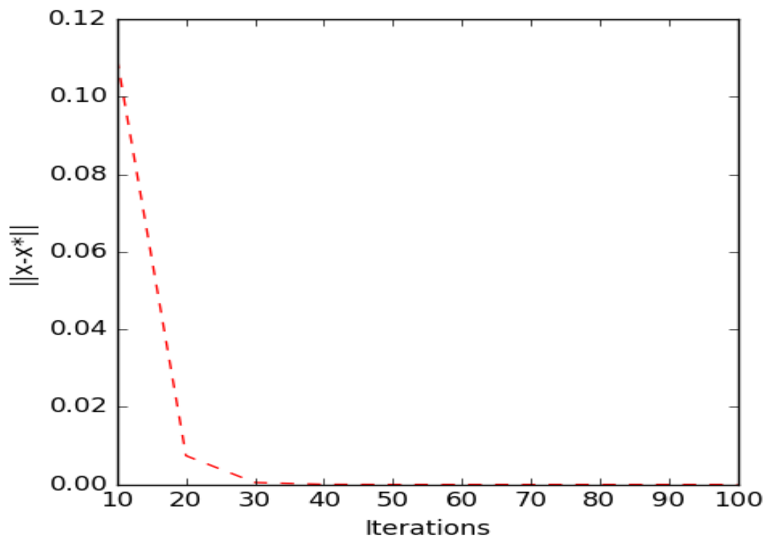

Figure 1. Relationship between $\|x - x^*\|$ - values and number of iterations.

4. Conclusion

In this paper, we have established the approximation of solution of general split inverse problem for multi-valued demi-contractive mappings in Hilbert spaces. Moreover, our result generalises many results in the literature. More precisely, theorem 3.3 generalises theorem 3.8 of [25]. Finally, lemma 2.11 and 3.1 are of special interest.

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