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CLOSEDNESS OF THE OPTIMAL SOLUTION SETS FOR GENERAL VECTOR ALPHA OPTIMIZATION PROBLEMS

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ABSTRACT. The aim of the paper is to study the closedness of the optimal solution sets for general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. Firstly, we present the relationships between the optimal solution sets of primal and dual general vector alpha optimization problems. Secondly, making use of the upper semicontinuity of a set-valued mapping, we discuss the results on closedness of the optimal solution sets for general vector alpha optimization problems in infinite-dimensional spaces.

KEYWORDS: Dual and primal general vector alpha optimization problems; Optimal solution sets; Upper $C-$ continuous set-valued mapping; Hausdorff locally convex topological vector spaces.
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1. Introduction

It is well known that the closedness, upper (lower) semicontinuity and connectedness or contractibility of optimal solution sets in set-valued optimization problems play an important role in the theory of set-valued analysis and applied analysis (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17] and the references therein). In recent years, Gong [5] studied the connectedness and path connectedness of efficient solution sets of vector equilibrium problems using the scalarization results; Gong and Yao [6, 7] discussed the results about the lower semicontinuity and connectedness of the efficient solution sets for parametric generalized systems which was introduced by Ding and Park [4] with monotone bifunction in real locally convex Hausdorff topological vector spaces; Khanh and Luu [9] obtained the result on the upper semicontinuity of solution set of quasivariational inequalities in Hausdorff topological vector spaces;

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Khanh and Anh [10] investigated the Holder continuity of solution to parametric multivalued vector equilibrium problems in metric linear spaces; Wu and Wu [17] have discussed the characterization of solution sets of a general convex program on a normed vector space using the Gateaux differentiable.

On characterizations of the solution sets for general alpha vector optimization problems have been extensively investigated in recent years because of their fields of applications (see, e.g., [11, 12, 13, 14, 15, 16] and the references therein). For example, Lin and Tan [11, 12] introduced and studied the solution existence results for the systems of quasivariational inclusion problems of type I and related problems in infinite dimensional spaces. On using the upper and lower semicontinuity of set-valued mappings, Tan [15, 16] together with Su [14] have received the result on existences of solution of generalized systems.

However, so far as we known, there are no results in the literature on the closedness of the efficient solutions for dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. The purpose of the article is to discuss the closedness for efficient solutions of this problems.

The organization of this paper is as follows. In Section 2, we recall some basic concepts and related properties. Section 3 is devoted to the relationships between the optimal solution sets of dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces. In this section, the closedness of optimal solution sets plays a central role in this paper. In Section 4, we make a conclusion to emphasis the obtained results again.

2. Preliminaries

Throughout this paper, let X and Y be two Hausdorff locally convex topological vector spaces in which Y be partially ordered by a convex cone C. We recall that C is a cone if $tc \in C$ for every $c \in C$ and every nonnegative number t. C is said to be a convex set if for any $c, d \in C$, the line segment $[c,d] = \{tc + (1-t)d : 0 \leq t \leq 1\}$ belongs to C. If C is a convex set then a cone C is called a convex cone. If C is a closed and convex set then a cone C is called a closed and convex cone. We set $l(C) := C \cap (-C)$. In this case, if $l(C) = \{0\}$ then a cone C is called a pointed cone. We denote D instead of a nonempty subset of X, and $F : D \rightrightarrows Y$ stands for a set-valued mapping F from D into Y. The domain and the graph of F are defined respectively by

$$\operatorname{dom} F = \{x \in D \,:\, F(x) \neq \emptyset\},$$
$$\operatorname{graph} F = \{(x,y) \in D \times Y \,:\, x \in \operatorname{dom} F,\, y \in F(x)\}.$$

For $A \subset X$, we denote as usual by $\text{int}A, cl\,A$ intead of the interior and the closure of A, respectively. The set of Ideal, Pareto, Proper and Weak minimal points of A with respect to C is denoted respectively as

$$IMin(A|C),\ \ PMin(A|C),\ \ PrMin(A|C) \text{ and } WMin(A|C).$$

The set of Ideal, Pareto, Proper and Weak maximal points of A with respect to C is denoted respectively as

$$IMax(A|C),\ \ PMax(A|C),\ \ PrMax(A|C) \text{ and } WMax(A|C).$$

The concepts of Ideal, Pareto, Proper and Weak minimal and maximal points can be found in Luc [13].

In this paper, the primal general vector alpha optimization problems corresponding to D, F and C (to short, $(GVOP)_{\alpha,\min}$) are defined as follows: finding $\overline{x} \in D$ such that

$$F(\overline{x}) \cap \alpha Min(F(D)|C) \neq \emptyset.$$

The set of such points $\overline{x}$ is said to be a solutions set of $(GVOP)_{\alpha,\min}$ which is denoted by $\alpha S_{\min}(D, F, C)$. The elements of $\alpha Min(F(D)|C)$ are called alpha optimal values of $(GVOP)_{\alpha,\min}$, where $\alpha = I$, $\alpha = P$, $\alpha = Pr$ and $\alpha = W$ instead of the case of Ideal, Pareto, Proper and Weak efficient points, respectively.

The dual general vector alpha optimization problems corresponding to D, F and C of problem $(GVOP)_{\alpha,\min}$, which is denoted by $(GVOP)_{\alpha,\max}$, can be defined as follows: finding $\overline{x} \in D$ such that

$$F(\overline{x}) \cap \alpha Max(F(D)|C) \neq \emptyset.$$

The set of such points $\overline{x}$ is said to be a solutions set of $(GVOP)_{\alpha,\max}$ which is denoted by $\alpha S_{\max}(D, F, C)$. The elements of $\alpha Max(F(D)|C)$ are called alpha optimal values of $(GVOP)_{\alpha,\max}$. The set D is sometimes called the set of alternatives and $F(D)$ is the set of outcomes.

We next recall the following definitions which will be needed in the paper.

Definition 2.1. ([13]) Let A be a nonempty subset of Y. We say that

(i) $x \in A$ is an ideal efficient (or ideal minimal) point of A with respect to C if $y - x \in C$ for every $y \in A$.
The set of ideal minimal points of A is denoted by $IMin(A|C)$.

(ii) $x \in A$ is an efficient (or Pareto-minimal, or nondominated) point of A with respect to C if there is no $y \in A$ with $x - y \in C \setminus l(C)$, where $l(C) := C \cap (-C)$.
The set of efficient points of A is denoted by $PMin(A|C)$.

(iii) $x \in A$ is a (global) proper efficient point of A with respect to C if there exists a convex cone $\widetilde{C}$ which is not the whole space and contains $C \setminus l(C)$ in its interior such that

$$x \in PMin(A|\widetilde{C}).$$

The set of proper efficient points of A is denoted by $PrMin(A|C)$.

(iv) Supposing that $\mathrm{int}C$ is nonempty, $x \in A$ is a weak efficient point of A with respect to C if

$$x \in PMin(A|\mathrm{int}\, C \cup \{0\}).$$

The set of weak efficient points of A is denoted by $WMin(A|C)$.

The concepts of upper and lower semicontinuity with a set-valued mapping play an important role in the paper.

Definition 2.2. ([15, 16]) Let $F : D \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be upper $C-$ continuous in $\overline{x} \in \mathrm{dom}F$ if for any neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ such that

$$F(x) \subset F(\overline{x}) + V + C \quad \forall\, x \in U \cap \mathrm{dom}F.$$

(ii) F is said to be lower $C-$ continuous in $\overline{x} \in \mathrm{dom}F$ if for any neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ such that

$$F(\overline{x}) \subset F(x) + V - C \quad \forall\, x \in U \cap \mathrm{dom}F.$$

(iii) If F is upper $C-$ continuous and lower $C-$ continuous in $\overline{x} \in \text{dom}F$ simultaneously, we say that F is $C-$ continuous in $\overline{x}$.
(iv) If F is upper (resp. lower) $C-$ continuous in any points of $\overline{x} \in \text{dom}F$, we say that F is upper (resp., lower) $C-$ continuous on D.

Let $\emptyset \neq A \subset Y$, $C \subset Y$ be a convex cone. By making use of the concepts in Definition 2.1, we receive the equivalences of efficiency, which can be stated as follows.

Proposition 2.3. *([13]) A equivalent definition of efficiency:*

(i) *$x \in IMin(A|C)$ if and only if $x \in A$ and $A \subset x + C$.*
(ii) *$x \in IMax(A|C)$ if and only if $x \in A$ and $A \subset x - C$.*
(iii) *$x \in PMin(A|C)$ if and only if $A \cap (x - C) \subset x + l(C)$, or equivalently, when C is pointed, $x \in PMin(A|C)$ if and only if $A \cap (x - C) = \{x\}$.*
(iv) *When C is not the whole space, $x \in WMin(A|C)$ if and only if $A \cap (x - intC) = \emptyset$, or equivalently, there is no $y \in A$ such that $x - y \in \{0\} \cup intC$ and not $y - x \in \{0\} \cup intC$.*

It can be easily seen that the following equalities hold

$$\alpha Min(A| - C) = \alpha Max(A|C),$$

$$\alpha Max(A| - C) = \alpha Min(A|C),$$

where α is one of the notions I, P, Pr and W. Moreover, it follows from Proposition 2.2 in Luc [13] that the following inclusions are true:

$$PrMin(A|C) \subset PMin(A|C) \subset WMin(A|C).$$

If, in addition, $IMin(A|C) \neq \emptyset$ then

$$PMin(A|C) = IMin(A|C).$$

Finally, the strict convexity of a set-valued mapping will be provided.

Definition 2.4. ([13]) Let D be a convex subset of domF with $F : D \rightrightarrows Y$. We say that

(i) F is called strictly $C-$ convex on D, when int$C \neq \emptyset$, if for $x_1, x_2 \in D,\ x_1 \neq x_2,\ t \in (0, 1)$, i.e. $0 < t < 1$,

$$F(tx_1 + (1 - t)x_2) \subset tF(x_1) + (1 - t)F(x_2) - \text{int}C.$$

(ii) F is called strictly $C-$ quasiconvex on a nonempty convex subset $D \subset X$, when int$C \neq \emptyset$, if for $y \in Y,\ x_1, x_2 \in D,\ x_1 \neq x_2,\ t \in (0, 1)$, i.e. $0 < t < 1$,

$$F(x_1), F(x_2) \subset y - C \ \text{ implies } \ F(tx_1 + (1 - t)x_2) \subset y - \text{int}C.$$

3. Closedness of the optimal solution sets for problems $(GVOP)_{\alpha,\min}$ and $(GVOP)_{\alpha,\max}$

In this section, we discuss the closedness and relationships between the optimal solution sets of dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces corresponding to D, F and C, where α is one of the qualifications: Pareto, Proper, Ideal and Weak.

Proposition 3.1. *Let $\alpha S_{\min}(D,F,C)$ be the solution set of $(GVOP)_{\alpha,\min}$, where α is one of the notions I, P, Pr and W. We have the following assertions hold.*

(i) $IS_{\min}(D,F,C) \subset PS_{\min}(D,F,C)$. *Moreover, if $IMin(F(D)|C) \neq \emptyset$ then*
$$IS_{\min}(D,F,C) = PS_{\min}(D,F,C),$$
and it is has at most a solution whenever C is pointed.

(ii) $PrS_{\min}(D,F,C) \subset PS_{\min}(D,F,C) \subset WS_{\min}(D,F,C)$.

Proof. Case (i): Let us assume that x be a solution of $(GVOP)_{I,\min}$, which yields that
$$F(x) \cap IMin(F(D)|C) \neq \emptyset.$$
By definitions, it can be easily seen that
$$F(x) \cap PMin(F(D)|C) \neq \emptyset.$$
Therefore, the vector x is an optimal solution of $(GVOP)_{P,\min}$. Making use of Proposition 2.2 [13] in the case $IMin(F(D)|C) \neq \emptyset$, and we obtain the result as required.

Case (ii): It is evident that
$$\begin{aligned} F(x) \cap PrMin(F(D)|C) &\subset F(x) \cap PMin(F(D)|C) \\ &\subset F(x) \cap WMin(F(D)|C) \quad \forall x \in D. \end{aligned}$$
Consequently,
$$PrS_{\min}(D,F,C) \subset PS_{\min}(D,F,C) \subset WS_{\min}(D,F,C),$$
which proves the claim. □

Proposition 3.2. *Let $\alpha S_{\max}(D,F,C)$ be the solution set of $(GVOP)_{\alpha,\max}$, where α is one of the notions I, P, Pr and W. We have the following assertions hold.*

(i) $IS_{\max}(D,F,C) \subset PS_{\max}(D,F,C)$. *Moreover, if $IMax(F(D)|C) \neq \emptyset$ then*
$$IS_{\max}(D,F,C) = PS_{\max}(D,F,C),$$
and it is has at most a solution whenever C is pointed.

(ii) $PrS_{\max}(D,F,C) \subset PS_{\max}(D,F,C) \subset WS_{\max}(D,F,C)$.

Proof. Case (i): Let x be a solution of $(GVOP)_{I,\max}$, which means that
$$F(x) \cap IMax(F(D)|C) \neq \emptyset.$$
By definitions, it is not hard to see that
$$F(x) \cap PMax(F(D)|C) \neq \emptyset.$$
Thus the vector x is a solution of problem $(GVOP)_{P,\max}$. If, in addition, $IMax(F(D)|C) \neq \emptyset$, taking into account of Proposition 2.2 [13], we arrive at the desired conclusion.

Case (ii): It is evident that

$$\begin{aligned} F(x) \cap PrMax(F(D)|C) &\subset F(x) \cap PMax(F(D)|C) \\ &\subset F(x) \cap WMax(F(D)|C) \ \ \forall x \in D. \end{aligned}$$

Therefore,

$$PrS_{\max}(D, F, C) \subset PS_{\max}(D, F, C) \subset WS_{\max}(D, F, C),$$

as was to be shown. □

Proposition 3.3. *Let D be a nonempty convex subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be either strictly $C-$ convex or strictly $C-$ quasiconvex on D. Assume, furthermore, that $F(x)$ is convex set for all $x \in D$. Then*

$$PS_{\min}(D, F, C) = WS_{\min}(D, F, C).$$

If, in addition, $IMin(F(D)|C) \neq \emptyset$, then

$$IS_{\min}(D, F, C) = WS_{\min}(D, F, C),$$

and it is has at most a solution whenever C is pointed.

Proof. Making use of the result obtained in Proposition 3.1 (ii), it suffices to prove that

$$WS_{\min}(D, F, C) \subset PS_{\min}(D, F, C).$$

Take arbitrary $x \in WS_{min}(D, F, C)$ and prove that $x \in PS_{\min}(D, F, C)$. In fact, we assume to the contrary, that $x \notin PS_{\min}(D, F, C)$. By definition, one finds an element $y \in D$ such that

$$F(y) \subset F(x) - C \setminus \{0\}.$$

It is well known that

$$\begin{aligned} &C \setminus \{0\} \subset C, \quad C \setminus \{0\} + C \subset C, \\ &\operatorname{int} C \subset C, \qquad \operatorname{int}C + C = \operatorname{int}C, \\ &\frac{1}{2}F(x) + \frac{1}{2}F(x) = F(x) \ \ \forall x \in D. \end{aligned}$$

We set

$$z = \frac{1}{2}x + \frac{1}{2}y.$$

Since D is convex set, it ensures that $z \in D$. Using the definition of strictly $C-$ quasiconvexity on D and the set $F(x)$ convex, it follows that

$$\begin{aligned} F(z) &\subset \frac{1}{2}F(x) + \frac{1}{2}F(y) - \operatorname{int}C \\ &\subset \frac{1}{2}F(x) + \frac{1}{2}F(x) - \frac{1}{2}C \setminus \{0\} - \operatorname{int}C \\ &\subset \frac{1}{2}F(x) + \frac{1}{2}F(x) - C - \operatorname{int}C \\ &\subset F(x) - \operatorname{int}C, \end{aligned}$$

which contradicting the condition $x \in WS_{min}(D, F, C)$. So, we have the following equality

$$PS_{\min}(D, F, C) = WS_{\min}(D, F, C).$$

The last case is due to preceding Proposition 3.1 and we get the required conclusion. □

Remark 3.4. It is worth noting that the results obtained in Proposition 3.3 are still holds for the senses

$$PS_{\max}(D, F, C) = WS_{\max}(D, F, C)$$

and

$$IS_{\max}(D, F, C) = WS_{\max}(D, F, C),$$

if the set-valued mapping F is strictly $(-C)-$ convex or strictly $(-C)-$ quasiconvex on D.

Theorem 3.1. *Let D be a nonempty closed subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be upper $C-$ continuous on D and assumming, in addition, that C be a closed convex cone in Y and $F(x)$ compact for any $x \in D$. Then $IS_{\min}(D, F, C)$ is closed.*

Proof. Assume to the contrary, that there exists sequence $(x_\alpha)_\alpha \subset IS_{\min}(D, F, C)$ such that

$$x_\alpha \to \overline{x}, \tag{3.1}$$

where $\overline{x} \notin IS_{min}(D, F, C)$. From the initial assumption, we have that D is a closed subset in X and $(x_\alpha)_\alpha \subset D$, and so, it follows that $\overline{x} \in D$. It is well known that F is upper $C-$ continuous on D, which yields that F is also upper $C-$ continuous at $\overline{x}$. Making use of Definition 2.2, for any open convex neighborhood V of the origin in Y, there exists a neighborhood U of $\overline{x}$ in X such that

$$F(x) \subset F(\overline{x}) + \frac{1}{2}V + C \ \ \forall x \in U \cap \mathrm{dom}F. \tag{3.2}$$

It follows from (3.1) that there exists $\alpha_0 > 0$ such that

$$x_\alpha \in U \cap \mathrm{dom}F \ \text{ for every } \ \alpha > \alpha_0.$$

From (3.2) we obtain the following inclusion

$$F(x_\alpha) \subset F(\overline{x}) + \frac{1}{2}V + C \ \text{ for every } \ \alpha > \alpha_0. \tag{3.3}$$

We arbitrarily take $\alpha > \alpha_0$. It is clear that F is upper $C-$ continuous at x_α. For the preceding open convex neighborhood V, there exists a neighborhood U_α of x_α satisfying

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(x_\alpha) + \frac{1}{2}V + C. \tag{3.4}$$

Since V is convex, it holds that

$$\frac{1}{2}V + \frac{1}{2}V \subset V.$$

Combining (3.3)-(3.4) yields that

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + V + C. \tag{3.5}$$

By the initial hypotheses, $F(\overline{x})$ is a compact set, C is a closed cone and V is an open neighborhood arbitrarily, and thus, it follows from (3.5) that

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + C. \tag{3.6}$$

Let us see that

$$F(x_\alpha) \cap IMin(F(D)|C) = \emptyset \ \ \forall \alpha > \alpha_0.$$

In fact, if it was not so, then there would exists an element $y_\alpha \in F(x_\alpha)$ with $\alpha > \alpha_0$ such that

$$F(D) \subset y_\alpha + C.$$

Because $\alpha > \alpha_0$, it follows from (3.6) that

$$y_\alpha \in F(\overline{x}) + C.$$

One finds an element $c_\alpha \in C$ such that $y_\alpha - c_\alpha \in F(\overline{x})$. On the other hand, for any $\alpha > \alpha_0$,

$$\begin{aligned} F(D) &\subset \big(y_\alpha - c_\alpha\big) + c_\alpha + C \\ &\subset \big(y_\alpha - c_\alpha\big) + C + C \\ &= \big(y_\alpha - c_\alpha\big) + C. \end{aligned}$$

By virtue of Definition 2.1 together with the fact that $F(\overline{x}) \subset F(D)$, one obtains

$$y_\alpha - c_\alpha \in F(\overline{x}) \cap IMin(F(D)|C) \ \ \forall\, \alpha > \alpha_0,$$

this means that $\overline{x} \in IS_{\min}(D, F, C)$, this is a contradiction. We thus will be allowed to say that the following relation is fulfilled

$$F(x_\alpha) \cap IMin(F(D)|C) = \emptyset \ \ \forall\, \alpha > \alpha_0,$$

it means that for any $\alpha > \alpha_0$, the vector x_α does not belong to the solution set $IS_{\min}(D, F, C)$, which conflicts with the initial assumptions. So the optimal solution set $IS_{\min}(D, F, C)$ is closed, and we get the desired conclusion. □

Proposition 3.5. *Let D be a nonempty closed subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be upper $(-C)-$ continuous on D and assumming, in addition, that C be a closed convex cone in Y and $F(x)$ compact for any $x \in D$. Then $IS_{\max}(D, F, C)$ is closed.*

Proof. We take $Q = -C$, then Q is a closed convex cone in Y and the set-valued mapping F is upper $Q-$ continuous on D. By using the obtained result in Theorem 3.1, we deduce that $IS_{\min}(D, F, Q)$ is closed. Therefore, the optimal solution set $IS_{max}(D, F, C)$ is also closed because the following equality holds

$$IS_{\max}(D, F, C) = IS_{\min}(D, F, Q),$$

which completing the proof. □

Theorem 3.2. *Let D be a nonempty closed subset in X and the set-valued mapping $F : D \rightrightarrows Y$ be upper $l(C)-$ continuous on D and assumming, in addition, that C be a closed convex cone in Y and $F(x)$ compact for any $x \in D$. Then $PS_{\min}(D, F, C)$ and $PS_{\max}(D, F, C)$ are closed.*

Proof. We prove only the case $PS_{\min}(D, F, C)$ is closed because the closedness of $PS_{\max}(D, F, C)$ is similarly proceeded. In fact, suppose to the contrary, that there exists sequence $(x_\alpha)_\alpha \subset PS_{\min}(D, F, C)$ such that

$$x_\alpha \to \overline{x},$$

where $\overline{x} \notin PS_{\min}(D, F, C)$. Arguing similarly as for proving Theorem 3.1, there exist neighborhoods U_α $(\alpha > \alpha_0)$ of x_α satisfying

$$F(U_\alpha \cap \mathrm{dom} F) \subset F(\overline{x}) + l(C). \tag{3.7}$$

We next have to show that

$$F(x_\alpha) \cap PMin(F(D)|C) = \emptyset \ \ \forall\, \alpha > \alpha_0. \tag{3.8}$$

Indeed, if (3.8) does not hold, it means that there is at least an element $z_\alpha \in F(x_\alpha)$ with $\alpha > \alpha_0$ such that

$$F(D) \cap (z_\alpha - C) \subset z_\alpha + l(C) \;\; \forall\, \alpha > \alpha_0.$$

It should be noted here that for every $\alpha > \alpha_0$, by using the proof of Theorem 3.1, we obtain $x_\alpha \in U_\alpha$, and moreover it leads to the following result holds

$$z_\alpha \in F(U_\alpha \cap \mathrm{dom}F).$$

This along with (3.7) lead to there exists $c_\alpha \in l(C)$ with $\alpha > \alpha_0$ satisfying

$$z_\alpha - c_\alpha \in F(\overline{x}).$$

It can be seen that

$$\begin{aligned}
&\mathrm{int}C \subset C, \;\; t\,\mathrm{int}C = \mathrm{int}C, \;\; tC = C, \;\; \forall\, t > 0,\\
&\mathrm{int}C + C = \mathrm{int}C, \;\; (-\mathrm{int}C) + (-C) \subset -(C + C) = -C,\\
&C + C \subset C \;\text{ implies }\; l(C) + l(C) \subset l(C).
\end{aligned}$$

We thus have the following relations

$$\begin{aligned}
F(D) \cap (z_\alpha - c_\alpha - C) &\subset F(D) \cap (z_\alpha - C - C)\\
&\subset F(D) \cap (z_\alpha - C) \subset z_\alpha + l(C)\\
&= z_\alpha - c_\alpha + c_\alpha + l(C)\\
&\subset z_\alpha - c_\alpha + l(C) + l(C)\\
&= z_\alpha - c_\alpha + l(C) \;\; \forall\, \alpha > \alpha_0.
\end{aligned}$$

We set

$$y_\alpha = z_\alpha - c_\alpha.$$

Obviously,

$$y_\alpha \in F(\overline{x}) \cap PMin(F(D)|C) \;\; \forall\, \alpha > \alpha_0.$$

So we conclude that $\overline{x}$ being an optimal solution of problem $(GVOP)_{P,\min}$, which conflicts with the fact that $\overline{x} \notin PS_{\min}(D, F, C)$. Therefore, the optimal solution set of problem $PS_{\min}(D, F, C)$ is closed in X, which completes the proof. □

Theorem 3.3. *Under the assumptions of Theorem 3.1. We have the following assertions hold.*

(i) *If $IS_{\min}(D, F, C) \neq \emptyset$ then $PS_{\min}(D, F, C)$ is closed.*
(ii) *If F is upper $(-C)-$ continuous on D and $IS_{\max}(D, F, C) \neq \emptyset$ then $PS_{\max}(D, F, C)$ is closed.*

Proof. By reasons of similarly, we prove only case (i). In fact, we may assume that the optimal solution set $IS_{\min}(D, F, C) \neq \emptyset$, then it is plain that

$$PS_{\min}(D, F, C) = IS_{\min}(D, F, C).$$

Adapting the result obtained in Theorem 3.1, we conclude that $PS_{\min}(D, F, C)$ is closed and the claim follows. □

Theorem 3.4. *Let D be a nonempty closed subset in X, the set-valued mapping $F : D \rightrightarrows Y$ and assumming, in addition, that C be a closed convex cone with its interior nonempty and be not the whole space in Y and $F(x)$ compact for any $x \in D$. Then*

(i) *$WS_{\min}(D, F, C)$ is closed if F is upper $C-$ continuous on D .*
(ii) *$WS_{\max}(D, F, C)$ is closed if F is upper $(-C)-$ continuous on D.*

Proof. We proof only case (i) by reasons of similarly. Assume to the contrary, that there exists sequence $(x_\alpha)_\alpha \subset WS_{\min}(D,F,C)$ and $\overline{x} \notin WS_{\min}(D,F,C)$ such that

$$x_\alpha \to \overline{x}.$$

Repeat the proof of preceding Theorem 3.1, then one finds $\alpha_0 > 0$, U_α is a neighborhood of x_α such that

$$F(U_\alpha \cap \mathrm{dom}F) \subset F(\overline{x}) + C \;\; \forall\, \alpha > \alpha_0. \tag{3.9}$$

It is not difficult to check that

$$F(x_\alpha) \cap PMin(F(D)|C) = \emptyset \;\; \forall\, \alpha > \alpha_0. \tag{3.10}$$

Indeed, if (3.10) does not hold, then there exists at least an element $w_\alpha \in F(x_\alpha)$ with $\alpha > \alpha_0$ such that

$$w_\alpha \in PMin(F(D)|C).$$

Because C is not the whole space, making use of Proposition 2.3 in Luc [13] to deduce that

$$F(D) \cap \Big(w_\alpha - \mathrm{int}C\Big) = \emptyset \;\; \forall\, \alpha > \alpha_0.$$

Note that for every $\alpha > \alpha_0$, then $x_\alpha \in U_\alpha$, which leads to the following result holds

$$w_\alpha \in F(U_\alpha \cap \mathrm{dom}F).$$

Together this with (3.9), it yields that there exists $c_\alpha \in C$ with $\alpha > \alpha_0$ satisfying

$$w_\alpha - c_\alpha \in F(\overline{x}).$$

Since C is a convex cone with its interior nonempty, it yields that the following equality holds

$$C + \mathrm{int}C = \mathrm{int}C.$$

Consequently,

$$w_\alpha - c_\alpha - \mathrm{int}C \subset w_\alpha - \mathrm{int}C.$$

Therefore,

$$w_\alpha - c_\alpha \in F(\overline{x}) \cap WMin(F(D)|C) \;\; \forall\, \alpha > \alpha_0,$$

which means that

$$\overline{x} \in WS_{\min}(D,F,C),$$

contradicting the fact that $\overline{x}$ is not an optimal solution of problem $(GVOP)_{W,\min}$. So the condition (3.10) holds, which leads to x_α with $\alpha > \alpha_0$ does not being optimal solutions of $(GVOP)_{W,min}$, a contradiction. From here we will be allowed to conclude that the optimal solution set $WS_{\min}(D,F,C)$ is closed and this completes the proof. □

Theorem 3.5. *Let D be a nonempty closed subset in X, the set-valued mapping $F : D \rightrightarrows Y$ and assumming, in addition, that C be a closed convex cone with its interior nonempty, $C \setminus l(C)$ be open, C be not the whole space in Y and $F(x)$ compact for any $x \in D$. Then*

- (i) *$PrS_{\min}(D,F,C)$ is closed if F is upper $C-$ continuous on D and the problem $(GVOP)_{I,\min}$ has solution.*
- (ii) *$PrS_{\max}(D,F,C)$ is closed if F is upper $(-C)-$ continuous on D and the problem $(GVOP)_{I,\max}$ has solution.*

Proof. Case (i): We take arbitrary sequence $(x_\alpha)_\alpha \subset PrS_{\min}(D,F,C)$ such that

$$x_\alpha \to \overline{x} \in X.$$

Since $x_\alpha \in D$ for every $\alpha \geq 1$ and the set D is closed, one gets $\overline{x} \in D$. By the initial assumption it yields that the problem $(GVOP)_{I,min}$ has solution. On one hand, it follows from Theorem 3.3 that the optimal solution set $PS_{min}(D,F,C)$ is closed. Making use of Proposition 3.1 to deduce that the following result holds

$$(x_\alpha)_\alpha \subset PS_{\min}(D,F,C).$$

Consequently,

$$\overline{x} \in PS_{\min}(D,F,C).$$

By definitions, we get

$$F(\overline{x}) \cap PMin(F(D)|C) \neq \emptyset.$$

Taking $\overline{y} \in F(\overline{x})$ such that

$$F(D) \cap (\overline{y} - C) \subset \{\overline{y}\} + l(C).$$

By picking

$$K = l(C) \cup \mathrm{int}C.$$

Then K is a convex cone which is not the whole space and contains $C \setminus l(C)$ in its interior. In fact, we get

$$C \setminus l(C) = \mathrm{int}C \subset \mathrm{int}\Big(l(C) \cup \mathrm{int}C\Big) = \mathrm{int}K.$$

On the other hand, it is evident that

$$F(D) \cap (\overline{y} - K) \subset \{\overline{y}\} + l(K),$$

which yields that

$$\overline{y} \in F(\overline{x}) \cap PMin(F(D)|K).$$

Consequently,

$$\overline{x} \in PrS_{\min}(D,F,C),$$

which completing the proof of case (i).

Case (ii): Arguing similarly as for the proof of case (i), where a cone C is replaced by a cone $-C$, we also arrive at the conclusion. □

To close this paper, we give an example illustrating Theorem 3.5.

Example 3.6. Let $X = \mathbb{R}^2 = \{x = (x_1, x_2) : x_1 \in \mathbb{R},\ \ x_2 \in \mathbb{R}\}$, $Y = \mathbb{R}$, $D = [-1,0] \times [-1,0] \subset \mathbb{R}^2$, $C = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. We consider the set-valued mapping $F : D \rightrightarrows \mathbb{R}$ is defined by

$$F(x_1, x_2) = \{x_1 + x_2\} \ \ (\forall\, (x_1, x_2) \in D).$$

It can be easily seen that the cone $C \neq Y$ is closed and convex with $\mathrm{int}C = \mathbb{R}_{++}$ (where $\mathbb{R}_{++} := \mathrm{int}\mathbb{R}_+$) and the cone $C \setminus l(C) = \mathbb{R}_{++}$ is open. Notice that for any $x \in D$, the set $F(x)$ is compact. Let us see that F be upper $C-$ continuous on D. In fact, take arbitrary $\overline{x} := (\overline{x}_1, \overline{x}_2) \in D$ and $\epsilon > 0$, define the neighborhood U of $\overline{x}$ by

$$U = \Big\{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2 \leq (\frac{\epsilon}{2})^2\Big\}.$$

For every $(x_1, x_2) \in U \cap D$ (note that $D = \mathrm{dom}F$), we obtain the following system

$$\begin{cases} (x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2 \leq (\frac{\epsilon}{2})^2 \\ x_1 + x_2 \leq 0. \end{cases}$$

We have that

$$F(x_1, x_2) \subset F(\overline{x}_1, \overline{x}_2) + (-\epsilon, \epsilon) + \mathbb{R}_+. \tag{3.11}$$

Indeed, (3.11) is equivalent to

$$x_1 + x_2 \in \overline{x}_1 + \overline{x}_2 + (-\epsilon, \epsilon) + \mathbb{R}_+,$$

or equivalently,

$$x_1 + x_2 - \overline{x}_1 - \overline{x}_2 \in (-\epsilon, +\infty).$$

Hence,

$$x_1 + x_2 - \overline{x}_1 - \overline{x}_2 > -\epsilon. \tag{3.12}$$

It is well-known that

$$\begin{aligned} |x_1 + x_2| &\le |x_1 + x_2 - \overline{x}_1 - \overline{x}_2| + |\overline{x}_1 + \overline{x}_2| \\ &\le \sqrt{2\big((x_1 - \overline{x}_1)^2 + (x_2 - \overline{x}_2)^2\big)} + |\overline{x}_1 + \overline{x}_2| \\ &< \epsilon + |\overline{x}_1 + \overline{x}_2|. \end{aligned}$$

So, (3.12) is fulfilled, which means that the set-valued mapping F is upper $C-$ continuous on D. We next check that $IS_{\min}(D, F, C) \neq \emptyset$. Indeed, we first pick $(x_1, x_2) \in D$, i.e., $x_i \in [-1, 0]$ for $i = 1, 2$, and one second considers $F(x_1, x_2) \cap IMin(F(D)|C) \neq \emptyset$. By definitions, $x_1 + x_2 \in F(D)$ and $F(D) \subset x_1 + x_2 + \mathbb{R}_+$. By directly calculating,

$$F(D) = \bigcup_{(x_1, x_2) \in D} F(x_1, x_2) = \Big[-2, 0\Big].$$

Thus,

$$\begin{cases} x_1 + x_2 \ge -2 \\ x_1 + x_2 \le -2. \end{cases}$$

This system is equivalent to $x_1 + x_2 = -2$, but $x_1 \ge -1$ and $x_2 \ge -1$, which leads to $x_1 = x_2 = -1$. So,

$$IS_{\min}(D, F, C) = \{(-1, -1)\} \neq \emptyset.$$

Thanks to Theorem 3.5 that the solution sets $PrS_{\min}(D, F, C)$ and $PrS_{\max}(D, F, -C)$ are closed. In fact, in this setting, it holds that $PS_{\min}(D, F, C) = \{(-1, -1)\}$ and further, it follows from Proposition 3.1 that

$$PrS_{\min}(D, F, C) \subset \{(-1, -1)\}.$$

We have to show that $(-1, -1) \in PrS_{\min}(D, F, C)$. In fact, we define a convex cone $\widetilde{C} = \mathbb{R}_+ = [0, +\infty)$. It is obvious that $\widetilde{C}$ is not the whole space $Y = \mathbb{R}$ and contains $C \setminus l(C) = \text{int}\mathbb{R}_+$ in its interior such that $(-1, -1) \in PS_{\min}(D, F, \widetilde{C})$, and so, $PrS_{\min}(D, F, C) = \{(-1, -1)\}$, which means that it is a closed set. Similarly, if we take $C = -\mathbb{R}_+$ then $PrS_{\max}(D, F, C) = \{(-1, -1)\}$ is a closed set, as it was checked.

We close this paper by making some comparisons between the results obtained in the paper and the existing one in the literature.

Remark 3.7. As far as we know, there have not been results on closedness of the optimal solution sets for general vector alpha optimization problems in Hausdorff locally convex topological vector spaces involving the upper (lower) C-continuity of set-valued mapping. The differences between our result in the paper with the well-known results of Cheraghi et al. [1], Farajzadeh et al. [2] and Farajzadeh and Shafie [3] are as follows. We study in this paper the relationships between the optimal solution sets of primal and dual general vector alpha optimization problems in which the closedness of optimal solution sets for the same plays a central role, while Cheraghi et al. [1] derived a link between subdifferential and Fréchet differential with $\epsilon-$generalized weak subdifferential and provided a necessary and sufficient condition for achieving a global minimum of a $\epsilon-$generalized weak subdifferential function; Farajzadeh et al. [2] formulated the relationship between the nonsmooth variational-like inequalities and vector optimization problems involving the existence of solution; Farajzadeh and Shafie [3] obtained some existence theorems of the solution of the system of vector quasi-equilibrium problems for a family of multivalued mappings in the setting of topological order spaces.

4. Conclusion

In this paper, we have shown that the optimal solution sets of dual and primal general vector alpha optimization problems in Hausdorff locally convex topological vector spaces are closed. In addition, some the relationships between the optimal solution sets of these problems are also obtained well.

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References

1. P. Cheraghi, A.P. Farajzadeh, S. Suantai, On Optimization via epsilon-Generalized Weak Subdifferentials, Thai Journal of Mathematics, 16 (1), 2018, 147-164.
2. A. Farajzadeh, M.A. Noor, K.I. Noor, Vector nonsmooth variational-like inequalities and optimization problems, Nonlinear Analysis: Theory, Methods & Applications, 71 (7-8), 2009, 3471-3476.
3. A.P. Farajzadeh, A. Shafie, On system of vector quasi-equilibrium problems for multivalued mappings, Kragujevac Journal of Mathematics, 42 (3), 2018, 357-369.
4. X. P. Ding and J. Y. Park, Generalized vector equilibrium problems in generalized convex spaces, J. Optim Theory Appl., 120, 2004, 327-353.
5. X. H. Gong, Connectedness of the solution sets and scalarization for vector equilibrium problems, J. Optim Theory Appl., 133, 2007, 151-161.
6. X. H. Gong and J. C. Yao, Lower semicontinuity of the set of efficient solutions for generalized systems, J. Optim Theory Appl., 138, 2008, 197-205.
7. X. H. Gong and J. C. Yao, Connectedness of the set of efficient solutions for generalized systems, J. Optim Theory Appl., 138, 2008, 189-196.
8. V. Jeyakumar and X. Q. Yang, On characterizing the solution sets of pseudolinear programs, J. Optim Theory Appl., 87, 1995, 747-755.
9. P. Q. Khanh and L. M. Luu, Upper semicontinuity of the solution set to parametric vector quasivariational inequalities, J. Global. Optim., 32, 2005, 569-560.
10. P. Q. Khanh and L. Q. Anh, On the Holder continuity of solution to parametric multivalued vector equilibrium problems, J. Math. Anal. Appl., 321, 2006, 308-315.
11. L. J. Lin and N. X. Tan, On systems of quasivariational inclusion problems of type I and related problems, Vietnam J. Math., 34, 2006, 423-440.
12. L. J. Lin and N. X. Tan, On quasivariational inclusion problems of type I and related problems, J. Global Optim., 39, 2007, 393-407.

13. D. T. Luc, Theory of vector optimization, Lect. Notes in Eco. and Math. System, Springer Verlag, Berlin, Germany, Vol. 319, 1989.
14. T.V. Su, On the upper $C-$ continuity on the solution set of general vector alpha optimization problems in infinite-dimensional spaces, Commun. Optim. Theory, Vol. 2018, 2018, 1-11.
15. N. X. Tan, On the existence of solutions of quasivariational inclusion problems, J. Optim. Theory Appl., 123, 2004, 619-638.
16. N. X. Tan, Quasi-equilibrium problems and fixed point theorems of separately l.s.c and u.s.c mappings, Numer. Funct. Anal. Optim., (online 2018), DOI: 10.1080/01630563.2017.1360346.
17. Z. L. Wu and S. Y. Wu, Characterizations of the solution sets of comvex programs and variational inequality problems, J. Optim. Theory Appl., 130 (2), 2006, 339-358.