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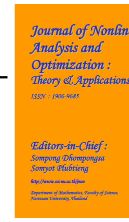

EXISTENCE AND UNIQUENESS OF COUPLED BEST PROXIMITY POINT IN PARTIALLY ORDERED METRIC SPACES

BINAYAK S. CHOUDHURY[1], N. METIYA[2,*] AND P. KONAR[3]

[1] Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India
[2] Department of Mathematics, Sovarani Memorial College, Jagatballavpur, Howrah-711408, West Bengal, India
[3] Department of Mathematics, NITMAS, South 24 Pargana, West Bengal, 743368, India

ABSTRACT. In this paper we utilize a generalized almost contractive mapping to establish some coupled best proximity point results which are global optimization results of finding the minimum distances between two sets. The results are obtained in metric spaces with a partial ordering defined therein. There is a blending of analytic and order theoretic approaches in the proofs. We illustrate the main theorem through an example.

KEYWORDS : Metric space; partial order; almost contraction; coupled best proximity point.
AMS Subject Classification: 47H10, 54H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Best proximity point results are related to the problem of finding minimum distances which is by itself a classical problem considered in many areas of mathematics. It occupies an important position in the calculus of variation [16]. In geometrical studies it is related to the concept of geodesic [3]. In our case the objects are subsets of metric spaces. The minimum distance between pairs of subsets are realized by utilizing best proximity points of non-self mappings.

Technically, $(X,\ d)$ denotes a metric space throughout the paper and $A,\ B \subseteq X$. We use the following notations.

$$D(x,\ B) = \inf\{d(x,\ b):\ b \in B\},\ \text{where } x \in X,$$
$$d(A,\ B) = \inf\{d(a,\ b): a \in A \text{ and } b \in B\},$$
$$A_0 = \{a \in A: d(a,\ b) = d(A,\ B) \text{ for some } b \in B\},$$
$$B_0 = \{b \in B: d(a,\ b) = d(A,\ B) \text{ for some } a \in A\}.$$

* *Corresponding author.*
Email address : binayak12@yahoo.co.in (Binayak S. Choudhury), metiya.nikhilesh@gmail.com (N. Metiya), pulakkonar@gmail.com (P. Konar).
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It is to be noted that for every $a \in A_0$ there exists $b \in B_0$ such that $d(a,\ b) = d(A,\ B)$ and conversely, for every $b^{'} \in B_0$ there exists $a^{'} \in A_0$ such that $d(a^{'},\ b^{'}) = d(A,\ B)$.

Let $T\colon A \longrightarrow B$ be a non-self mapping. Then $x^* \in A$ is a best proximity point of T if $d(x^*,\ Tx^*) = d(A,\ B)$ [10]. A best proximity point reduces to a fixed point in case where $A = B$. A best proximity point problem can be described as the problem of finding an optimal approximate solution of the fixed point equation $x = Tx$ although the exact solution need not exist. This is an approach to the global optimization problem of finding minimum distance between two sets by minimizing globally the quantity $d(x,\ Tx)$ such that the minimum value $d(A,\ B)$ is attained at some point.

At this, point it is pertinent to point out the difference between proximity point results and best approximation results. Unlike former, the best approximation results are not necessarily optimal. For instance, the famous Ky Fan's best approximation result is not an optimality result [11].

Eldred et al [10] introduced best proximity points. Interest in results associated with this concept increased rapidly which has resulted in the publication of a good number of papers on this topic. Side by side, coupled fixed point theorems also occupied large research interest in recent times with the publication of results like [5, 6, 12, 15, 17]. Coupled mapping was utilized in research on best proximity pairs in the work of [20] and was followed by works like [13, 14, 18, 19]. Our purpose is to establish coupled best proximity point theorems in a metric space where a partial order is defined. In the sequel we use an almost contraction like inequality. These inequalities featured in the study of generalized contractions originated by Berinde [4]. This category of inequalities has been utilized in a good number of papers which are predominantly on fixed point studies, some instances being [1, 2, 7, 8, 9]. We utilize this idea for finding best proximity pairs through coupled maps. Precisely, we utilize a generalized almost contraction mapping for the purpose of obtaining coupled best proximity points. The above mentioned mapping is assumed to be defined from one set $A \times A$ to the other set B. Then under suitable conditions, by applying fixed point methodologies, we obtained a coupled best proximity point of the above mentioned mapping which realizes the minimum distance. The main result has one corollary and an illustrative example. Separate order theoretic condition is imposed to ensure the uniqueness of the coupled best proximity point in the main result.

The following are the requisite mathematical concepts for the discussions in this paper.

Throughout this paper, $(X,\ d,\ \preceq)$ denotes a partially ordered metric space where $\preceq$ is a partially order on the metric space $(X,\ d)$.

Definition 1.1 ([12]). A mapping $g\colon A \times A \longrightarrow A$ is said to have the mixed monotone property if

$$u,\ v \in A,\ \ u \preceq v \Longrightarrow g(u,\ y) \preceq g(v,\ y), \quad \text{for all } y \in A;$$

and

$$p,\ q \in A,\ \ p \preceq q \Longrightarrow g(x,\ p) \succeq g(x,\ q), \quad \text{for all } x \in A.$$

Definition 1.2 ([14]). A mapping $g\colon A \times A \longrightarrow B$ is said to have proximal mixed monotone property if for all $x,\ y \in A$

$$\left.\begin{array}{l} u \preceq v \\ d(a,\ g(u,\ y)) = d(A,\ B), \\ d(b,\ g(v,\ y)) = d(A,\ B) \end{array}\right\} \Rightarrow a \preceq b$$

and

$$\left.\begin{array}{l} p \preceq q \\ d(c,\ g(x,\ p)) = d(A,\ B), \\ d(d,\ g(x,\ q)) = d(A,\ B) \end{array}\right\} \Rightarrow c \succeq d,$$

where $u,\ v,\ p,\ q,\ a,\ b,\ c,\ d \in A$.

If $A = B$ in the above definition, the notion of the proximal mixed monotone property reduces to that of the mixed monotone property.

Definition 1.3. A mapping $F\colon A \times A \longrightarrow B$ is said to have proximal mixed monotone property on $A_0 \times A_0$ if for all $x,\ y \in A_0$

$$\left.\begin{array}{l} u \preceq v \\ d(a,\ g(u,\ y)) = d(A,\ B), \\ d(b,\ g(v,\ y)) = d(A,\ B) \end{array}\right\} \Rightarrow a \preceq b$$

and

$$\left.\begin{array}{l} p \preceq q \\ d(c,\ g(x,\ p)) = d(A,\ B), \\ d(d,\ g(x,\ q)) = d(A,\ B) \end{array}\right\} \Rightarrow c \succeq d,$$

where $u,\ v,\ p,\ q,\ a,\ b,\ c,\ d \in A_0$.

Lemma 1.4 ([14]). *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $A,\ B$ are nonempty subsets of X. Assume A_0 is non-empty. Let $g\colon A \times A \longrightarrow B$ be a mapping such that $g(A_0 \times A_0) \subseteq B_0$ and g has proximal mixed monotone property. Then for all $u,\ v,\ p,\ q,\ w \in A_0$*

$$\left.\begin{array}{l} u \preceq v \text{ and } p \succeq q \\ d(v,\ g(u,\ p)) = d(A,\ B), \\ d(w,\ g(v,\ q)) = d(A,\ B) \end{array}\right\} \Rightarrow v \preceq w.$$

Lemma 1.5 ([14]). *Let $(X,\ d,\ \preceq)$ be a partially ordered metric space and $A,\ B$ are nonempty subsets of X. Assume A_0 is non-empty. Let $g\colon A \times A \longrightarrow B$ be a mapping such that $g(A_0 \times A_0) \subseteq B_0$ and g has proximal mixed monotone property. Then for all $u,\ v,\ p,\ q,\ z \in A_0$*

$$\left.\begin{array}{l} u \preceq v \text{ and } p \succeq q \\ d(q,\ g(p,\ u)) = d(A,\ B), \\ d(z,\ g(q,\ v)) = d(A,\ B) \end{array}\right\} \Rightarrow q \succeq z.$$

Definition 1.6 ([20]). A point $(a,\ b) \in A \times A$ is said to be a coupled best proximity point of the mapping $g\colon A \times A \longrightarrow B$ if $d(a,\ g(a,\ b)) = d(A,\ B)$ and $d(b,\ g(b,\ a)) = d(A,\ B)$.

2. MAIN RESULTS

Theorem 2.1. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $(A,\ B)$ be a pair of non-empty closed subsets of X such that A_0 is non-empty and closed. Let $F\colon A \times A \longrightarrow B$ be a mapping such that $F(A_0 \times A_0) \subseteq B_0$ and F has proximal mixed monotone property on $A_0 \times A_0$. Suppose that there exist nonnegative real numbers a, b and L with $a + b < 1$ such that for all $x,\ y,\ u,\ v,\ p,\ q \in A_0$*

$$\left.\begin{array}{l} x \preceq u \text{ and } y \succeq v, \\ d(p,\ F(x,\ y)) = d(A,\ B), \\ d(q,\ F(u,\ v)) = d(A,\ B) \end{array}\right\} \Longrightarrow d(p,\ q) \leq N(x,\ y,\ u,\ v,\ p,\ q),$$

where

$N(x,\ y,\ u,\ v,\ p,\ q) = a\, d(x,\ u) + b\, d(y,\ v) + L\ min\{d(p,\ u),\ d(q,\ x),\ d(p,\ x),\ d(q,\ u)\}.$

Suppose either

(a) F is continuous or

(b) X has the following properties:

(i) if a nondecreasing sequence $\{x_n\} \longrightarrow x$, then $x_n \preceq x$, for all $n \geq 0$;

(ii) if a nonincreasing sequence $\{y_n\} \longrightarrow y$, then $y \preceq y_n$, for all $n \geq 0$.

Also, suppose that there exist elements $(x_0,\ y_0)$, $(x_1,\ y_1) \in A_0 \times A_0$ such that $d(x_1,\ F(x_0,\ y_0)) = d(A,\ B)$ and $d(y_1,\ F(y_0,\ x_0)) = d(A,\ B)$ with $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Then F has a coupled best proximity point in $A_0 \times A_0$.

Proof. By the hypothesis of the theorem there exist elements $(x_0,\ y_0)$, $(x_1,\ y_1) \in A_0 \times A_0$ such that $x_0 \preceq x_1$ and $y_0 \succeq y_1$ and

$$d(x_1,\ F(x_0,\ y_0)) = d(A,\ B) \text{ and } d(y_1,\ F(y_0,\ x_0)) = d(A,\ B). \tag{2.1}$$

Since $F(A_0 \times A_0) \subseteq B_0$, there exists an element $(x_2,\ y_2) \in A_0 \times A_0$ such that

$$d(x_2,\ F(x_1,\ y_1)) = d(A,\ B) \text{ and } d(y_2,\ F(y_1,\ x_1)) = d(A,\ B). \tag{2.2}$$

Hence by the Lemmas 1.4 and 1.5, we have $x_1 \preceq x_2$ and $y_1 \succeq y_2$.

Continuing this process, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in A_0 such that

$$x_n \preceq x_{n+1} \text{ and } y_n \succeq y_{n+1} \text{ for all } n \geq 0 \tag{2.3}$$

and

$$d(x_{n+1},\ F(x_n,\ y_n)) = d(A,\ B) \text{ and } d(y_{n+1},\ F(y_n,\ x_n)) = d(A,\ B). \tag{2.4}$$

Now,

$$\left.\begin{array}{l} x_n \preceq x_{n+1} \text{ and } y_n \succeq y_{n+1}, \\ d(x_{n+1},\ F(x_n,\ y_n)) = d(A,\ B), \\ d(x_{n+2},\ F(x_{n+1},\ y_{n+1})) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(x_{n+1},\ x_{n+2}) \leq N(x_n,\ y_n,\ x_{n+1},\ y_{n+1},\ x_{n+1},\ x_{n+2}), \tag{2.5}$$

where

$$\begin{aligned} N(x_n,\ y_n,\ x_{n+1},\ y_{n+1},\ x_{n+1},\ x_{n+2}) &= a\ d(x_n,\ x_{n+1}) + b\ d(y_n,\ y_{n+1}) \\ &\quad + L \min\{d(x_{n+1},\ x_{n+1}),\ d(x_{n+2},\ x_n),\ d(x_{n+1},\ x_n),\ d(x_{n+2},\ x_{n+1})\} \\ &= a\ d(x_n,\ x_{n+1}) + b\ d(y_n,\ y_{n+1}). \end{aligned}$$

So, we have

$$d(x_{n+1},\ x_{n+2}) \leq a\ d(x_n,\ x_{n+1}) + b\ d(y_n,\ y_{n+1}). \tag{2.6}$$

Similarly, it can be obtained that

$$d(y_{n+1},\ y_{n+2}) \leq a\ d(y_n,\ y_{n+1}) + b\ d(x_n,\ x_{n+1}). \tag{2.7}$$

Combination of (2.6) and (2.7) implies that

$$d(x_{n+1},\ x_{n+2}) + d(y_{n+1},\ y_{n+2}) \leq (a+b)\ [d(x_n,\ x_{n+1}) + d(y_n,\ y_{n+1})]. \tag{2.8}$$

Let $r_n = d(x_n,\ x_{n+1}) + d(y_n,\ y_{n+1})$ and $k = a + b$. By repeated application of (2.8), we get

$$0 \leq r_n \leq k\ r_{n-1} \leq k^2\ r_{n-2} \leq \ldots \leq k^n\ r_0. \tag{2.9}$$

Let $m,\ n \in \mathbb{N}$ with $m < n$. Then

$$\begin{aligned} d(x_m,\ x_n) + d(y_m,\ y_n) &\leq d(x_m,\ x_{m+1}) + d(y_m,\ y_{m+1}) + d(x_{m+1},\ x_{m+2}) + \\ & d(y_{m+1},\ y_{m+2}) + \cdots + d(x_{n-1},\ x_n) + d(y_{n-1},\ y_n) \\ &\leq r_m + r_{m+1} + \ldots + r_{n-1} \leq [k^m + k^{m+1} + \ldots + k^{n-1}] r_0 \end{aligned}$$

$$\leq \frac{k^m}{1-k}\, r_0 \longrightarrow 0 \text{ as } m \longrightarrow \infty \text{ (since } 0 \leq k < 1).$$

Then it follows that

$$\lim_{m,\ n \to \infty} d(x_m,\ x_n) = 0 \text{ and } \lim_{m,\ n \to \infty} d(y_m,\ y_n) = 0,$$

which implies that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in A_0. Since A_0 is a closed subset of the complete metric space $(X,\ d)$, A_0 is also complete. Now from the completeness of A_0, there exists $x^*,\ y^* \in A_0$ such that

$$\lim_{n \to \infty} x_n = x^*; \text{ that is, } \lim_{n \to \infty} d(x_n,\ x^*) = 0 \tag{2.10}$$

and

$$\lim_{n \to \infty} y_n = y^*; \text{ that is, } \lim_{n \to \infty} d(y_n,\ x^*) = 0. \tag{2.11}$$

Let the condition (a) holds.

Taking limit as $n \longrightarrow \infty$ in (2.4) and using (2.10), (2.11) and the continuity of F, we have

$$d(x^*,\ F(x^*,\ y^*)) = d(A,\ B) \text{ and } d(y^*,\ F(y^*,\ x^*)) = d(A,\ B).$$

Therefore, $(x^*,\ y^*)$ is a coupled best proximity point of F.

Next we suppose that the condition (b) holds.

Using the condition (b) of the theorem, (2.3), (2.10) and (2.11), we have

$$x_n \preceq x^* \text{ and } y_n \succeq y^*, \text{ for all } n \geq 0. \tag{2.12}$$

Since $x^*,\ y^* \in A_0$ and $F(A_0 \times A_0) \subseteq B_0$, there exists $u,\ v \in A_0$ such that

$$d(u,\ F(x^*,\ y^*)) = d(A,\ B) \text{ and } d(v,\ F(y^*,\ x^*)) = d(A,\ B). \tag{2.13}$$

By (2.4), (2.12) and (2.13)

$$\left.\begin{array}{l} x_n \preceq x^* \text{ and } y_n \succeq y^*, \\ d(x_{n+1},\ F(x_n,\ y_n)) = d(A,\ B), \\ d(u,\ F(x^*,\ y^*)) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(x_{n+1},\ u) \leq N(x_n,\ y_n,\ x^*,\ y^*,\ x_{n+1},\ u), \tag{2.14}$$

where

$$N(x_n,\ y_n,\ x^*,\ y^*,\ x_{n+1},\ u) = a\, d(x_n,\ x^*) + b\, d(y_n,\ y^*) + L \min \{d(x_{n+1},\ x^*),\ d(u,\ x_n),\ d(x_{n+1},\ x_n),\ d(u,\ x^*)\}.$$

Using (2.10) and (2.11), we obtain

$$\lim_{n \to \infty} N(x_n,\ y_n,\ x^*,\ y^*,\ x_{n+1},\ u) = 0. \tag{2.15}$$

Taking the limit as $n \longrightarrow \infty$ in (2.14), using (2.10) and (2.15), we have $d(x^*,\ u) \leq 0$, which implies that $d(x^*,\ u) = 0$; that is, $u = x^*$.

Again, by (2.4), (2.12) and (2.13)

$$\left.\begin{array}{l} y^* \preceq y_n \text{ and } x^* \succeq x_n, \\ d(v,\ F(y^*,\ x^*)) = d(A,\ B), \\ d(y_{n+1},\ F(y_n,\ x_n)) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(v,\ y_{n+1}) \leq N(y^*,\ x^*,\ y_n,\ x_n,\ v,\ y_{n+1}), \tag{2.16}$$

where

$$N(y^*,\ x^*,\ y_n,\ x_n,\ v,\ y_{n+1}) = a\, d(y^*,\ y_n) + b\, d(x^*,\ x_n) + L \min \{d(v,\ y_n),\ d(y_{n+1},\ y^*),\ d(v,\ y^*),\ d(y_{n+1},\ y_n)\}.$$

Using (2.10) and (2.11), we obtain

$$\lim_{n \to \infty} N(y^*,\ x^*,\ y_n,\ x_n,\ v,\ y_{n+1}) = 0. \tag{2.17}$$

Taking the limit as $n \longrightarrow \infty$ in (2.16), using (2.11) and (2.17), we have $d(v,\ y^*) \leq 0$ which implies that $d(v,\ y^*) = 0$; that is, $v = y^*$.
Since $u = x^*$ and $v = y^*$, we have from (2.13) that

$$d(x^*,\ F(x^*,\ y^*)) = d(A,\ B) \ \text{ and } \ d(y^*,\ F(y^*,\ x^*)) = d(A,\ B).$$

Hence $(x^*,\ y^*)$ is a coupled best proximity point of F. □

With the help of partially ordered set $(X,\ \preceq)$ we endow the product space $X \times X$ with the following partial order:
for $(x,\ y),\ (u,\ v) \in X \times X,\ \ (u,\ v) \preceq (x,\ y) \Leftrightarrow x \succeq u,\ y \preceq v.$

Theorem 2.2. *In addition to the hypotheses of Theorems 2.1, suppose that for every $(x,\ y),\ (x^*,\ y^*) \in A_0 \times A_0$ there exists $(u,\ v) \in A_0 \times A_0$ such that $(u,\ v)$ is comparable to $(x,\ y)$ and $(x^*,\ y^*)$. Then F has a unique coupled best proximity point.*

Proof. From Theorem 2.1, the set of coupled best proximity points F is non-empty. Suppose that $(x,\ y)$ and $(x^*,\ y^*)$ are coupled best proximity points of F. So

$$d(x,\ F(x,\ y)) = d(A,\ B), \quad d(y,\ F(y,\ x) = d(A,\ B), \tag{2.18}$$

and

$$d(x^*,\ F(x^*,\ y^*)) = d(A,\ B), \ \ d(y^*,\ F(y^*,\ x^*) = d(A,\ B). \tag{2.19}$$

Now, we show that $(x,\ y) = (x^*,\ y^*)$.
By the assumption, there exists $(u,\ v) \in A_0 \times A_0$ such that $(u,\ v)$ is comparable with $(x,\ y)$ and $(x^*,\ y^*)$.
Put $(u_0,\ v_0) = (u,\ v)$.
Suppose that

$$(u_0,\ v_0) \preceq (x,\ y); \ \text{that is,} \ \ u_0 \preceq x, \ \ v_0 \succeq y \ \text{(the proof is similar in other case).} \tag{2.20}$$

Since $u = u_0,\ v = v_0 \in A_0$ and $F(A_0 \times A_0) \subseteq B_0$, there exists $(u_1,\ v_1) \in A_0 \times A_0$ such that

$$d(u_1,\ F(u_0,\ v_0)) = d(A,\ B) \ \text{ and } \ d(v_1,\ F(v_0,\ u_0)) = d(A,\ B). \tag{2.21}$$

From (2.18), (2.19), (2.20) and (2.21), we have

$$\left.\begin{array}{lcl} u_0 \preceq x \text{ and } v_0 \succeq y & & u_0 \preceq x \text{ and } v_0 \succeq y \\ d(u_1,\ F(u_0,\ v_0)) = d(A,\ B) & \text{and} & d(v_1,\ F(v_0,\ u_0)) = d(A,\ B) \\ d(x,\ F(x,\ y)) = d(A,\ B), & & d(y,\ F(y,\ x)) = d(A,\ B). \end{array}\right\}$$

Since $F(A_0 \times A_0) \subseteq B_0$ and $x,\ v_0 \in A_0$, there exists $x_1 \in A_0$ such that

$$d(x_1,\ F(x,\ v_0)) = d(A,\ B).$$

Now we have

$$\left.\begin{array}{lcl} u_0 \preceq x, & & y \preceq v_0, \\ d(u_1,\ F(u_0,\ v_0)) = d(A,\ B) & \text{and} & d(x,\ F(x,\ y)) = d(A,\ B) \\ d(x_1,\ F(x,\ v_0)) = d(A,\ B), & & d(x_1,\ F(x,\ v_0)) = d(A,\ B). \end{array}\right\}$$

Using the proximal mixed monotone property of F, we have

$$u_1 \preceq x_1 \text{ and } x_1 \preceq x \text{ which implies that } u_1 \preceq x.$$

Again, since $F(A_0 \times A_0) \subseteq B_0$ and $v_0,\ x \in A_0$, there exists $y_1 \in A_0$ such that

$$d(y_1,\ F(v_0,\ x)) = d(A,\ B).$$

Now we have

$$\left.\begin{array}{lcl} u_0 \preceq x, & & y \preceq v_0, \\ d(v_1,\ F(v_0,\ u_0)) = d(A,\ B) & \text{and} & d(y,\ F(y,\ x)) = d(A,\ B) \\ d(y_1,\ F(v_0,\ x)) = d(A,\ B), & & d(y_1,\ F(v_0,\ x)) = d(A,\ B). \end{array}\right\}$$

Using the proximal mixed monotone property of F, we have

$$v_1 \succeq y_1 \text{ and } y_1 \succeq y \text{ which implies that } v_1 \succeq y.$$

Therefore, we have

$$(u_1,\ v_1) \preceq (x,\ y). \tag{2.22}$$

Continuing this process, we have sequences $\{u_n\}$ and $\{v_n\}$ in A_0 such that

$$d(u_{n+1},\ F(u_n,\ v_n)) = d(A,\ B),\ \ d(v_{n+1},\ F(v_n,\ u_n)) = d(A,\ B) \text{ and } (u_n,\ v_n) \preceq (x,\ y) \text{ for all } n \geq 0. \tag{2.23}$$

By (2.18) and (2.23)

$$\left.\begin{array}{l} u_n \preceq x \text{ and } v_n \succeq y \\ d(u_{n+1},\ F(u_n,\ v_n)) = d(A,\ B), \\ d(x,\ F(x,\ y)) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(u_{n+1},\ x) \leq N(u_n,\ v_n,\ x,\ y,\ u_{n+1},\ x), \tag{2.24}$$

where

$$\begin{aligned} N(u_n,\ v_n,\ x,\ y,\ u_{n+1},\ x) &= a\ d(u_n,\ x) + b\ d(v_n,\ y) \\ &\quad + L \min\{d(u_{n+1},\ x),\ d(x,\ u_n),\ d(u_{n+1},\ u_n),\ d(x,\ x)\} \\ &= a\ d(u_n,\ x) + b\ d(v_n,\ y). \end{aligned}$$

Therefore, from (2.24), we have

$$d(u_{n+1},\ x) = a\ d(u_n,\ x) + b\ d(v_n,\ y). \tag{2.25}$$

Again, by (2.18) and (2.23)

$$\left.\begin{array}{l} y \preceq v_n \text{ and } x \succeq u_n \\ d(y,\ F(y,\ x)) = d(A,\ B), \\ d(v_{n+1},\ F(v_n,\ u_n)) = d(A,\ B) \end{array}\right\} \Longrightarrow$$

$$d(y,\ v_{n+1}) \leq N(y,\ x,\ v_n,\ u_n,\ y,\ v_{n+1}), \tag{2.26}$$

where

$$\begin{aligned} N(y,\ x,\ v_n,\ u_n,\ y,\ v_{n+1}) &= a\ d(y,\ v_n) + b\ d(x,\ u_n) \\ &\quad + L \min\{d(y,\ v_n),\ d(v_{n+1},\ y),\ d(y,\ y),\ d(v_{n+1},\ v_n)\} \\ &= a\ d(y,\ v_n) + b\ d(x,\ u_n). \end{aligned}$$

From (2.26) it follows that

$$d(y,\ v_{n+1}) \leq a\ d(y,\ v_n) + b\ d(x,\ u_n). \tag{2.27}$$

Combining (2.25) and (2.27), we have

$$d(u_{n+1},\ x) + d(v_{n+1},\ y) \leq (a+b)\ [d(u_n,\ x) + d(v_n,\ y)]. \tag{2.28}$$

Since $a + b < 1$, it follows from (2.28) that

$$d(u_{n+1},\ x) + d(v_{n+1},\ y) \leq d(u_n,\ x) + d(v_n,\ y).$$

Therefore, $\{d(u_n,\ x) + d(v_n,\ y)\}$ is a monotonically decreasing sequence of non-negative real numbers and hence there exists a $p \geq 0$ such that

$$\lim_{n\to\infty} [d(u_n,\ x) + d(v_n,\ y)] = p. \tag{2.29}$$

We show that $p = 0$. If possible, let $p > 0$.
Taking limit as $n \longrightarrow \infty$ in (2.28) and using (2.29), we have

$$p \leq (a+b)\ p < p, \quad (\text{since } (a+b) < 1)$$

which is a contradiction. Therefore, $p = 0$. Hence

$$\lim_{n\to\infty} [d(u_n,\ x) + d(v_n,\ y)] = 0, \tag{2.30}$$

which implies that

$$\lim_{n\to\infty} d(u_n,\ x) = 0 \text{ and } \lim_{n\to\infty} d(v_n,\ y) = 0. \tag{2.31}$$

Similarly, we can prove that

$$\lim_{n\to\infty} [d(u_n,\ x^*) + d(v_n,\ y^*)] = 0, \tag{2.32}$$

and hence

$$\lim_{n\to\infty} d(u_n,\ x^*) = 0 \text{ and } \lim_{n\to\infty} d(v_n,\ y^*) = 0. \tag{2.33}$$

Using triangle inequality, (2.31) and (2.33), we have

$$d(x,\ x^*)+d(y,\ y^*) \leq [d(x,\ u_n)+d(u_n,\ x^*)+d(y,\ v_n)+d(v_n,\ y^*)] \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Hence $d(x,\ x^*) + d(y,\ y^*) = 0$, which implies that $d(x,\ x^*) = 0$ and $d(y,\ y^*) = 0$; that is, $x = x^*$ and $y = y^*$; that is, $(x,\ y) = (x^*,\ y^*)$. Therefore, the coupled best proximity point of F is unique. □

Example 2.3. Let $X = R^2$ (R denotes the set of real numbers) and d be the Euclidean metric on X. We define a partial order $\preceq$ on X such that $(x,\ y) \preceq (u,\ v)$ if and only if $x \leq u$ and $y \leq v$, for all $(x,\ y),\ (u,\ v) \in X$. Let
$A = \{(2,\ 0),\ (0,\ 2)\} \cup \{(x,\ 0) : 2 \leq x \leq 3\}$,
$B = \{(-2,\ 0),\ (0,\ -2)\} \cup \{(0,\ x) : -3 \leq x \leq -2\}$,
$A_0 = \{(2,\ 0),\ (0,\ 2)\}$ and $B_0 = \{(-2,\ 0),\ (0,\ -2)\}$.
Let $F : A \times A \to B$ be defined as

$$F((x_1,\ x_2),\ (y_1,\ y_2)) = (-x_2,\ -x_1) \text{ for all } (x_1,\ x_2),\ (y_1,\ y_2) \in A \times A.$$

Let a, b and L be three nonnegative real numbers with $a + b < 1$.

Here all the conditions of theorem 2.1 are satisfied and it is seen that ((2, 0), (0, 2)) and ((0, 2), (2, 0)) are two coupled best proximity points of F.

Considering $L = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.4. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $(A,\ B)$ be a pair of non-empty closed subsets of X such that A_0 is non-empty and closed. Let $F\colon A\times A \longrightarrow B$ be a mapping such that $F(A_0\times A_0) \subseteq B_0$ and F has proximal mixed monotone property on $A_0 \times A_0$. Suppose that there exist nonnegative real numbers a and b with $a + b < 1$ such that for all $x,\ y,\ u,\ v,\ p,\ q \in A_0$*

$$\left.\begin{array}{l} x \preceq u \text{ and } y \succeq v, \\ d(p,\ F(x,\ y)) = d(A,\ B), \\ d(q,\ F(u,\ v)) = d(A,\ B) \end{array}\right\} \Longrightarrow d(p,\ q) \leq a\, d(x,\ u) + b\, d(y,\ v).$$

Suppose that the condition (a) or (b) of the theorem 2.1 holds. Also, suppose that there exist elements $(x_0,\ y_0),\ (x_1,\ y_1) \in A_0 \times A_0$ such that $d(x_1,\ F(x_0,\ y_0)) = d(A,\ B)$ and $d(y_1,\ F(y_0,\ x_0)) = d(A,\ B)$ with $x_0 \preceq x_1$ and $y_0 \succeq y_1$. Then F has a coupled best proximity point in $A_0 \times A_0$.

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