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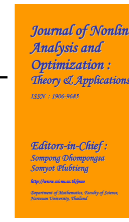

BEST APPROXIMATION AND BEST COAPPROXIMATION IN METRIC LINEAR SPACES

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ABSTRACT. In this paper, we discuss best approximation and best coapproximation in metric linear spaces. We obtain some results on the characterizations, existence and uniqueness of elements of best approximation and best coapproximation in metric linear spaces. We also study single-valuedness and linearity of metric projection and metric coprojection.

KEYWORDS : Proximinal set; coproximinal set; Chebyshev set; co-Chebyshev set; quasi-orthogonality.
AMS Subject Classification: 41A50, 41A52, 41A65.

1. INTRODUCTION AND PRELIMINARIES

A new tool in approximation theory, called best coapproximation by Papini and Singer [13], was introduced in normed linear spaces by C. Franchetti and M. Furi [1]. Subsequently, this theory has been developed to a large extent in normed linear spaces and in Hilbert spaces by C. Franchetti and M. Furi, H. Mazaheri, T.D. Narang, P.L. Papini and I. Singer, Geetha S. Rao and her coworkers, and by many others (see e.g. [1], [3], [4], [8], [13]-[16] and references cited therein). However, the situation in case of metric linear spaces and metric spaces is somewhat different. Although, some attempts have been made in this direction (see e.g. [9]-[12]) but still the theory is less developed as compared to the theory of best approximation. The present paper is also a step in this direction. This paper mainly deals with the characterizations of elements of best approximation and best coapproximation in metric linear spaces. Some results concerning the existence and uniqueness of elements of best approximation and best coapproximation have been discussed. We also study single-valuedness and linearity of metric projection and metric coprojection.

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Let G be a closed subset of a metric space (X, d). An element $g_0 \in G$ is called a *best approximation* (*best coapproximation*) to $x \in X$ if

$$d(x, g_0) \leq d(x, g)\ (d(g_0, g) \leq d(x, g))$$

for all $g \in G$. The set of all such $g_0 \in G$ is denoted by $P_G(x)(R_G(x))$. The set G is called *proximinal* (*coproximinal*) if $P_G(x)$ $(R_G(x))$ contains at least one element for every $x \in X$. If for each $x \in X$, $P_G(x)(R_G(x))$ has exactly one element, then the set G is called $Chebyshev$ $(co - Chebyshev)$.

We shall denote the set $\{x \in X : g_0 \in P_G(x)\}$ $(\{x \in X : g_0 \in R_G(x)\})$ by $P_G^{-1}(g_0)$ $(R_G^{-1}(g_0))$.

For a proximinal (coproximinal) subset G of X, the mapping $P_G(R_G) : X \to 2^G(\equiv$ the collection of all subsets of $G)$ defined by $P_G(x) = \{g_0 \in G : d(x, g_0) \leq d(x, g)\ for\ every\ g \in G\}$ $(R_G(x) = \{g_0 \in G : d(g_0, g) \leq d(x, g)\ for\ every\ g \in G\})$ is called *metric projection* (*metric coprojection*).

A linear space X together with a translation invariant metric d (i.e., $d(x+z, y+z) = d(x, y)\ for\ all\ x, y, z \in X)$ such that addition and scalar multiplication are continuous in (X, d) is called a *metric linear space*.

Every normed linear space is a metric linear sapce but a metric linear space need not be normable (see [17], p.31-36).

Remarks 1.1.

(i) A proximinal subset of a metric space need not be coproximinal:
Let $X = \mathbb{R}^2$ and $G = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then G is a compact subset of $\mathbb{R}^2$ and hence proximinal. However, G is not coproximinal as $(0, 0) \in \mathbb{R}^2$ does not have any best coapproximation in G.

(ii) A coproximinal subset of a metric space need not be proximinal:
Let $X = \mathbb{R} - \{1\}$ and $M = (1, 2]$, then M is a coproximinal subset of X but is not proximinal.

(iii) A Chebyshev subset of a metric space need not be co-Chebyshev:
Let $X = \mathbb{R}$ and $G = [1, 2]$, then G is Chebyshev but not co-Chebyshev.

(iv) A co-Chebyshev subset of a metric space need not be Chebyshev:
Let $X = R^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and $G = \{(x, y) \in R^2 : x = y\}$. Then G is a proximinal subset of X. We have $P_G(x, y) = \{\alpha(x, x) + (1 - \alpha)(y, y) : 0 \leq \alpha \leq 1\}$, i.e., G is not Chebyshev, but $R_G(x, y) = \{(\frac{x+y}{2}, \frac{x+y}{2})\}$, i.e., G is co-Chebyshev.

(v) The set $P_G^{-1}(g_0)(R_G^{-1}(g_0))$ is a closed set for every $g_0 \in G$.

(vi) If G is a subspace of a metric linear space (X, d) then $P_G^{-1}(0) \bigcap G = \{0\}$ and $R_G^{-1}(0) \bigcap G = \{0\}$, where $P_G^{-1}(0) = \{x \in X : 0 \in P_G(x)\}$ and $R_G^{-1}(0) = \{x \in X : 0 \in R_G(x)\}$.

(vii) If G is subspace of a metric linear space (X, d), then $g_0 \in P_G(x)$ $(g_0 \in R_G(x))$ if and only if $x - g_0 \in P_G^{-1}(0)$ $(x - g_0 \in R_G^{-1}(0))$ and $P_G(x + g) = P_G(x) + g$ $(R_G(x + g) = R_G(x) + g)$ for every $g \in G$.

(viii) If G is subspace of a metric linear space (X, d), then $d(g, 0) = d(g, R_G^{-1}(0))$ for every $g \in G$.

For a closed linear subspace G of a metric linear space (X, d), the *canonical mapping* π of X onto X/G is defined as $\pi(x) = x + G,\ x \in X$. This mapping π is linear, continuous and open (see [17], p.29).

Let (X, d) be a metric linear space and $x, y \in X$. We say that x is *orthogonal* to y, $x \perp y$ if $d(x, 0) \leq d(x, \alpha y)$ for every scalar α. For a subset G of X, we say that $G \perp x$ if $g \perp x$ for every $g \in G$.

2. MAIN RESULTS

This section mainly deals with the characterizations, existence and uniqueness of elements of best approximation and best coapproximation in metric linear spaces. We start with the following theorem which gives equivalent conditions under which coproximinal subspaces are co-Chebyshev:

Theorem 2.1. *Let G be a coproximinal subspace of a metric linear space (X, d), then the following are equivalent:*
(i) R_G is one-valued and linear.
(ii) $R_G^{-1}(0)$ is a linear subspace of X.

Proof. (i)⇒ (ii). Let $x, y \in R_G^{-1}(0)$ and $\alpha,\ \beta$ be scalars. Then $R_G(x) = \{0\}\ and\ R_G(y) = \{0\}$. Since R_G is linear, we have $R_G(\alpha x + \beta y) = \alpha R_G(x) + \beta R_G(y) = \{0\}$. This implies that $\alpha x + \beta y \in R_G^{-1}(0)$.
(ii)⇒ (i). Let $g_1, g_2 \in R_G(x)$. This gives $x - g_1, x - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a subspace, we have $(x - g_1) - (x - g_2) \in R_G^{-1}(0)$, i.e., $g_2 - g_1 \in R_G^{-1}(0)$. This gives $g_2 - g_1 \in R_G^{-1}(0) \bigcap G = \{0\}$ and so $g_1 = g_2$. Hence R_G is one-valued.

Let $x, y \in X$ and α, β be scalars. Suppose $g_1 \in R_G(x)$ and $g_2 \in R_G(y)$. This gives $x - g_1,\ y - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a subspace, we have $\alpha(x - g_1) + \beta(y - g_2) \in R_G^{-1}(0)$, i.e., $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in R_G^{-1}(0)$. Now, $R_G(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) = R_G(\alpha x + \beta y - (\alpha g_1 + \beta g_2)) = \{0\}$, as R_G is single-valued. Hence $R_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha R_G(x) + \beta R_G(y)$. □

Proceeding on similar lines, we obtain the following theorem which gives equivalent conditions under which proximinal subspaces are Chebyshev:

Theorem 2.2. *Let G be a proximinal subspace of a metric linear space (X, d), then the following are equivalent:*
(i) P_G is one-valued and linear.
(ii) $P_G^{-1}(0)$ is a linear subspace of X.

Remarks 2.3. *(i) For normed linear spaces, Theorem 2.1 was proved in* [12] *and Theorem 2.2 in* [2].
(ii) If we take G to be only a proximinal subset containing zero, then $P_G^{-1}(0)$ is a subspace but G need not be Chebyshev.

Example 2.4. *Let $X = \mathbb{R}$ with usual metric and $G = (-\infty, 1] \bigcup [2, \infty)$. Then $P_G^{-1}(0) = \{0\}$ is a subspace but G is not a Chebyshev set.*

Example 2.4 also shows that one of the main result (Theorem 2.6) proved in [5] is not valid.

We require the following lemmas proved in [12] ([10]) for our next results:

Lemma 2.5. *Let G be a linear subspace of a metric linear space (X, d), then the following are equivalent:*
(i) G is proximinal (coproximinal).
(ii) $X = G + P_G^{-1}(0)$ $(X = G + R_G^{-1}(0))$.

Lemma 2.6. *Let G be a linear subspace of a metric linear space (X, d) then the following are equivalent:*
(i) G is Chebyshev (co-Chebyshev).
(ii) $X = G \bigoplus P_G^{-1}(0)$ $(X = G \bigoplus R_G^{-1}(0))$, where $\bigoplus$ means that the sum decomposition of each $x \in X$ is unique.

The following theorem gives necessary and sufficient condition for the metric coprojection to be linear.

Theorem 2.7. *Let G be a co-Chebyshev subspace of a metric linear space (X, d), then the following are equivalent:*
(i) R_G is linear.
(ii) $R_G^{-1}(0)$ is a subspace.
(iii) $R_G^{-1}(0)$ contains a subspace N for which $X = G \bigoplus N$.

Proof. (i)⇒ (ii) follows from Theorem 2.1.
(ii)⇒(iii) is obvious by Lemma 2.6.
(iii)⇒ (i). Let $x, y \in X$ and α, β be scalars.. Then $x = g_1 + n_1$ and $y = g_2 + n_2$ for some $g_1, g_2 \in G$ and $n_1, n_2 \in N$. Therefore, $x - g_1, y - g_2 \in N$. Since N is a subspace, we have $\alpha(x - g_1) + \beta(y - g_2) \in N \subseteq R_G^{-1}(0)$ for all scalars α, β. This gives $(\alpha x + \beta y) - (\alpha g_1 + \beta g_2) \in R_G^{-1}(0)$, i.e., $R_G(\alpha x + \beta y - (\alpha g_1 + \beta g_2)) = \{0\}$(as G is co-Chebyshev), i.e., $R_G(\alpha x + \beta y) = \alpha g_1 + \beta g_2 = \alpha R_G(x) + \beta R_G(y)$. □

Proceeding on similar lines, we obtain the following theorem which gives necessary and sufficient conditions for the metric projection to be linear.

Theorem 2.8. *Let G be a Chebyshev subspace of a metric linear space (X, d), then the following are equivalent:*
(i) P_G is linear.
(ii) $P_G^{-1}(0)$ is a subspace.
(iii) $P_G^{-1}(0)$ contains a subspace N for which $X = G \bigoplus N$.

Remarks 2.9. *For normed linear spaces, Theorem 2.7 was proved in* [15] *and Theorem 2.8 in* [2].

The following theorem gives necessary and sufficient conditions for the set $R_G^{-1}(0)$ to be Chebyshev:

Theorem 2.10. *Let G be a coproximinal subspace of a metric linear space (X, d). If $R_G^{-1}(0)$ is an additive group then the following are equivalent:*
(i) $R_G^{-1}(0)$ is a Chebyshev set.
(ii) $G = \{x \in X : d(x, R_G^{-1}(0)) = d(x, 0)\}$.

Proof. (i)⇒ (ii). Suppose $x \in X$ is such that

$$d(x, R_G^{-1}(0)) = d(x, 0). \tag{2.1}$$

Then there exist $g \in G$ such that $\breve{g} = x - g \in R_G^{-1}(0)$ (as G is a coproximinal subspace and so by Lemma 2.5, $X = G + R_G^{-1}(0)$). Therefore $d(x, \breve{g}) = d(g, 0) = d(g, R_G^{-1}(0)) = d(x - \breve{g}, R_G^{-1}(0)) = d(x, R_G^{-1}(0))$, as $R_G^{-1}(0)$ is an additive group. This implies $\breve{g} \in P_{R_G^{-1}(0)}(x)$. From (2.1), we have $0 \in P_{R_G^{-1}(0)}(x)$. Since $R_G^{-1}(0)$ is Chebyshev, we have $\breve{g} = 0$ and so $x = g \in G$. Also, by Remark 1.1 (8), we have $d(g, 0) = d(g, R_G^{-1}(0))$ for all $g \in G$. Consequently, the result follows.
(ii)⇒ (i). Let $x \in X$. Then there exist $g \in G$ such that $\breve{g} = x - g \in R_G^{-1}(0)$. Therefore

$$d(x, R_G^{-1}(0)) = d(x - \breve{g}, R_G^{-1}(0)) = d(x, \breve{g}) \ (as\ x - \breve{g} \in G)$$

i.e., $R_G^{-1}(0)$ is proximinal.

Suppose that for some $x \in X$, there exist $\breve{g_1}, \breve{g_2} \in P_{R_G^{-1}(0)}(x)$, i.e.,

$$d(x, \breve{g_1}) = d(x, R_G^{-1}(0)) = d(x - \breve{g_1}, R_G^{-1}(0))$$

and

$$d(x, \breve{g_2}) = d(x, R_G^{-1}(0)) = d(x - \breve{g_2}, R_G^{-1}(0))$$

Then by hypothesis, $x - \breve{g_1}, x - \breve{g_2} \in G$. Since G is a subspace, $(x - \breve{g_1}) - (x - \breve{g_2}) \in G$, i.e., $\breve{g_1} - \breve{g_2} \in G$. Also $R_G^{-1}(0)$ is an additive group, we have $\breve{g_1} - \breve{g_2} \in$

$R_G^{-1}(0)$. Therefore, $\breve{g}_1 - \breve{g}_2 \in R_G^{-1}(0) \bigcap G = \{0\}$ and so $\breve{g}_2 = \breve{g}_1$. Hence $R_G^{-1}(0)$ is Chebyshev. □

Let G be a linear subspace of a metric linear space (X, d). We say that G is a $quasi - orthogonal$ set if $G \perp P_G^{-1}(0)$, i.e., $g \perp g'$ for all $g \in G$ and $g' \in P_G^{-1}(0)$.

Concerning quasi-orthogonality of subspaces, we have

Lemma 2.11. *Let G be a quasi-orthogonal subspace of a metric linear space (X, d) then $d(g, 0) = d(g, P_G^{-1}(0))$.*

Proof. Since G is a quasi-orthogonal subspace, $G \perp P_G^{-1}(0)$, i.e., $g \perp g'$ for all $g \in G$ and $g' \in P_G^{-1}(0)$, i.e., $d(g, 0) \leq d(g, \alpha g')$ for all $g \in G,\ g' \in P_G^{-1}(0)$ and all scalars α. Taking $\alpha = 1$, we obtain $d(g, 0) \leq d(g, g')$ for all $g \in G,\ g' \in P_G^{-1}(0)$. This implies $d(g, 0) \leq \inf_{g' \in P_G^{-1}(0)} d(g, g') = d(g, P_G^{-1}(0))$ for all $g \in G$. Also, $d(g, P_G^{-1}(0)) = \inf_{g' \in P_G^{-1}(0)} d(g, g') \leq d(g, 0)$ for all $g \in G$. Consequently, $d(g, 0) = d(g, P_G^{-1}(0))$. □

Using Lemma 2.11, we have the following result which characterizes Chebyshevity of $P_G^{-1}(0)$:

Theorem 2.12. *Let G be a proximinal, quasi-orthogonal subspace of a metric linear space (X, d). If $P_G^{-1}(0)$ is an additive group then the following are equivalent:*
(i) $P_G^{-1}(0)$ is a Chebyshev set.
(ii) $G = \{x \in X : d(x, P_G^{-1}(0)) = d(x, 0)\}$.

Proof. The proof runs on similar lines as that of Theorem 2.10. □

Remarks 2.13.

(i) For normed linear spaces, Theorem 2.12 was proved in [5].
(ii) It was shown in [11] that if G is a proximinal (coproximinal) subspace of a metric linear space (X, d) and $P_G^{-1}(0)$ $(R_G^{-1}(0))$ is a convex set, then G is Chebyshev (co-Chebyshev). If we take G to be a proximinal (coproximinal) subset containing zero instead of a subspace then G need not be Chebyshev (co-Chebyshev). Example 2.4 and the following example confirm these facts.

Example 2.14. *Let $X = R$ and $G = [0, \infty)$, then $R_G^{-1}(0) = (-\infty, 0]$ and $R_G(-1) = [0, 1]$, i.e., $R_G^{-1}(0)$ is a convex set but G is not co-Chebyshev.*

Concerning the coproximinality of quotient spaces, we have

Lemma 2.15. *Let G be a closed linear subspace of a metric linear space (X, d) and F a coproximinal subspace of X containing G. Then F/G is coproximinal in X/G.*

Proof. Let $x + G \in X/G,\ x \in X$, and f be a best coapproximation to x. We prove that $f + G$ is a best coapproximation to $x + G$. Suppose it is not, then there exist $f' + G \in F/G$ such that $d(f + G, f' + G) > d(x + G, f' + G),\ i.e.,\ \inf_{g \in G} d(x - f', g) < d(f - f', G)$. Then there exist some $g_0 \in G$ such that

$$d(x - f', g_0) < d(f - f', G) \leq d(f - f', g_0)$$

$i.e.,\ d(x, f' + g_0) < d(f, f' + g_0)$. Thus f is not a best coapproximation to x from F, a contradiction. Hence $f + G$ is a best coapproximation to $x + G$ and consequently, F/G is coproximinal in X/G. □

Concerning the coproximinality of F, we have

Lemma 2.16. *Let G be a proximinal subspace of a metric linear space (X,d) and F a subspace of X containing G. If F/G is coproximinal in X/G then F is coproximinal in X.*

Proof. Let $x \in X$ be arbitrary, then $x+G \in X/G$. Since F/G is coproximinal in X/G, there is some $f+G \in R_{F/G}(x+G)$, i.e., $d(f+G, f'+G) \leq d(x+G, f'+G)$ $for\ evrey$ $f'+G \in F/G$. Since G is proximinal, there exist $g_0 \in G$ such that $d(f-f', g_0) \leq d(x-f', G) \leq d(x-f', 0)$ $for\ every$ $f' \in F$. This gives $f-g_0 \in R_F(x)$. Hence F is coproximinal in X. □

Using Lemmas 2.15 and 2.16, we obtain the following result:

Theorem 2.17. *Let G be a proximinal subspace of a metric linear space (X,d) and F a coproximinal subspace of X containing G. If $\pi : X \to X/G$ is the canonical map, then $\pi(R_F(x)) = R_{F/G}(x+G)$.*

Concerning the co-Chebyshevity of quotient spaces, we have

Theorem 2.18. *Let G be a proximinal subspace of a metric linear space (X,d) and F a coproximinal subspace containing G. If $R_F^{-1}(0)$ is a convex set then F/G is a co-Chebyshev subspace of X/G.*

Proof. Using Theorem 2.17, we have $\pi(R_F(x)) = R_{F/G}(x+G)$.

In view of Remark 2.13, it is sufficient to prove that $R_{F/G}^{-1}(G)$ is convex. For this, let $x+G, y+G \in R_{F/G}^{-1}(G)$ and $0 < \lambda < 1$. Since $G \in R_{F/G}(x+G)$ and $G \in R_{F/G}(y+G)$, there exist $g \in R_F(x)$ and $h \in R_F(y)$ such that $\pi(g) = G = \pi(h)$. Therefore, $x-g, y-h \in R_F^{-1}(0)$ (as $g \in R_F(x),\ h \in R_F(y)$).

Since $R_F^{-1}(0)$ is a convex set, we have $\lambda(x-g) + (1-\lambda)(y-h) \in R_F^{-1}(0)$ $i.e.,$ $d(0,f) \leq d(\lambda(x-g) + (1-\lambda)(y-h), f)$ $for\ all$ $f \in F$. This implies $d(\lambda g + (1-\lambda)h, \lambda g + (1-\lambda)h + f) \leq d(\lambda x + (1-\lambda)y, f + \lambda g + (1-\lambda)h)$ $for\ all$ $f \in F$. Therefore, $\lambda g + (1-\lambda)h \in R_F(\lambda x + (1-\lambda)y)$.

Also $\pi(\lambda g + (1-\lambda h)) = G$. Therefore, $G \in R_{F/G}(\lambda x + (1-\lambda)y + G)$, i.e., $\lambda(x+G) + (1-\lambda)(y+G) \in R_{F/G}^{-1}(G)$ and so $R_{F/G}^{-1}(G)$ is convex. Hence F/G is co-Chebyshev in X/G. □

Remarks 2.19. *For normed linear spaces, Lemmas 2.15, 2.16 and Theorems 2.17, 2.18 were proved in* [3].

Proceeding on similar lines, we obtain the following results on best approximation in quotient spaces. For normed linear spaces these results were proved in [5] and [6]:

(i) Let G be a closed linear subspace of a metric linear space (X,d) and F a proximinal subspace of X containing G. Then F/G is proximinal in X/G.
(ii) Let G be a proximinal subspace of a metric linear space (X,d) and F a subspace of X containing G. If F/G is proximinal in X/G then F is proximinal in X.
(iii) Let G be a proximinal subspace of a metric linear space (X,d) and F a proximinal subspace of X containing G. If $\pi : X \to X/G$ is the canonical map then $\pi(P_F(x)) = P_{F/G}(x+G)$.
(iv) Let G be a proximinal subspace and F a proximinal subspace of X containing G. If $P_F^{-1}(0)$ is a convex set then F/G is a Chebyshev subspace of X/G.

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