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DUALITY RESULTS FOR SECOND-ORDER MULTI-OBJECTIVE PROGRAMMING PROBLEM

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ABSTRACT. The purpose of this paper is to study a class of non-differentiable multi objective fractional programming problem in which every component of objective functions contains a term involving the support function of a compact convex set. For a differentiable function, we use the definition of second-order $(C, \alpha, \rho, d) - V$-type-I convex function. Further, Wolfe type dual has been formulated for this problem and appropriate duality results have been proved under second-order $(C, \alpha, \rho, d) - V$-type-I convexity assumptions.

KEYWORDS : Duality; non-differentiable; multi-objective fractional programming; generalized convexity.
AMS Subject Classification: 90C26 . 90C30 . 90C32 . 90C46

1. INTRODUCTION

Second-order duality has even greater significance over first order duality, since it provides tighter bounds for the value of the objective function when approximation are used. Second-order duality for non-linear programming was the first to study in Mond [1]. Then, the concept of second-order duality for non-linear programming problems introduced by Mangasarian [2].

In the recent years attempts have been made by several authors to define various classes of differentiable non-convex functions and to study their duality and optimality criteria for solving such problems ([4], [5], [8], [9], [10], [11], and others). One such generalization of convex function is the invexity notion introduced by Hanson [5]. The term invex (which means invariant convex) was suggested by Craven [7]. Over the years, many generalizations of this concept have been given in the literature. For example, the concept of invexity of functions was also generalized to B-invex functions by Suneja et al.[6]. The class of (p, r)-invex functions

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with respect to η is an extension of the class of invex functions with respect to η introduced by Hanson [5].

Recently, Kim et al. [12] delineated a group of non-differentiable multi-objective fractional programs and founded necessary and sufficient optimality conditions and duality results for weakly efficient solutions of non-differentiable multi-objective fractional programming problems. Later on, Kim et al. [13] considered two pairs of non-differentiable multi-objective symmetric dual problems of second order with cone constraints over arbitrary closed convex cones, which are Mond-Weir type and Wolfe type and weak, strong, converse and self-duality theorems were founded under the assumptions of pseudo-invex functions of second order. We have to maximize the efficiency of an economic system resulting optimization problems whose objective function is a ratio. Including maximization of productivity, maximization of return on investment, maximization of risk, minimization of cost. Ojha [14] derived a pair of second order symmetric non-differentiable multi-objective fractional problems, also derived weak and strong duality theorems.

In this paper, we use the concept of second-order $(C, \alpha, \rho, d) - V$-type-I functions for a non-differentiable multi-objective second-order fractional programming problem. A numerical non-trivial example illustrates the existence of such functions. In this dual, we generalize the models already existing in the literature. Using $(C, \alpha, \rho, d) - V$-type-I function, we set up weak, strong and strict converse duality results for Wolfe-type dual program.

2. DEFINITIONS AND NUMERICAL ILLUSTRATION

Definition 2.1. A function $C : X \times X \times R^n \longrightarrow R$ $(X \subset R^n)$ is said to be convex on R^n iff for any fixed $(x,\ u) \in X \times X$ and for any $x_1,\ x_2 \in R^n$, such that

$$C_{x,u}(\lambda x_1 + (1-\lambda)x_2) \leq \lambda C_{x,u}(x_1) + (1-\lambda)C_{x,u}(x_2),\ \forall \lambda \in (0,1).$$

Suppose the real valued function $d : X \times X \to R$ satisfy the property $d(x,u) = 0 \Leftrightarrow x = u$ and let $C_{x,u}(0) = 0$, for any $(x,u) \in X \times X$.

The general multi-objective programming problem can be expressed in the following form :

(P) Minimize $f(x) = (f_1(x), f_2(x), ..., f_k(x))$

subject to $x \in X^0 = \{x \in X : g(x) \leq 0\}$,

where $X \subset R^n$ is open, $f : X \to R^k$, $g : X \to R^m$, are differentiable functions on X.

Definition 2.2 . A point $\bar{x} \in X^0$ is said to be an efficient solution of (P) if there exists no $x \in X^0$ such that $f_i(x) \leq f_i(\bar{x}),\ i = 1,2,...,k$.

We write the definition of second-order $(C, \alpha, \rho, d) - V$ -type-I functions. Let C be a convex function with respect to third variable.

Definition 2.3. For any $i = 1,2,...,k,\ j = 1,2,...,m$, the function (f_i, h_j) is said to be second-order $(C, \alpha, \rho, d) - V$-type-I at $u \in X$ if there exist $\alpha_i^1,\ \alpha_j^2 : X \times X \to$

$R_+ \setminus \{0\}$ and $\rho_i^1,\ \rho_j^2 \in R$, such that for each $x \in X^0$ and $p \in R^n$, we have

$$\frac{1}{\alpha_i^1(x,u)}[f_i(x) - f_i(u) + \frac{1}{2}p^T\nabla^2 f_i(u)p] \geq C_{x,u}\big(\nabla f_i(u) + \nabla^2 f_i(u)p\big) + \frac{\rho_i^1 d^2(x,u)}{\alpha_i^1(x,u)}$$

and

$$\frac{1}{\alpha_j^2(x,u)}[-h_j(u) + \frac{1}{2}p^T\nabla^2 h_j(u)p] \geq C_{x,u}\Bigg(\nabla h_j(u) + \nabla^2 h_j(u)p\Bigg) + \frac{\rho_j^2 d^2(x,u)}{\alpha_j^2(x,u)}.$$

Definition 2.4. For any $i = 1,2,...,k,\ j = 1,2,...,m$, the functions (f_i, h_j) are said to be second-order semi-strictly $(C,\alpha,\rho,d) - V$-type-I at $u \in X$ if there exist $\alpha_i^1,\ \alpha_j^2 : X \times X \to R_+ \setminus \{0\}$ and $\rho_i^1,\ \rho_j^2 \in R$, such that for each $x \in X^0$ and $p \in R^n$, we have

$$\frac{1}{\alpha_i^1(x,u)}[f_i(x) - f_i(u) + \frac{1}{2}p^T\nabla^2 f_i(u)p] > C_{x,u}\big(\nabla f_i(u) + \nabla^2 f_i(u)p\big) + \frac{\rho_i^1 d^2(x,u)}{\alpha_i^1(x,u)}$$

and

$$\frac{1}{\alpha_j^2(x,u)}[-h_j(u) + \frac{1}{2}p^T\nabla^2 h_j(u)p] \geq C_{x,u}\Bigg(\nabla h_j(u) + \nabla^2 h_j(u)p\Bigg) + \frac{\rho_j^2 d^2(x,u)}{\alpha_j^2(x,u)}.$$

Every (F,α,ρ,d) type-I function is (C,α,ρ,d) type-I. But the converse need not be true. This is seen from the following example.

Example 1. Let $X = R$. Let $f : X \to R$ and $g : X \to R$ where $f(x) = x^2 - cos2x - 1$ and $g(x) = \dfrac{cos2x + 1}{2} - 2x$. Suppose $\alpha_1^1,\ \alpha_2^1 \in R_+ \setminus \{0\}$ and $\alpha_1^1 = \dfrac{1}{20},\ \alpha_2^2 = \dfrac{1}{3}$ and $C_{x,u}(a) = \frac{a^2}{24}$. Let $d : X \times X \to R_+$ be $d(x,u) = (x-u)^2$.

Let $p = -1,\ \rho_1^1 = \frac{-1}{20},\ \rho_1^2 = -1$.

By definition of (C,α,ρ,d) type-I at $u = -0.5\pi$, we have

$$\frac{1}{\alpha_1^1(x,u)}[f(x) - f(u) + \frac{1}{2}p^T\nabla^2 f(u)p] - C_{x,u}\big(\nabla f(u) + \nabla^2 f(u)p\big) - \frac{\rho^1 d^2(x,u)}{\alpha_1^1(x,u)},$$

$$20x^2 - 40(1 + cos2x) + 60 - 5\pi^2 - \frac{1}{24}(\pi - 6)^2 + (x + 5\pi)^2 \geq 0,$$

Similarly,

$$\frac{1}{\alpha_1^2(x,u)}[-h(u) + \frac{1}{2}p^T\nabla^2 h(u)p] - C_{x,u}\Bigg(\nabla h(u) + \nabla^2 h(u)p\Bigg) - \frac{\rho_1^2 d^2(x,u)}{\alpha_1^2(x,u)} \geq 0.$$

for all $x \in X$. Hence, (f,h) is second order (C,α,ρ,d) type-I but (f,h) is not second order (F,α,ρ,d) type-I at $u = 0.5\pi$ as C is not sub-linear with respect to the third argument.

Definition 2.5. Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a sub-differential, that is, there exists a $z \in R^n$ such that

$$s(y|C) \geq s(x|C) + z^T(y - x), \forall x \in C.$$

The sub-differential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n : y^T(z - x) \leq 0, \forall z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

3. Problem formulation

A general multi-objective fractional programming problem can be expressed in the following form :

(MFP) Min $G(x) = \left(\dfrac{f_1(x) + S(x|C_1)}{g(x) - S(x|D)}, \dfrac{f_2(x) + S(x|C_2)}{g(x) - S(x|D)}, ..., \dfrac{f_k(x) + S(x|C_k)}{g(x) - S(x|D)} \right)$

Subject to $x \in X^0 = \{x \in X : h_j(x) + S(x|E_j) \leq 0,\ j = 1, 2, ..., m\}$,

where $X \subset R^n$ is a closed convex set, $f_i,\ g : X \to R\ (i = 1, 2, ..., k)$ and $h_j : X \to R\ (j = 1, 2, ..., m)$ are continuously differentiable functions. $f_i(.) + S(.|C_i) \geq 0$ and $g(.) - S(.|D) > 0$, and $S(x|C_i),\ S(x|D)$ and $S(x|E_j)$ denote the support functions of compact convex sets, $C_i,\ D$ and E_j, for all $i = 1, 2, ..., k,\ j = 1, 2, ..., m$, respectively.

For every $\nu \in R^k_+$, consider the following auxiliary problem:

(MFP)$_\nu$ Minimize $G(x) = \Big(f_1(x) + S(x|C_1) - \nu_1(g(x) - S(x|D)),$

$, ..., f_k(x) + S(x|C_k) - \nu_k(g(x) - S(x|D)) \Big).$

Lemma 3[16]. Let $\bar{x} \in X$ is an efficient solution for (MFP) if and only if there exists $\bar{\nu} \in R^k_+$ such that $\bar{x}$ is an efficient solution for (MFP)$_{\bar{\nu}}$ and $\bar{\nu}_i = \dfrac{f_i(\bar{x}) + S(\bar{x}|C_i)}{g(\bar{x}) - S(\bar{x}|D)}$, $i = 1, 2, ..., k$.

Theorem 3.1 (Necessary Condition)[15]. Assume that $\bar{x}$ is an efficient solution of (MFP) and let the Slater's constraint qualification be satisfied on X. Then there exist $0 < \bar{\lambda} \in R^k$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$, $0 \leq \bar{y} \in R^m$, $\bar{z_i} \in R^n$, $\bar{v} \in R^n$ and $\bar{w_j} \in R^n$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$ such that

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left(\frac{f_i(\bar{x}) + \bar{x}^T \bar{z}_i}{g_i(\bar{x}) - \bar{x}^T \bar{v}} \right) + \sum_{j=1}^{m} \bar{y_j} \nabla (h_j(\bar{x}) + \bar{x}^T \bar{w_j}) = 0,$$

$$\sum_{j=1}^{m} \bar{y_j} (h_j(\bar{x}) + \bar{x}^T \bar{w_j}) = 0,$$

$$\bar{x}^T \bar{z_i} = S(\bar{x}|C_i),\ \bar{x}^T \bar{v} = S(\bar{x}|D_i),\ \bar{x}^T \bar{w_j} = S(\bar{x}|E_j),$$

$$\bar{z_i} \in C_i,\ \bar{v} \in D_i,\ \bar{w_j} \in E_j,\ i = 1, 2, ..., k,\ j = 1, 2, ..., m.$$

4. Duality model-I

Now, in this section we study the duality for (MFP)$_\nu$, for some $\nu \in R^k_+$. We first consider the following auxiliary problem:

(MFD) maximize
$$\Big[f_1(u)+u^Tz_1-\nu_1(g(u)-u^Tv)-\frac{1}{2}p^T\nabla^2\Big(f_1(u)+u^Tz_1-\nu_1(g(u)$$
$$-u^Tv)\Big)p+\sum_{j=1}^m y_j(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2 h_j(u)p)$$
$$,...,f_k(u)+u^Tz_k-\nu_k(g(u)-u^Tv)-\frac{1}{2}p^T\nabla^2\Big(f_k(u)+u^Tz_k-\nu_k(g(u)$$
$$-u^Tv)\Big)p+\sum_{j=1}^m y_j(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2 h_j(u)p)\Big]$$

Subject to

$$\sum_{i=1}^k \lambda_i\nabla\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)+\sum_{j=1}^m y_j\nabla(h_j(u)+u^Tw_j)$$
$$+\sum_{i=1}^k \lambda_i\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)p+\sum_{j=1}^m y_j\nabla^2 h_j(u)p=0, \tag{4.1}$$
$$z_i\in C_i,\ v\in D,\ i=1,2,...,k,\ w_j\in E_j,\ j=1,2,...,m,$$
$$y_j\geq 0,\ j=1,2,...,m,\ \lambda_i>0,\ i=1,2,...,k,\ \sum_{i=1}^k\lambda_i=1.$$

We now discuss the duality results for (MFP)$_\nu$ and (MFD).

Theorem 4.1 (Weak Duality Theorem). Let x be a feasible solution for (MFP)$_\nu$, for some $\nu\in R^k_+$ and for each feasible solution $(u,z,v,y,\lambda,\nu,w,p)$ of (MFD), for the same $\nu\in R^k_+$. Suppose that, for any $i=1,2,...,k,\ j=1,2,...,m$,

(*i*) $\Big[f_i(.)+(.)^Tz_i-\nu_i(g(.)-(.)^Tv),\ h_j(.)+(.)^Tw_j\Big]$ is second-order $(C,\alpha,\rho,d)-V$-type-I at u,

(*ii*) $\alpha_i^1(x,u)=\alpha_j^2(x,u)=\alpha(x,u)$, for all i and j,

(*iii*) $\sum_{i=1}^k\lambda_i\rho_i^1+\sum_{j=1}^m y_j\rho_j^2\geq 0.$

Then the following cannot hold

$$f_i(x)+S(x|C_i)-\nu_i(g(x)-S(x|D))\leq f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)$$
$$-\frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv)\Big)p$$
$$+\sum_{j=1}^m y_j\Big(h_j(u)+u^Tw_j-\frac{1}{2}p^T\nabla^2 h_j(u)p\Big),\ \forall\ i=1,2,...,k \tag{4.2}$$

and

$$f_r(x) + S(x|C_r) - \nu_r(g(x) - S(x|D)) < f_r(u) + u^T z_r - \nu_r(g(u) - u^T v)$$
$$- \frac{1}{2} p^T \nabla^2 \Big(f_r(u) + u^T z_r - \nu_r(g(u) - u^T v) \Big) p$$
$$+ \sum_{j=1}^{m} y_j \Big(h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u) p \Big), \text{ for some } r = 1, 2, ..., k. \tag{4.3}$$

Proof. Suppose that (4.2) and (4.3) hold, then using $\lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ x^T z_i \leq S(x|C_i)$

$,\ x^T v \leq S(x|D),\ i = 1, 2, ..., k,$ we have

$$\sum_{i=1}^{k} \lambda_i \Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v_i) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v))$$
$$+ \frac{1}{2} p^T \nabla^2 \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v) \Big) p \Big]$$
$$- \sum_{j=1}^{m} y_j \Big(h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u) p \Big) < 0. \tag{4.4}$$

For any $i = 1, 2, ..., k,\ j = 1, 2, ..., m,$ $\Big[f_i(.) + (.)^T z_i - \nu_i(g(.) - (.)^T v),\ h_j(.) + (.)^T w_j \Big]$ is second-order $(C, \alpha, \rho, d) - V$-type-I, we have

$$\frac{1}{\alpha_i^1(x, u)} \Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v))$$
$$+ \frac{1}{2} p^T \nabla^2 \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v) \Big) p \Big] - \frac{\rho_i^1 d^2(x, u)}{\alpha_i^1(x, u)}$$
$$\geq C_{x,u} \Big[\nabla \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v) \Big) + \nabla^2 \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v) \Big) p \Big] \tag{4.5}$$

and

$$\frac{1}{\alpha_j^2(x, u)} \Big[- (h_j(u) + u^T w_j) + \frac{1}{2} p^T \nabla^2 h_j(u) p \Big]$$
$$\geq C_{x,u} \Big[\nabla (h_j(u) + u^T w_j) + \nabla^2 h_j(u) p \Big] + \frac{\rho_j^2 d^2(x, u)}{\alpha_j^2(x, u)}. \tag{4.6}$$

Let $\tau = 1 + \sum_{j=1}^{m} y_j > 0$. Adding the two inequalities after multiplying (4.5) by $\frac{\lambda_i}{\tau}$ and (4.6) by $\frac{y_j}{\tau}$ and summing them over all i and j, we obtain

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x, u)\tau} \Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)) +$$
$$\frac{1}{2} p^T \nabla^2 \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v) p \Big] - \sum_{j=1}^{m} \frac{y_j}{\alpha_j^2(x, u)\tau} \Big[h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u) p \Big]$$

$$\geq \sum_{i=1}^{k} \frac{\lambda_i}{\tau} C_{x,u}\Big\{\nabla\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)\Big) + \nabla^2\Big(f_i(u) + u^T z_i - \nu_i(g(u)$$

$$- u^T v)\Big)p\Big\} + \sum_{j=1}^{m} \frac{y_j}{\tau} C_{x,u}\Big\{\nabla(h_j(u) + u^T w_j) + \nabla^2 h_j(u)p\Big\}$$

$$+\Big(\sum_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha_i^1(x,u)\tau} + \sum_{j=1}^{m} \frac{y_j \rho_j^2}{\alpha_j^2(x,u)\tau}\Big) d^2(x,u). \tag{4.7}$$

Using hypothesis (ii) and convexity of C, it gives

$$\frac{1}{\alpha(x,u)\tau} \sum_{i=1}^{k} \lambda_i \Big[f_i(x) + x^T z_i - \nu_i(g(x) - x^T v) - (f_i(u) + u^T z_i - \nu_i(g(u) - u^T v)) +$$

$$\frac{1}{2} p^T \nabla^2\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v\Big)p\Big] - \frac{1}{\alpha(x,u)\tau} \sum_{j=1}^{m} y_j \Big[h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u)p\Big]$$

$$\geq C_{x,u}\Big[\sum_{i=1}^{k} \frac{\lambda_i}{\tau}\Big\{\nabla\Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v\Big) + \nabla^2\Big(f_i(u) + u^T z_i - \nu_i(g(u)$$

$$- u^T v)\Big)p\Big\} + \sum_{j=1}^{m} \frac{y_j}{\tau}\Big\{\nabla(h_j(u) + u^T w_j) + \nabla^2 h_j(u)p\Big\}\Big]$$

$$+ \frac{1}{\alpha(x,u)\tau}\Big(\sum_{i=1}^{k} \lambda_i \rho_i^1 + \sum_{j=1}^{m} y_j \rho_j^2\Big) d^2(x,u). \tag{4.8}$$

Finally using feasibility condition (4.1), hypothesis (iii) and using $C_{x,u}(0) = 0$, it follows that

$$\sum_{i=1}^{k} \lambda_i \Big(f_i(x) + x^T z_i - \nu_i(g(x) - x^T v_i) - f_i(u) + u^T z_i - \nu_i(g(u) - u^T v) + \frac{1}{2} p^T \nabla^2\Big(f_i(u) +$$

$$u^T z_i - \nu_i(g(u) - u^T v\Big)p\Big)$$

$$- \sum_{j=1}^{m} y_j \Big(h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u)p\Big) \geq 0.$$

which contradict (4.4). Hence the result. □

Theorem 4.2 (Strong Duality Theorem). If $\bar{u} \in X^0$ is an efficient solution of (MFP)$_\nu$ and let the Slater constraint qualification be satisfied. Then there exist $\bar{\lambda} \in R^k$, $\bar{y} \in R^m$, $\bar{z}_i \in R^n$, $\bar{v} \in R^n$ and $\bar{w}_j \in R^n$, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$, such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is a feasible solution of (MFD) and the objective function values of (MFP)$_\nu$ and (MFD) are equal. Moreover, if the conditions of Theorem 3.1 holds for all feasible solutions of (MFP)$_\nu$ and (MFD), then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is an efficient solution of (MFD).

Proof. Since $\bar{u}$ is an efficient solution of (MFP)$_\nu$ and the Slater constraint qualification is satisfied, from Theorem 2.1, there exist $\bar{\mu} \in R^k, \bar{y} \in R^m, \bar{z}_i \in R^n$, $\bar{v} \in R^n$ and $\bar{w}_j \in R^n, i = 1, 2, ..., k, j = 1, 2, ..., m$, such that

$$\sum_{i=1}^{k} \bar{\mu}_i \nabla\Big(\frac{f_i(\bar{u}) + \bar{u}^T \bar{z}_i}{g(\bar{u}) - \bar{u}^T \bar{v}}\Big) + \sum_{j=1}^{m} \bar{y}_j \nabla(h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0, \tag{4.9}$$

$$\sum_{j=1}^{m} \bar{y_j}(h_j(\bar{u}) + \bar{u}^T \bar{w_j}) = 0, \tag{4.10}$$

$$\bar{u}^T \bar{z_i} = S(\bar{u}|C_i),\ \bar{u}^T \bar{v} = S(\bar{u}|D_i),\ \bar{u}^T \bar{w_j} = S(\bar{u}|E_j), \tag{4.11}$$

$$\bar{z_i} \in C_i,\ \bar{v} \in D_i,\ \bar{w_j} \in E_j, \tag{4.12}$$

$$\bar{\mu_i} > 0,\ \bar{y_j} \geq 0,\ i = 1, 2, ..., k,\ j = 1, 2, ..., m. \tag{4.13}$$

Equation (4.9) can be written as,

$$\sum_{i=1}^{k} \frac{\bar{\mu_i}}{g(\bar{u}) - \bar{u}^T \bar{v}} \left(\nabla(f_i(\bar{u}) + \bar{u}^T \bar{z_i}) - \frac{f_i(\bar{u}) + \bar{u}^T \bar{z_i}}{g(\bar{u}) - \bar{u}^T \bar{v}} \nabla(g(\bar{u}) - \bar{u}^T \bar{v}) \right) + \sum_{j=1}^{m} \nabla \bar{y_j}(h_j(\bar{u}) + \bar{u}^T \bar{w_j}) = 0. \tag{4.14}$$

Letting $\bar{\lambda}_i = \dfrac{\bar{\mu_i}}{g(\bar{u}) - \bar{u}^T \bar{v}}$ and $\bar{\nu_i} = \dfrac{f_i(\bar{u}) + \bar{u}^T \bar{z_i}}{g(\bar{u}) - \bar{u}^T \bar{v}},\ i = 1, 2, ..., k,$ we have

$$\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left(f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) \right) + \sum_{j=1}^{m} \bar{y_j} \nabla(h_j(\bar{u}) + \bar{u}^T \bar{w_j}) = 0 \tag{4.15}$$

$$f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) = 0. \tag{4.16}$$

Thus, $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu_i}, \bar{w}, \bar{p} = 0)$ is feasible for (MFD) and the objective function values of (MFP)$_\nu$ and (MFD) are equal. This complete the proof of theorem 4.2. □

Theorem 4.3 (Strict Converse Duality Theorem). Let $\bar{x}$ be a feasible solution for (MFP)$_\nu$ and $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ be feasible for (MFD) Suppose that, for any $i = 1, 2, ..., k,\ j = 1, 2, ..., m,$

(i) $\sum_{i=1}^{k} \bar{\lambda}_i \Big(f_i(\bar{x}) + \bar{x}^T \bar{z_i} - \bar{\nu_i}(g(\bar{x}) - \bar{x}^T \bar{v}) - (f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) + \frac{1}{2} p^T \nabla^2 \Big(f_i(u) + u^T z_i - \nu_i(g(u) - u^T v \Big) p \Big) - \sum_{j=1}^{m} y_j \Big(h_j(u) + u^T w_j - \frac{1}{2} p^T \nabla^2 h_j(u) p \Big) \leq 0,$

(ii) $\Big[f_i(.) + (.)^T z_i - \bar{\nu_i}(g(.) - (.)^T v),\ h_j(.) + (.)^T w_j \Big]$ is second-order semi-strictly $(C, \alpha, \rho, d) - V$-type-I at $\bar{u}$,

(iii) $\alpha_i^1(\bar{x}, \bar{u}) = \alpha_j^2(\bar{x}, \bar{u}) = \alpha(\bar{x}, \bar{u})$, for all i and j,

(iv) $\sum_{i=1}^{k} \bar{\lambda}_i \rho_i^1 + \sum_{j=1}^{m} \bar{y}_j \rho_j^2 \geq 0.$

Then, $\bar{x} = \bar{u}$.

Proof. We suppose on contrary $\bar{x} \neq \bar{u}$. Since $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ is feasible solution for (MFD), we have

$$C_{\bar{x},\bar{u}} \Bigg[\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left(f_i(\bar{u}) + \bar{u}^T \bar{z_i} - \bar{\nu_i}(g(\bar{u}) - \bar{u}^T \bar{v}) \right) + \sum_{j=1}^{m} \bar{y_j} \nabla(h_j(\bar{u}) + \bar{u}^T \bar{w_j})$$

$$+\sum_{i=1}^{k}\bar{\lambda}_i\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})\Big)\bar{p}+\sum_{j=1}^{m}\bar{y}_j\nabla^2 h_j(\bar{u})\bar{p}\Big]=0. \quad (4.17)$$

By hypothesis (ii), we get

$$\frac{1}{\alpha_i^1(\bar{x},\bar{u})}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v})-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})$$
$$+\frac{1}{2}\bar{p}^T\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big]-\frac{\rho_i d^2(\bar{x},\bar{u})}{\alpha_i^1(\bar{x},\bar{u})}$$

$$> C_{\bar{x},\bar{u}}\Big[\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big] \quad (4.18)$$

and

$$\frac{1}{\alpha_j^2(\bar{x},\bar{u})}\Big[-(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big]$$

$$\geq C_{\bar{x},\bar{u}}\Big[\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\nabla^2 h_j(\bar{u})\bar{p}\Big]+\frac{\rho_j d^2(\bar{x},\bar{u})}{\alpha_j^2(\bar{x},\bar{u})}. \quad (4.19)$$

Let $\bar{\tau}=1+\sum_{j=1}^{m}\bar{y}_j>0$. Multiplying (4.18) by $\frac{\bar{\lambda}_i}{\bar{\tau}}$ and (4.19) by $\frac{\bar{y}_j}{\bar{\tau}}$, summing over $i=1,2,...,k$ and $j=1,2,...,m$, we have

$$\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\alpha_i^1(\bar{x},\bar{u})\bar{\tau}}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v}))-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))+$$
$$\frac{1}{2}\bar{p}^T\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big]-\sum_{j=1}^{m}\frac{\bar{y}_j}{\alpha_j^2(\bar{x},\bar{u})\bar{\tau}}\Big[h_j(\bar{u})+\bar{u}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big]$$
$$>\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\bar{\tau}}C_{\bar{x},\bar{u}}\Big\{\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i$$
$$-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}\Big)\bar{p}\Big\}+\sum_{j=1}^{m}\frac{\bar{y}_j}{\bar{\tau}}C_{\bar{x},\bar{u}}\Big\{\nabla(h_j(\bar{u})+\bar{u}^T\bar{w}_j)+\nabla^2 h_j(\bar{u})\bar{p}\Big\}+$$
$$\Big(\sum_{i=1}^{k}\frac{\bar{\lambda}_i\rho_i^1}{\alpha_i^1(\bar{x},\bar{u})\bar{\tau}}+\sum_{j=1}^{m}\frac{\bar{y}_j\rho_j^2}{\alpha_j^2(\bar{x},\bar{u})\bar{\tau}}\Big)d^2(\bar{x},\bar{u}).$$

Using hypothesis (iii) and convexity of C, it yield

$$\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big[f_i(\bar{x})+\bar{x}^T\bar{z}_i-\bar{\nu}_i(g(\bar{x})-\bar{x}^T\bar{v}))-(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))+$$
$$\frac{1}{2}\bar{p}^T(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v}))\bar{p}\Big]-\sum_{j=1}^{m}\frac{\bar{y}_j}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big(h_j(\bar{u})+\bar{u}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2 h_j(\bar{u})\bar{p}\Big)$$
$$> C_{\bar{x},\bar{u}}\Big[\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\bar{\tau}}\Big\{\nabla\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-\bar{u}^T\bar{v})\Big)+\nabla^2\Big(f_i(\bar{u})+\bar{u}^T\bar{z}_i-\bar{\nu}_i(g(\bar{u})-$$

$$\bar{u}^T\bar{v})\Big)\bar{p}\Big\} + \sum_{j=1}^{m} \frac{\bar{y}_j}{\bar{\tau}}\Big\{\nabla(h_j(\bar{u}) + \bar{u}^T\bar{w}_j) + \nabla^2 h_j(\bar{u})\bar{p}\Big\}\Big]$$

$$+\frac{1}{\tau\alpha(\bar{x},\bar{u})}\Big(\sum_{i=1}^{k}\bar{\lambda}_i\rho_i^1 + \sum_{j=1}^{m}\bar{y}_j\rho_j^2\Big)d^2(\bar{x},\bar{u}). \tag{4.20}$$

Finally, using feasibility conditions (4.1) and hypothesis (iv) in (4.20), we get

$$\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big[f_i(\bar{x}) + \bar{x}^T\bar{z}_i - \bar{\nu}_i(g(\bar{x}) - \bar{x}^T\bar{v})) - (f_i(\bar{u}) + \bar{u}^T\bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T\bar{v})) + \frac{1}{2}p^T\nabla^2\Big(f_i(u)+u^Tz_i-\nu_i(g(u)-u^Tv\Big)p\Big] - \sum_{j=1}^{m}\frac{\bar{y}_j}{\alpha(\bar{x},\bar{u})\bar{\tau}}\Big(h_j(\bar{u})+\bar{u}^T\bar{w}_j-\frac{1}{2}\bar{p}^T\nabla^2h_j(\bar{u})\bar{p}\Big)$$

$$> C_{\bar{x},\bar{u}}\Big[\sum_{i=1}^{k}\frac{\bar{\lambda}_i}{\bar{\tau}}\Big\{\nabla\Big(f_i(\bar{u}) + \bar{u}^T\bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T\bar{v})\Big) + \nabla^2\Big(f_i(\bar{u}) + \bar{u}^T\bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T\bar{v})\Big)p\Big\} + \sum_{j=1}^{m}\frac{\bar{y}_j}{\bar{\tau}}[\nabla(h_j(\bar{u}) + \bar{u}^T\bar{w}_j) + \nabla^2h_j(\bar{u})\bar{p}]\Big].$$

Using (4.17), above equation gives

$$\sum_{i=1}^{k}\bar{\lambda}_i\Big(f_i(\bar{x}) + \bar{x}^T\bar{z}_i - \bar{\nu}_i(g(\bar{x}) - \bar{x}^T\bar{v}) - (f_i(\bar{u}) + \bar{u}^T\bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T\bar{v})) + \frac{1}{2}p^T\nabla^2\Big(f_i(u) + u^Tz_i - \nu_i(g(u) - u^Tv\Big)p\Big) - \sum_{j=1}^{m}y_j\Big(h_j(u) + u^Tw_j - \frac{1}{2}p^T\nabla^2h_j(u)p\Big) > 0.$$

this contradicts the hypothesis (i). Hence proved. □

5. Duality model-II

Now, in this section we study the duality for (MFP)$_\nu$, for some $\nu \in R_+^k$. We first consider the following auxiliary problem:

(MFD1)

maximize $\Big[f_1(u) + u^Tz_1 - \nu_1(g(u) - u^Tv), ..., f_k(u) + u^Tz_k - \nu_k(g(u) - u^Tv)\Big]$

Subject to

$$\sum_{i=1}^{k}\lambda_i\nabla\Big(f_i(u) + u^Tz_i - \nu_i(g(u) - u^Tv)\Big) + \sum_{j=1}^{m}y_j\nabla(h_j(u) + u^Tw_j) + \sum_{i=1}^{k}\lambda_i\nabla^2\Big(f_i(u) + u^Tz_i - \nu_i(g(u) - u^Tv)\Big)p + \sum_{j=1}^{m}y_j\nabla^2h_j(u)p = 0, \tag{5.1}$$

$$\sum_{i=1}^{k}\lambda_ip^T\nabla^2\Big(f_i(u) + u^Tz_i - \nu_i(g(u) - u^Tv)\Big)p \leq 0, \tag{5.2}$$

$$p^T\nabla^2 h_j(u)p \leq 0, \tag{5.3}$$

$$z_i \in C_i,\ v \in D,\ i = 1,2,...,k,\ w_j \in E_j,\ j = 1,2,...,m,$$

$$y_j \geq 0,\ j = 1,2,...,m,\ \lambda_i > 0,\ i = 1,2,...,k,\ \sum_{i=1}^{k}\lambda_i = 1.$$

We now discuss the duality results for (MFP)$_\nu$ and (MFD1).

Theorem 5.1 (Weak Duality Theorem). Let $x \in X^0$ be a feasible solution for (MFP)$_\nu$, for some $\nu \in R^k_+$ and for each feasible solution $(u, z, v, y, \lambda, \nu, w, p)$ of (MFD1), for the same $\nu \in R^k_+$. Suppose that, for any $i = 1,2,...,k,\ j = 1,2,...,m,$

(i) $\left[f_i(.)+(.)^T z_i-\nu_i(g(.)-(.)^T v),\ h_j(.)+(.)^T w_j\right]$ is second-order $(C, \alpha, \rho, d)-V$-type-I at u,

(ii) $\alpha^1_i(x, u) = \alpha^2_j(x, u) = \alpha(x, u)$, for all i and j,

(iii) $\sum_{i=1}^{k}\lambda_i\rho^1_i + \sum_{j=1}^{m} y_j\rho^2_j \geq 0.$

Then the following cannot hold

$$f_i(x) + S(x|C_i) - \nu_i(g(x) - S(x|D)) \\ \leq f_i(u) + u^T z_i - \nu_i(g(u) - u^T v),\ \forall\ i = 1,2,...,k \tag{5.4}$$

and

$$f_r(x) + S(x|C_r) - \nu_r(g(x) - S(x|D)) \\ < f_r(u) + u^T z_r - \nu_r(g(u) - u^T v),\ \text{for some } r = 1,2,...,k. \tag{5.5}$$

Proof. The proof follows on the lines of Theorem 4.1. □

Theorem 5.2 (Strong Duality Theorem). If $\bar{u} \in X^0$ is an efficient solution of (MFP)$_\nu$ and let the Slater constraint qualification be satisfied. Then there exist $\bar{\lambda} \in R^k,\ \bar{y} \in R^m,\ \bar{z}_i \in R^n,\ \bar{v} \in R^n$ and $\bar{w}_j \in R^n,\ i = 1,2,...,k,\ j = 1,2,...,m,$ such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is a feasible solution of (MFD1) and the objective function values of (MFP)$_\nu$ and (MFD1) are equal. Moreover, if the conditions of Theorem 3.1 holds for all feasible solutions of (MFP)$_\nu$ and (MFD1), then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{\nu}_i, \bar{w}, \bar{p} = 0)$ is an efficient solution of (MFD1).

Proof. The proof follows on the lines of Theorem 4.2. □

Theorem 5.3 (Strict Converse Duality Theorem). Let $\bar{x} \in X^0$ be a feasible solution for (MFP)$_\nu$ and $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p})$ be feasible for (MFD1). Suppose that, for any $i = 1,2,...,k,\ j = 1,2,...,m,$

(i) $\sum_{i=1}^{k}\bar{\lambda}_i\Bigg(f_i(\bar{x}) + \bar{x}^T\bar{z}_i - \bar{\nu}_i(g(\bar{x}) - \bar{x}^T\bar{v}) - (f_i(\bar{u}) + \bar{u}^T\bar{z}_i - \bar{\nu}_i(g(\bar{u}) - \bar{u}^T\bar{v})\Bigg) \leq 0,$

(ii) $\left[f_i(.)+(.)^T z_i-\bar{\nu}_i(g(.)-(.)^T v),\ h_j(.)+(.)^T w_j\right]$ is second-order semi-strictly $(C, \alpha, \rho, d) - V$-type-I at $\bar{u}$,

(iii) $\alpha^1_i(\bar{x}, \bar{u}) = \alpha^2_j(\bar{x}, \bar{u}) = \alpha(\bar{x}, \bar{u})$, for all i and j,

$$(iv)\ \sum_{i=1}^{k} \bar{\lambda}_i \rho_i^1 + \sum_{j=1}^{m} \bar{y}_j \rho_j^2 \geq 0.$$

Then, $\bar{x} = \bar{u}$.

Proof. The proof follows on the lines of Theorem 4.3. □

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