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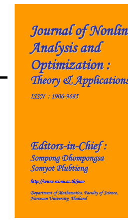

ON A NEW SYSTEM OF NONLINEAR REGULARIZED NONCONVEX VARIATIONAL INEQUALITIES IN HILBERT SPACES

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ABSTRACT. The aim of this work is to suggest a new *system of nonlinear regularized nonconvex variational inequalities in a real Hilbert spaces* and established an equivalence relation between this system and fixed point problems. By using the equivalence relation we construct a new perturbed projection iterative algorithms with mixed errors for finding a solution set of *system of nonlinear regularized nonconvex variational inequalities* and discussed the convergence of the sequences generated by an iterative algorithm.

KEYWORDS : System of nonlinear regularized nonconvex variational inequalities; uniformly r-prox regular sets; iterative sequences, convergence analysis; mixed errors.
AMS Subject Classification: 49J40, 47H06.

1. Historical Background

Variational inequalities which were introduced by Stampacchia [23] provided us with a powerful tools to study a wide class of problems arising in mechanic, physics, optimization and control theory, linear programming, economics and engineering sciences, *see* [8, 9, 12]. In recent years, several authors studied different type of system of variational inequalities and suggested iterative algorithms to find the approximate solutions of such system, *see* [7, 10, 11, 14, 15, 19, 20, 22, 26, 27]. We remark that the almost all results concerning the system of solutions of iterative scheme for solving the system of variational inequalities and related problems are being considered in the setting of convex sets. Consequently the techniques are based on the projections of operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the unified prox-regular sets are nonconvex and included the convex sets as special cases, *see* [4, 18, 28].
Inspired by the recent works going on this fields, *see* [1, 2, 3, 5, 6, 13, 16, 17, 24, 25], in this paper, we introduced and studied a new *system of nonlinear regularized nonconvex variational inequalities in a real Hilbert spaces*. We established

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the equivalence between the *system of nonlinear regularized nonconvex variational inequalities* and the fixed point problems. By using the equivalence relation, we construct a perturbed projection iterative algorithms with mixed errors for finding a solution set of the aforementioned system. Also we proved the convergence of the defined iterative algorithms under suitable assumptions.

2. Basic Foundation

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{K}$ be a nonempty convex subset of $\mathcal{H}$, and $CB(\mathcal{H})$ denote the family of all closed and bounded subsets of $\mathcal{H}$.

Definition 2.1. The proximal normal cone of $\mathcal{K}$ at a point $u \in \mathcal{H}$ is given by

$$N_{\mathcal{K}}^{P}(u) = \{\zeta \in \mathcal{H} : u \in P_{\mathcal{K}}(u + \alpha\zeta)\},$$

where $\alpha > 0$ is a constant and $P_{\mathcal{K}}$ is projection operator of $\mathcal{H}$ onto $\mathcal{K}$, that is,

$$P_{\mathcal{K}}(u) = \{v \in \mathcal{K} : d_{\mathcal{K}}(u) = \|u - v\|\},$$

where $d_{\mathcal{K}}(u)$ is the usual distance function to the subset $\mathcal{K}$, that is,

$$d_{\mathcal{K}}(u) = \inf_{v\in\mathcal{K}} \|u - v\|.$$

Lemma 2.2. *Let $\mathcal{K}$ be a nonempty closed subset in $\mathcal{H}$. Then $\zeta \in N_{\mathcal{K}}^{P}(u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle\zeta, v - u\rangle \le \alpha\|v - u\|^2 \ \forall v \in \mathcal{K}.$$

Definition 2.3. The Clarke normal cone, denoted by $N_{\mathcal{K}}^{C}(u)$, is defined as

$$N_{\mathcal{K}}^{C}(u) = \overline{co}[N_{\mathcal{K}}^{P}(u)],$$

where $\overline{co}\mathcal{A}$ means the closure of the convex hull of $\mathcal{A}$. It is clear that $N_{\mathcal{K}}^{P}(x) \subseteq N_{\mathcal{K}}^{C}(x)$. The converse is not true in general. Note that $N_{\mathcal{K}}^{C}(x)$ is always closed and convex cone where as $N_{\mathcal{K}}^{P}(x)$ is always convex but may not be closed, *see* [4, 9, 18, 24].

Definition 2.4. For any $r \in (0, +\infty]$, a subset $\mathcal{K}_r$ of $\mathcal{H}$ is called the normalized uniformly prox-regular (or uniformly r-prox-regular) if every nonzero proximal normal to $\mathcal{K}_r$ can be realized by an r-ball. This mean that for all $\overline{x} \in \mathcal{K}_r$ and all $0 \neq \zeta \in N_{\mathcal{K}_r}^{P}(\overline{x})$ with $\|\zeta\| = 1$,

$$\langle\zeta, x - \overline{x}\rangle \le \frac{1}{2r}\|x - \overline{x}\|^2, \ x \in \mathcal{K}.$$

Lemma 2.5. [8] *A closed set $\mathcal{K} \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.*

Proposition 2.6. *Let $r > 0$ and $\mathcal{K}_r$ be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set*

$$\mathcal{U}(r) = \{u \in \mathcal{H} : 0 \le d_{\mathcal{K}_r}(u) < r\}.$$

Then the following statements are hold:

(a) *for all $x \in \mathcal{U}(r), P_{\mathcal{K}_r}(x) \neq \emptyset$;*
(b) *for all $r' \in (0, r)$, $P_{\mathcal{K}_r}$ is Lipschitz continuous mapping with constant $\frac{r}{r-r'}$ on*

$$\mathcal{U}(r') = \{u \in \mathcal{H} : 0 \le d_{\mathcal{K}_r}(u) < r'\};$$

(c) *the proximal normal cone is closed as a set valued mapping.*

From Proposition 2.6 (c), we have $N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$. Therefore we define $N_{\mathcal{K}_r}(x) = N^C_{\mathcal{K}_r}(x) = N^P_{\mathcal{K}_r}(x)$ for a class of sets.

Definition 2.7. The single-valued operator $p : \mathcal{H} \longrightarrow \mathcal{H}$ is called

(i) monotone if

$$\langle p(x) - p(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H},$$

(ii) β-strongly monotone if there exists a constant $\beta > o$ such that

$$\langle p(x) - p(y), x - y\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in \mathcal{H},$$

(iii) σ-Lipschitz continuous mapping if there exists a constant $\sigma > o$ such that

$$\|p(x) - p(y)\| \leq \sigma\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.8. Let $p : \mathcal{H} \longrightarrow \mathcal{H}$ be a single valued mapping and let $T : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ be a set valued mapping. Then T is said to be

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H}, u \in T(x), v \in T(y),$$

(ii) κ-strongly monotone with respect to p if there exists a constant $\kappa > 0$ such that

$$\langle u - v, p(x) - p(y)\rangle \geq \kappa\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in T(x), v \in T(y).$$

Definition 2.9. A two-variable set-valued operator $T : \mathcal{H} \times \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is $\xi - \widehat{D}$-Lipschitz continuous in the first variable, if there exists a constant $\xi > o$ such that, for all $x, x' \in \mathcal{H}$,

$$\widehat{\mathcal{D}}(T(x, y), T(x', y')) \leq \xi\|x - x'\|, \quad \forall y, y' \in \mathcal{H}$$

where $\widehat{\mathcal{D}}$ is the Hausdorff pseudo metric that is for any two nonempty subsets A and B of $\mathcal{H}$

$$\widehat{\mathcal{D}}(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}.$$

3. System of Nonlinear Regularized Nonconvex Variational Inequalities

In this section, we introduce a new *system of nonlinear regularized nonconvex variational inequalities* in a real Hilbert space and investigated their relations.
Let $T_1, \ldots, T_N : \mathcal{H} \times \mathcal{H} \longrightarrow CB(\mathcal{H})$ be the nonlinear set valued mappings and let $g_1, \ldots, g_N, h_1, \ldots, h_N : \mathcal{H} \longrightarrow \mathcal{H}$ be the nonlinear single valued mappings such that $\mathcal{K}_r \subseteq g_i(\mathcal{H})$ for $i = 1, \ldots, N$. For any constants $\eta_1, \ldots, \eta_N > 0$, we consider the problem of finding $x_1, \ldots, x_N \in \mathcal{H}$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ such that $h_1(x_1), \ldots, h_N(x_N) \in \mathcal{K}_r$ and

$$\begin{cases} \langle \eta_1 u_1 + h_1(x_1) - g_1(x_2), g_1(x) - h_1(x_1)\rangle + \frac{1}{2r}\|g_1(x) - h_1(x_1)\|^2 \geq 0, \\ \langle \eta_2 u_2 + h_2(x_2) - g_2(x_3), g_2(x) - h_2(x_2)\rangle + \frac{1}{2r}\|g_2(x) - h_2(x_2)\|^2 \geq 0, \\ \qquad\vdots \\ \langle \eta_{N-1} u_{N-1} + h_{N-1}(x_{N-1}) - g_{N-1}(x_N), g_{N-1}(x) - h_{N-1}(x_{N-1})\rangle \\ \quad + \frac{1}{2r}\|g_{N-1}(x) - h_{N-1}(x_{N-1})\|^2 \geq 0, \\ \langle \eta_N u_N + h_N(x_N) - g_N(x_1), g_N(x) - h_N(x_N)\rangle + \frac{1}{2r}\|g_N(x) - h_N(x_N)\|^2 \geq 0, \\ \quad \forall x \in \mathcal{K}_r : g_1(x), \ldots, g_N(x) \in \mathcal{K}_r. \end{cases} \tag{3.1}$$

The problem (3.1) is called the *system of nonlinear regularized nonconvex variational inequalities.*

Lemma 3.1. *Let $\mathcal{K}_r$ be a uniformly r-prox-regular set then the problem (3.1) is equivalent to finding $x_1, \ldots, x_N \in \mathcal{H}$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ such that*

$$\begin{cases} 0 \in \eta_1 u_1 + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)), \\ 0 \in \eta_2 u_2 + h_2(x_2) - g_2(x_3) + N^P_{\mathcal{K}_r}(h_2(x_2)), \\ \qquad\qquad \vdots \\ 0 \in \eta_{N-1} u_{N-1} + h_{N-1}(x_{N-1}) - g_{N-1}(x_N) + N^P_{\mathcal{K}_r}(h_{N-1}(x_{N-1})), \\ 0 \in \eta_N u_N + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.2}$$

where $N^P_{\mathcal{K}_r}(s)$ denotes the P-normal cone of $\mathcal{K}_r$ at s in the sense of nonconvex analysis.

Proof. Let $(x_1, \ldots, x_N, u_1, \ldots, u_N)$ with $x_1, \ldots, x_N \in \mathcal{H}$, $h_1(x_1), \ldots, h_N(x_N) \in \mathcal{K}_r$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ be a solution set of the system (3.1). If

$$\eta_1 u_1 + h_1(x_1) - g_1(x_2) = 0$$

because the vector zero always belongs to any normal cone, then

$$0 \in \eta_1 u_1 + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)).$$

If

$$\eta_1 u_1 + h_1(x_1) - g_1(x_2) \neq 0$$

then for all $x \in \mathcal{H}$ with $g_1(x) \in \mathcal{K}_r$

$$\langle -(\eta_1 u_1 + h_1(x_1) - g_1(x_2)), g_1(x) - h_1(x_1)\rangle \leq \frac{1}{2r}\|g_1(x) - h_1(x_1)\|^2. \tag{3.3}$$

From Lemma 2.2 we have

$$-(\eta_1 u_1 + h_1(x_1) - g_1(x_2)) \in N^P_{\mathcal{K}_r}(h_1(x_1))$$

and

$$0 \in \eta_1 u_1 + h_1(x_1) - g_1(x_2) + N^P_{\mathcal{K}_r}(h_1(x_1)). \tag{3.4}$$

Similarly

$$\begin{cases} 0 \in \eta_2 u_2 + h_2(x_2) - g_2(x_3) + N^P_{\mathcal{K}_r}(h_2(x_2)), \\ 0 \in \eta_3 u_3 + h_3(x_3) - g_3(x_4) + N^P_{\mathcal{K}_r}(h_3(x_3)), \\ \qquad\qquad \vdots \\ 0 \in \eta_{N-1} u_{N-1} + h_{N-1}(x_{N-1}) - g_{N-1}(x_N) + N^P_{\mathcal{K}_r}(h_{N-1}(x_{N-1})), \\ 0 \in \eta_N u_N + h_N(x_N) - g_N(x_1) + N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases} \tag{3.5}$$

Conversely, if $(x_1, \ldots, x_N, u_1, \ldots, u_N)$ with $x_1, \ldots, x_N \in \mathcal{H}$, $h_1(x_1), \ldots, h_N(x_N) \in \mathcal{K}_r$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ is a solution set of the system (3.2) then from Definition 2.4, $x_1, \ldots, x_N \in \mathcal{H}$ and $u_1 \in T_1(x_2, x_1)$, $u_2 \in T_2(x_3, x_2)$, …,$u_{N-1} \in T_{N-1}(x_N, x_{N-1})$, $u_N \in T_N(x_1, x_N)$ with $h_1(x_1), \ldots, h_N(x_N) \in \mathcal{K}_r$ is a solution set of the system (3.1). ☐

The problem (3.2) is called *system of nonlinear regularized nonconvex variational inclusions.*

4. MAIN RESULTS

Lemma 4.1. *Let $T_1,\dots,T_N$, $g_1,\dots,g_N$, $h_1,\dots,h_N$, $\eta_1,\dots,\eta_N$ be the same as in the system (3.1). Then $(x_1,\dots,x_N,u_1,\dots,u_N)$ with $x_i\in\mathcal{H}$,$h_i(x_i)\in\mathcal{K}_r$ for all $i=1,\dots,N$ and $u_1\in T_1(x_2,x_1)$, $u_2\in T_2(x_3,x_2)$, …,$u_{N-1}\in T_{N-1}(x_N,x_{N-1})$, $u_N\in T_N(x_1,x_N)$ is a solution set of the system (3.1) if and only if*

$$\begin{cases} h_1(x_1)=P_{\mathcal{K}_r}[g_1(x_2)-\eta_1u_1],\\ h_2(x_2)=P_{\mathcal{K}_r}[g_2(x_3)-\eta_2u_2],\\ \qquad\vdots\\ h_{N-1}(x_{N-1})=P_{\mathcal{K}_r}[g_{N-1}(x_N)-\eta_{N-1}u_{N-1}],\\ h_N(x_N)=P_{\mathcal{K}_r}[g_N(x_1)-\eta_Nu_N], \end{cases}\tag{4.1}$$

where $P_{\mathcal{K}_r}$ is the projection of $\mathcal{H}$ onto the uniformly r-prox-regular set $\mathcal{K}_r$.

Proof. Let $(x_1,\dots,x_N,u_1,\dots,u_N)$ with $x_i\in\mathcal{H}$,$h_i(x_i)\in\mathcal{K}_r$ for all $i=1,\dots,N$ and $u_1\in T_1(x_2,x_1)$, $u_2\in T_2(x_3,x_2)$, …,$u_{N-1}\in T_{N-1}(x_N,x_{N-1})$, $u_N\in T_N(x_1,x_N)$ is a solution set of the system (3.1). Then from Lemma 3.1, we have

$$\begin{cases} 0\in\eta_1u_1+h_1(x_1)-g_1(x_2)+N^P_{\mathcal{K}_r}(h_1(x_1)),\\ 0\in\eta_2u_2+h_2(x_2)-g_2(x_3)+N^P_{\mathcal{K}_r}(h_2(x_2)),\\ \qquad\vdots\\ 0\in\eta_{N-1}u_{N-1}+h_{N-1}(x_{N-1})-g_{N-1}(x_N)+N^P_{\mathcal{K}_r}(h_{N-1}(x_{N-1})),\\ 0\in\eta_Nu_N+h_N(x_N)-g_N(x_1)+N^P_{\mathcal{K}_r}(h_N(x_N)), \end{cases}\tag{4.2}$$

$$\Leftrightarrow\begin{cases} g_1(x_2)-\eta_1u_1\in(I+N^P_{\mathcal{K}_r})(h_1(x_1)),\\ g_2(x_3)-\eta_2u_2\in(I+N^P_{\mathcal{K}_r})(h_2(x_2)),\\ \qquad\vdots\\ g_{N-1}(x_N)-\eta_{N-1}u_{N-1}\in(I+N^P_{\mathcal{K}_r})(h_{N-1}(x_{N-1})),\\ g_N(x_1)-\eta_Nu_N\in(I+N^P_{\mathcal{K}_r})(h_N(x_N)), \end{cases}\tag{4.3}$$

$$\Leftrightarrow\begin{cases} h_1(x_1)=P_{\mathcal{K}_r}[g_1(x_2)-\eta_1u_1],\\ h_2(x_2)=P_{\mathcal{K}_r}[g_2(x_3)-\eta_2u_2],\\ \qquad\vdots\\ h_{N-1}(x_{N-1})=P_{\mathcal{K}_r}[g_{N-1}(x_N)-\eta_{N-1}u_{N-1}],\\ h_N(x_N)=P_{\mathcal{K}_r}[g_N(x_1)-\eta_Nu_N], \end{cases}\tag{4.4}$$

where I is an identity mapping and $P_{\mathcal{K}_r}=(I+N^P_{\mathcal{K}_r})^{-1}$. □

Remark 4.2. The inequality (4.1) can be written as follows

$$\begin{cases} q_1=g_1(x_2)-\eta_1u_1,\quad h_1(x_1)=P_{\mathcal{K}_r}[q_1]\\ q_2=g_2(x_3)-\eta_2u_2,\quad h_2(x_2)=P_{\mathcal{K}_r}[q_2]\\ \qquad\vdots\\ q_{N-1}=g_{N-1}(x_N)-\eta_{N-1}u_{N-1},\quad h_{N-1}(x_{N-1})=P_{\mathcal{K}_r}[q_{N-1}]\\ q_N=g_N(x_1)-\eta_Nu_N,\quad h_N(x_N)=P_{\mathcal{K}_r}[q_N], \end{cases}\tag{4.5}$$

where $\eta_i>0, i=1,\dots,N$ are constants.

The fixed point formulation (4.5) enables us to construct the following perturbed iterative algorithms with mixed errors.

Algorithm 4.1. *Let $T_1,\dots,T_N$, $g_1,\dots,g_N$, $h_1,\dots,h_N$, $\eta_1,\dots,\eta_N$ be the same as in the system (3.1) such that $h_1,\dots,h_N:\mathcal{H}\longrightarrow\mathcal{H}$ be onto operators. Let $e_1^0,\dots,e_N^0$, $r_1^0,\dots,r_N^0\in\mathcal{H}$, $\alpha_0\in\mathbb{R}$ and $\eta_0>0$. For given $q_1^0,\dots,q_N^0\in\mathcal{H}$, we let $x_1^0,\dots,x_N^0\in\mathcal{H}$, $u_1^0\in T_1(x_2^0,x_1^0)$, $u_2^0\in T_2(x_3^0,x_2^0)$, …,$u_{N-1}^0\in T_{N-1}(x_N^0,x_{N-1}^0)$, $u_N^0\in T_N(x_1^0,x_N^0)$ such that*

$$\begin{cases} h_1(x_1^0)=P_{\mathcal{K}_r}(q_1^0);\ \ q_1^1=(1-\alpha_0)q_1^0+\alpha_0(g_1(x_2^0)-\eta_0u_1^0+e_1^0)+r_1^0,\\ h_2(x_2^0)=P_{\mathcal{K}_r}(q_2^0);\ \ q_2^1=(1-\alpha_0)q_2^0+\alpha_0(g_2(x_3^0)-\eta_0u_2^0+e_2^0)+r_2^0,\\ \qquad\qquad\vdots\\ h_{N-1}(x_{N-1}^0)=P_{\mathcal{K}_r}(q_{N-1}^0);\ \ q_{N-1}^1=(1-\alpha_0)q_{N-1}^0+\alpha_0(g_{N-1}(x_N^0)-\eta_0u_{N-1}^0\\ \qquad\qquad +e_{N-1}^0)+r_{N-1}^0,\\ h_N(x_N^0)=P_{\mathcal{K}_r}(q_N^0);\ \ q_N^1=(1-\alpha_0)q_N^0+\alpha_0(g_N(x_1^0)-\eta_0u_N^0+e_N^0)+r_N^0.\end{cases}\tag{4.6}$$

We choose $x_1^1,\dots,x_1^N\in\mathcal{H}$ such that $h_1(x_1^1)=P_{\mathcal{K}_r}(q_1^1)$,…,$h_N(x_1^N)=P_{\mathcal{K}_r}(q_N^1)$. By Nadler's theorem [21], there exists

$$\begin{cases} u_1^1\in T_1(x_2^0,x_1^0);\ \ \|u_1^0-u_1^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_1(x_2^0,x_1^0),T_1(x_2^1,x_1^1)),\\ u_2^1\in T_2(x_3^0,x_2^0);\ \ \|u_2^0-u_2^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_2(x_3^0,x_2^0),T_2(x_3^1,x_2^1)),\\ \qquad\qquad\vdots\\ u_{N-1}^1\in T_{N-1}(x_N^0,x_{N-1}^0);\ \|u_{N-1}^0-u_{N-1}^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_{N-1}(x_N^0,x_{N-1}^0),\\ \qquad\qquad\qquad T_{N-1}(x_N^1,x_{N-1}^1)),\\ u_N^1\in T_N(x_1^0,x_N^0);\ \ \|u_N^0-u_N^1\|\le(1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^0,x_N^0),T_N(x_1^1,x_N^1)).\end{cases}\tag{4.7}$$

Continuing the above process inductively, we can obtain the sequences $\{x_1^n\}_{n=0}^\infty$, …, $\{x_N^n\}_{n=0}^\infty$, $\{u_1^n\}_{n=0}^\infty,\dots,\{u_N^n\}_{n=0}^\infty$ by using

$$\begin{cases} h_1(x_1^n)=P_{\mathcal{K}_r}(q_1^n);\ \ q_1^{n+1}=(1-\alpha_n)q_1^n+\alpha_n(g_1(x_2^n)-\eta_1u_1^n+e_1^n)+r_1^n,\\ h_2(x_2^n)=P_{\mathcal{K}_r}(q_2^n);\ \ q_2^{n+1}=(1-\alpha_n)q_2^n+\alpha_n(g_2(x_3^n)-\eta_2u_2^n+e_2^n)+r_2^n,\\ \qquad\qquad\vdots\\ h_{N-1}(x_{N-1}^n)=P_{\mathcal{K}_r}(q_{N-1}^n);\ \ q_{N-1}^{n+1}=(1-\alpha_n)q_{N-1}^n+\alpha_n(g_{N-1}(x_N^n)\\ \qquad\qquad -\eta_{N-1}u_{N-1}^n+e_{N-1}^n)+r_{N-1}^n,\\ h_N(x_N^n)=P_{\mathcal{K}_r}(q_N^n);\ \ q_N^{n+1}=(1-\alpha_n)q_N^n+\alpha_n(g_N(x_1^n)-\eta_Nu_N^n+e_N^n)+r_N^n,\end{cases}\tag{4.8}$$

and

$$\begin{cases} u_1^{n+1} \in T_1(x_2^{n+1}, x_1^{n+1}); \;\; \|u_1^n - u_1^{n+1}\| \leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_1(x_2^{n+1}, x_1^{n+1})), \\ u_2^{n+1} \in T_2(x_3^{n+1}, x_2^{n+1}); \;\; \|u_2^n - u_2^{n+1}\| \leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_2(x_3^n, x_2^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_2(x_3^{n+1}, x_2^{n+1})), \\ \qquad\qquad \vdots \\ u_{N-1}^{n+1} \in T_{N-1}(x_N^{n+1}, x_{N-1}^{n+1}); \;\; \|u_{N-1}^n - u_{N-1}^{n+1}\| \leq (1+(1+n)^{-1})\times \\ \qquad\qquad \widehat{\mathcal{D}}(T_{N-1}(x_N^n, x_{N-1}^n), T_{N-1}(x_N^{n+1}, x_{N-1}^{n+1})), \\ u_N^{n+1} \in T_N(x_1^{n+1}, x_N^{n+1}); \;\; \|u_N^n - u_N^{n+1}\| \leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n, x_N^n), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T_N(x_1^{n+1}, x_N^{n+1})), \end{cases} \tag{4.9}$$

where $0 \leq \alpha_n \leq 1$ is a parameter and $\{e_1^n\}_{n=0}^{\infty}, \ldots, \{e_N^n\}_{n=0}^{\infty}$, $\{r_1^n\}_{n=0}^{\infty}, \ldots, \{r_N^n\}_{n=0}^{\infty}$ are sequences in $\mathcal{H}$ to take into account of a possible inexact computation of the resolvent operator satisfying the following conditions:

$$\lim_{n\longrightarrow\infty} e_i^n = \lim_{n\longrightarrow\infty} r_i^n = 0;$$

$$\sum_{n=1}^{\infty} \|e_i^n - e_i^{n-1}\| < \infty, \;\; \sum_{n=1}^{\infty} \|r_i^n - r_i^{n-1}\| < \infty, \tag{4.10}$$

for all $i = 1, \ldots, N$.

Theorem 4.3. *Let T_i, g_i, h_i, η_i, for $i = 1, \ldots, N$ be the same as in the system (3.1) such that, for each $i = 1, \ldots, N$,*

- (i) *T_i is κ_i-strongly monotone with respect to g_i and $\xi_i - \widehat{\mathcal{D}}$-Lipschitz continuous mapping in the first variables;*
- (ii) *h_i is β_i-strongly monotone and σ_i-Lipschitz continuous mapping;*
- (iii) *g_i is μ_i-Lipschitz continuous mapping.*

If the constants $\eta_i > 0$ satisfying the following conditions:

$$\begin{aligned} \left|\eta_1 - \frac{\kappa_1}{\xi_1^2}\right| &< \frac{\sqrt{r^2\kappa_1^2 - \xi_1^2\Big(r^2\mu_1^2 - (r-r')^2(1-\pi_2)^2\Big)}}{r\xi_1^2}, \\ &\;\;\vdots \\ \left|\eta_N - \frac{\kappa_N}{\xi_N^2}\right| &< \frac{\sqrt{r^2\kappa_N^2 - \xi_N^2\Big(r^2\mu_N^2 - (r-r')^2(1-\pi_1)^2\Big)}}{r\xi_N^2}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} r\kappa_1 &> \xi_1\sqrt{r^2\mu_1^2 - (r-r')^2(1-\pi_2)^2}, \\ &\;\;\vdots \\ r\kappa_N &> \xi_N\sqrt{r^2\mu_N^2 - (r-r')^2(1-\pi_1)^2}, \end{aligned} \tag{4.12}$$

$$r\mu_1 > (r-r')(1-\pi_2), \ldots, r\mu_N > (r-r')(1-\pi_1), \tag{4.13}$$

and

$$\pi_i = \sqrt{1 - 2\beta_i + \sigma_i^2}, \;\; 2\pi_i < 1 + \sigma_i^2, \tag{4.14}$$

for each $i = 1, \ldots, N$, where $r' \in (0, r)$, then there exists $x_1^*, \ldots, x_N^* \in \mathcal{H}$ with $h_1(x_1^*), \ldots, h_N(x_N^*) \in \mathcal{K}_r$ and $u_1^* \in T_1(x_2^*, x_1^*)$, $u_2^* \in T_2(x_3^*, x_2^*)$, $\ldots$, $u_{N-1}^* \in T_{N-1}(x_N^*, x_{N-1}^*)$,$u_N^* \in T_N(x_1^*, x_N^*)$ such that $(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*)$ is a solution set of system (3.1) and sequences $\{(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*)\}_{n=0}^{\infty}$ suggested by Algorithm 4.1 converges strongly to $(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*)$.

Proof. From (4.8), we have

$$\begin{aligned}\|q_1^{n+1} - q_1^n\| \leq\ & (1-\alpha_n)\|q_1^n - q_1^{n-1}\| + \alpha_n\|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(u_1^n - u_1^{n-1})\| \\ & +\alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\|. \end{aligned} \tag{4.15}$$

Since T_1 is κ_1-strongly monotone with respect to g_1 and $\xi_1 - \widehat{\mathcal{D}}$-Lipschitz continuous mapping in the first variables and g_1 is μ_1-Lipschitz continuous, we get

$$\begin{aligned} & \|g_1(x_2^n) - g_1(x_2^{n-1}) - \eta_1(u_1^n - u_1^{n-1})\|^2 \\ & = \|g_1(x_2^n) - g_1(x_2^{n-1})\|^2 - 2\eta_1\langle u_1^n - u_1^{n-1}, g_1(x_2^n) - g_1(x_2^{n-1})\rangle \\ & \quad + \eta_1^2\|u_1^n - u_1^{n-1}\|^2 \\ & \leq \mu_1^2\|x_2^n - x_2^{n-1}\|^2 - 2\eta_1\kappa_1\|x_2^n - x_2^{n-1}\|^2 \\ & \qquad + \eta_1^2(1+n^{-1})^2(\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})))^2 \\ & = (\mu_1^2 - 2\eta_1\kappa_1)\|x_2^n - x_2^{n-1}\|^2 \\ & \quad + \eta_1^2(1+n^{-1})^2(\widehat{\mathcal{D}}(T_1(x_2^n, x_1^n), T_1(x_2^{n-1}, x_1^{n-1})))^2 \\ & \leq (\mu_1^2 - 2\eta_1\kappa_1)\|x_2^n - x_2^{n-1}\|^2 + \eta_1^2\xi_1^2(1+n^{-1})^2\|x_2^n - x_2^{n-1}\|^2 \\ & = (\mu_1^2 - 2\eta_1\kappa_1 + \eta_1^2\xi_1^2(1+n^{-1})^2)\|x_2^n - x_2^{n-1}\|^2. \end{aligned} \tag{4.16}$$

It follows from (4.15) and (4.16), we obtain that

$$\begin{aligned}\|q_1^{n+1} - q_1^n\| \leq\ & (1-\alpha_n)\|q_1^n - q_1^{n-1}\| + \alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\| \\ & +\alpha_n\sqrt{\mu_1^2 - 2\eta_1\kappa_1 + \eta_1^2\xi_1^2(1+n^{-1})^2}\|x_2^n - x_2^{n-1}\|. \end{aligned} \tag{4.17}$$

Similarly, we can prove that

$$\begin{aligned}\|q_2^{n+1} - q_2^n\| \leq\ & (1-\alpha_n)\|q_2^n - q_2^{n-1}\| + \alpha_n\|e_2^n - e_2^{n-1}\| + \|r_2^n - r_2^{n-1}\| \\ & +\alpha_n\sqrt{\mu_2^2 - 2\eta_2\kappa_2 + \eta_2^2\xi_2^2(1+n^{-1})^2}\|x_3^n - x_3^{n-1}\| \\ \|q_3^{n+1} - q_3^n\| \leq\ & (1-\alpha_n)\|q_3^n - q_3^{n-1}\| + \alpha_n\|e_3^n - e_3^{n-1}\| + \|r_3^n - r_3^{n-1}\| \\ & +\alpha_n\sqrt{\mu_3^2 - 2\eta_3\kappa_3 + \eta_1^2\xi_3^2(1+n^{-1})^2}\|x_4^n - x_4^{n-1}\| \\ & \vdots \\ \|q_{N-1}^{n+1} - q_{N-1}^n\| \leq\ & (1-\alpha_n)\|q_{N-1}^n - q_{N-1}^{n-1}\| \\ & +\alpha_n\sqrt{\mu_{N-1}^2 - 2\eta_{N-1}\kappa_{N-1} + \eta_{N-1}^2\xi_{N-1}^2(1+n^{-1})^2} \times \\ & \|x_N^n - x_N^{n-1}\| + \alpha_n\|e_{N-1}^n - e_{N-1}^{n-1}\| + \|r_{N-1}^n - r_{N-1}^{n-1}\| \\ \|q_N^{n+1} - q_N^n\| \leq\ & (1-\alpha_n)\|q_N^n - q_N^{n-1}\| \\ & +\alpha_n\sqrt{\mu_N^2 - 2\eta_N\kappa_N + \eta_N^2\xi_N^2(1+n^{-1})^2}\|x_1^n - x_1^{n-1}\| \\ & +\alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|. \end{aligned} \tag{4.18}$$

By using (4.8), we get that

$$\begin{aligned}\|x_1^n - x_1^{n-1}\| & \leq \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|h_1(x_1^n) - h_1(x_1^{n-1})\| \\ & = \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \|P_{\mathcal{K}_r}(q_1^n) - P_{\mathcal{K}_r}(q_1^{n-1})\| \end{aligned}$$

$$\leq \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\| + \tfrac{r}{r-r'}\|q_1^n - q_1^{n-1}\|. \quad (4.19)$$

Since h_1 is β_1-strongly monotone and σ_1-Lipschitz continuous, we have

$$\begin{aligned} \|x_1^n - x_1^{n-1} - (h_1(x_1^n) - h_1(x_1^{n-1}))\|^2 &= \|x_1^n - x_1^{n-1}\|^2 + \|h_1(x_1^n) - h_1(x_1^{n-1})\|^2 \\ &\quad -2\langle h_1(x_1^n) - h_1(x_1^{n-1}), x_1^n - x_1^{n-1}\rangle \\ &\leq \|x_1^n - x_1^{n-1}\|^2 - 2\beta_1\|x_1^n - x_1^{n-1}\|^2 \\ &\quad +\sigma_1^2\|x_1^n - x_1^{n-1}\|^2 \\ &= (1 - 2\beta_1 + \sigma_1^2)\|x_1^n - x_1^{n-1}\|^2. \end{aligned} \quad (4.20)$$

By (4.19) and (4.20), we obtain that

$$\|x_1^n - x_1^{n-1}\| \leq \sqrt{1 - 2\beta_1 + \sigma_1^2}\|x_1^n - x_1^{n-1}\| + \frac{r}{r - r'}\|q_1^n - q_1^{n-1}\| \quad (4.21)$$

that is

$$\|x_1^n - x_1^{n-1}\| \leq \frac{r}{(r - r')(1 - \sqrt{1 - 2\beta_1 + \sigma_1^2})}\|q_1^n - q_1^{n-1}\|. \quad (4.22)$$

Similarly, we can prove that

$$\begin{aligned} \|x_2^n - x_2^{n-1}\| &\leq \frac{r}{(r - r')(1 - \sqrt{1 - 2\beta_2 + \sigma_2^2})}\|q_2^n - q_2^{n-1}\|, \\ \|x_3^n - x_3^{n-1}\| &\leq \frac{r}{(r - r')(1 - \sqrt{1 - 2\beta_3 + \sigma_3^2})}\|q_3^n - q_3^{n-1}\|, \\ &\vdots \\ \|x_{N-1}^n - x_{N-1}^{n-1}\| &\leq \frac{r}{(r - r')(1 - \sqrt{1 - 2\beta_{N-1} + \sigma_{N-1}^2})}\|q_{N-1}^n - q_{N-1}^{n-1}\|, \\ \|x_N^n - x_N^{n-1}\| &\leq \frac{r}{(r - r')(1 - \sqrt{1 - 2\beta_N + \sigma_N^2})}\|q_N^n - q_N^{n-1}\|. \end{aligned} \quad (4.23)$$

It follows from (4.17), (4.18), (4.22) and (4.23) that

$$\begin{aligned} \|q_1^{n+1} - q_1^n\| &\leq (1 - \alpha_n)\|q_1^n - q_1^{n-1}\| + \alpha_n\frac{r\Omega_1(n)}{(r - r')(1 - \pi_2)}\|q_2^n - q_2^{n-1}\| \\ &\quad +\alpha_n\|e_1^n - e_1^{n-1}\| + \|r_1^n - r_1^{n-1}\|, \\ \|q_2^{n+1} - q_2^n\| &\leq (1 - \alpha_n)\|q_2^n - q_2^{n-1}\| + \alpha_n\frac{r\Omega_2(n)}{(r - r')(1 - \pi_3)}\|q_3^n - q_3^{n-1}\| \\ &\quad +\alpha_n\|e_2^n - e_2^{n-1}\| + \|r_2^n - r_2^{n-1}\|, \\ &\vdots \\ \|q_{N-1}^{n+1} - q_{N-1}^n\| &\leq (1 - \alpha_n)\|q_{N-1}^n - q_{N-1}^{n-1}\| + \alpha_n\frac{r\Omega_{N-1}(n)}{(r - r')(1 - \pi_N)}\|q_N^n - q_N^{n-1}\| \\ &\quad +\alpha_n\|e_{N-1}^n - e_{N-1}^{n-1}\| + \|r_{N-1}^n - r_{N-1}^{n-1}\|, \\ \|q_N^{n+1} - q_N^n\| &\leq (1 - \alpha_n)\|q_N^n - q_N^{n-1}\| + \alpha_n\frac{r\Omega_N(n)}{(r - r')(1 - \pi_1)}\|q_1^n - q_1^{n-1}\| \\ &\quad +\alpha_n\|e_N^n - e_N^{n-1}\| + \|r_N^n - r_N^{n-1}\|, \end{aligned} \quad (4.24)$$

where

$$\Omega_i(n) = \sqrt{\mu_i^2 - 2\eta_i\kappa_i + \eta_i^2\xi_i^2(1 + n^{-1})^2}, \text{ and } \pi_i = \sqrt{1 - 2\beta_i + \sigma_i^2},$$

for all $i = 1, 2, \ldots, N$.

Now we define $\|\cdot\|_*$ on $\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}$ by

$$\|(x_1,\ldots,x_N)\|_* = \|x_1\|_* + \ldots + \|x_N\|_*, \quad \text{for all } (x_1,\ldots,x_N) \in \underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}.$$

It is obvious that $(\underbrace{\mathcal{H}\times\ldots\times\mathcal{H}}_{N-\text{times}}, \|\cdot\|_*)$ is a Hilbert space, applying (4.24) we have

$$\begin{aligned}\|(q_1^{n+1},\ldots,q_N^{n+1}) - (q_1^n,\ldots,q_N^n)\|_* \leq\ & (1-\alpha_n)\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* \\ &+\alpha_n\Theta(n)\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* + \alpha_n\|(e_1^n,\ldots,e_N^n) \\ &- (e_1^{n-1},\ldots,e_N^{n-1})\|_* + \|(r_1^n,\ldots,r_N^n) - (r_1^{n-1},\ldots,r_N^{n-1})\|_*. \end{aligned} \tag{4.25}$$

Put

$$\Theta(n) = \max\left\{\frac{r\Omega_1(n)}{(r-r')(1-\pi_2)},\ldots,\frac{r\Omega_N(n)}{(r-r')(1-\pi_1)}\right\}. \tag{4.26}$$

Let $\Theta(n) \longrightarrow \Theta$, as $n \longrightarrow \infty$

$$\Theta = \max\left\{\frac{r\Omega_1}{(r-r')(1-\pi_2)},\ldots,\frac{r\Omega_N}{(r-r')(1-\pi_1)}\right\}. \tag{4.27}$$

By (4.11),we know that $0 \leq \Theta < 1$. For $\Theta = \frac{1}{2}(\Theta+1) \in (\Theta, 1)$ there exists $n_0 \geq 1$ such that $\Theta(n) = \widehat{\Theta}$ for each $n \geq n_0$. So it follows from (4.25) that, for each $n \geq n_0$,

$$\begin{aligned}\|(q_1^{n+1},\ldots,q_N^{n+1}) - (q_1^n,\ldots,q_N^n)\|_* \leq\ & (1-\alpha_n)\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* \\ &+ \alpha_n\widehat{\Theta}\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* \\ &+ \alpha_n\|(e_1^n,\ldots,e_N^n) - (e_1^{n-1},\ldots,e_N^{n-1})\|_* + \|(r_1^n,\ldots,r_N^n) - (r_1^{n-1},\ldots,r_N^{n-1})\|_* \\ =\ & (1-\alpha_n(1-\widehat{\Theta}))\|(q_1^n,\ldots,q_N^n) - (q_1^{n-1},\ldots,q_N^{n-1})\|_* \\ &+ \alpha_n\|(e_1^n,\ldots,e_N^n) - (e_1^{n-1},\ldots,e_N^{n-1})\|_* + \|(r_1^n,\ldots,r_N^n) - (r_1^{n-1},\ldots,r_N^{n-1})\|_* \\ \leq\ & (1-\alpha_n(1-\widehat{\Theta}))\Big((1-\alpha_n(1-\widehat{\Theta}))\|(q_1^{n-1},\ldots,q_N^{n-1}) - (q_1^{n-2},\ldots,q_N^{n-2})\|_* \\ &\quad + \alpha_n\|(e_1^{n-1},\ldots,e_N^{n-1}) - (e_1^{n-2},\ldots,e_N^{n-2})\|_* + \|(r_1^{n-1},\ldots,r_N^{n-1}) \\ &\quad - (r_1^{n-2},\ldots,r_N^{n-2})\|_*\Big) + \alpha_n\|(e_1^n,\ldots,e_N^n) - (e_1^{n-1},\ldots,e_N^{n-1})\|_* \\ &\quad + \|(r_1^n,\ldots,r_N^n) - (r_1^{n-1},\ldots,r_N^{n-1})\|_* \\ =\ & (1-\alpha_n(1-\widehat{\Theta}))^2\|(q_1^{n-1},\ldots,q_N^{n-1}) - (q_1^{n-2},\ldots,q_N^{n-2})\|_* \\ &\quad + \alpha_n\Big((1-\alpha_n(1-\widehat{\Theta}))\|(e_1^{n-1},\ldots,e_N^{n-1}) - (e_1^{n-2},\ldots,e_N^{n-2})\|_* \\ &\qquad + \|(e_1^n,\ldots,e_N^n) - (e_1^{n-1},\ldots,e_N^{n-1})\|_*\Big) \\ &\quad + (1-\alpha_n(1-\widehat{\Theta}))\|(r_1^{n-1},\ldots,r_N^{n-1}) - (r_1^{n-2},\ldots,r_N^{n-2})\|_* \\ &\quad + \|(r_1^n,\ldots,r_N^n) - (r_1^{n-1},\ldots,r_N^{n-1})\|_* \\ \leq\ & \\ & \vdots \\ \leq\ & (1-\alpha_n(1-\widehat{\Theta}))^{n-n_0}\|(q_1^{n_0+1},\ldots,q_N^{n_0+1}) - (q_1^{n_0},\ldots,q_N^{n_0})\|_* \\ &+\alpha_n\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\ldots,e_N^{n-(i-1)}) - (e_1^{n-i},\ldots,e_N^{n-i})\|_* \end{aligned}$$

$$+\sum_{i=1}^{n-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\ldots,r_N^{n-(i-1)})-(r_1^{n-i},\ldots,r_N^{n-i})\|_*. \quad (4.28)$$

Thus, for any $m \geq n > n_0$, we get that

$$\begin{aligned}
&\|(q_1^m,\ldots,q_N^m)-(q_1^n,\ldots,q_N^n)\|_* \\
&\quad\leq \sum_{j=n}^{m-1}\|(q_1^{j+1},\ldots,q_N^{j+1})-(q_1^j,\ldots,q_N^j)\|_* \\
&\quad\leq \sum_{j=n}^{m-1}(1-\alpha_n(1-\widehat{\Theta}))^{j-n_0}\|(q_1^{n_0+1},\ldots,q_N^{n_0+1})-(q_1^{n_0},\ldots,q_N^{n_0})\|_* \\
&\quad+\alpha_n\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(e_1^{n-(i-1)},\ldots,e_N^{n-(i-1)})-(e_1^{n-i},\ldots,e_N^{n-i})\|_*
\end{aligned}$$

$$+\sum_{j=n}^{m-1}\sum_{i=1}^{j-n_0}(1-\alpha_n(1-\widehat{\Theta}))^{i-1}\|(r_1^{n-(i-1)},\ldots,r_N^{n-(i-1)})-(r_1^{n-i},\ldots,r_N^{n-i})\|_*. \quad (4.29)$$

Since $(1-\alpha_n(1-\widehat{\Theta}))\in(0,1)$, it follows from (4.10) and (4.29) that

$$\|(q_1^m,\ldots,q_N^m)-(q_1^n,\ldots,q_N^n)\|_*=\|q_1^m-q_1^n\|+\ldots+\|q_N^m-q_N^n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

So $\{q_1^n\},\ldots,\{q_N^n\}$ are Cauchy sequences in $\mathcal{H}$ and thus there exists $q_1^*,\ldots,q_N^*\in\mathcal{H}$ such that $q_1^n\longrightarrow q_1^*$,..., $q_N^n\longrightarrow q_N^*$ as $n\longrightarrow\infty$. By (4.22) and (4.23), it follows that the sequences $\{x_1^n\}$,...,$\{x_N^n\}$ are also Cauchy sequences in $\mathcal{H}$. Hence there exists $x_1^*,\ldots,x_N^*\in\mathcal{H}$ such that $x_1^n\longrightarrow x_1^*,\ldots,x_N^n\longrightarrow x_N^*$ as $n\longrightarrow\infty$. Since for each $i=1,\ldots,N$, T_i is $\xi_i-\widehat{\mathcal{D}}$-Lipschitz continuous mapping in the first variable, therefore it follow from (4.7) that

$$\begin{aligned}
\|u_1^n-u_1^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_1(x_2^n,x_1^n),T_1(x_2^{n+1},x_1^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_1\|x_2^n-x_2^{n+1}\|\longrightarrow 0, \\
\|u_2^n-u_2^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_2(x_3^n,x_2^n),T_2(x_3^{n+1},x_2^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_2\|x_3^n-x_3^{n+1}\|\longrightarrow 0, \\
&\vdots \\
\|u_{N-1}^n-u_{N-1}^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_{N-1}(x_N^n,x_{N-1}^n),T_{N-1}(x_N^{n+1},x_{N-1}^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_{N-1}\|x_N^n-x_N^{n+1}\|\longrightarrow 0, \\
\|u_N^n-u_N^{n+1}\| &\leq (1+(1+n)^{-1})\widehat{\mathcal{D}}(T_N(x_1^n,x_N^n),T_1(x_1^{n+1},x_N^{n+1})) \\
&\leq (1+(1+n)^{-1})\xi_N\|x_1^n-x_1^{n+1}\|\longrightarrow 0, \quad (4.30)
\end{aligned}$$

as $n\longrightarrow\infty$. Hence $\{u_1^n\},\ldots,\{u_N^n\}$ are Cauchy sequences in $\mathcal{H}$ and so there exists $u_1^*,\ldots,u_N^*\in\mathcal{H}$ such that $u_1^n\longrightarrow u_1^*,\ldots,u_N^n\longrightarrow u_N^*$ as $n\longrightarrow\infty$. Further $u_1^n\in T_1(x_2^n,x_1^n)$ we have

$$\begin{aligned}
d(u_1^*,T_1(x_2^*,x_1^*)) &= \inf\{\|u_1^*-t\|:t\in T_1(x_2^*,x_1^*)\} \\
&\leq \|u_1^*-u_1^n\|+d(u_1^n,T_1(x_2^*,x_1^*)) \\
&\leq \|u_1^*-u_1^n\|+\widehat{\mathcal{D}}(T_1(x_2^n,x_1^n),T_1(x_2^{n+1},x_1^{n+1})) \\
&\leq \|u_1^*-u_1^n\|+\xi_1\|x_2^n-x_2^*\|\longrightarrow 0, \quad (4.31)
\end{aligned}$$

as $n \longrightarrow \infty$. Hence $d(u_1^*, T_1(x_2^*, x_1^*)) = 0$ and so $u_1^n \longrightarrow u_1^* \in T_1(x_2^*, x_1^*)$.
By the same method, we can prove that

$$\begin{aligned} d(u_2^*, T_2(x_3^*, x_2^*)) &\leq \|u_2^* - u_2^n\| + \xi_2\|x_3^n - x_3^*\| \longrightarrow 0, \\ d(u_3^*, T_3(x_4^*, x_3^*)) &\leq \|u_3^* - u_3^n\| + \xi_3\|x_4^n - x_4^*\| \longrightarrow 0, \\ &\vdots \\ d(u_{N-1}^*, T_{N-1}(x_N^*, x_{N-1}^*)) &\leq \|u_{N-1}^* - u_{N-1}^n\| + \xi_{N-1}\|x_N^n - x_N^*\| \longrightarrow 0, \\ d(u_N^*, T_N(x_1^*, x_N^*)) &\leq \|u_N^* - u_N^n\| + \xi_N\|x_1^n - x_1^*\| \longrightarrow 0, \end{aligned} \tag{4.32}$$

as $n \longrightarrow \infty$. Therefore $u_2^* \in T_2(x_3^*, x_2^*)$,...,$u_N^* \in T_N(x_1^*, x_N^*)$. Since g_i for $i = 1, \ldots, N$ is continuous, it follows from (4.8) and (4.10) that

$$q_1^* = g_1(x_2^*) - \eta_1 u_1^*, \ \ \ldots, \ \ q_N^* = g_N(x_1^*) - \eta_N u_N^*. \tag{4.33}$$

Since $h_1, \ldots, h_N$ and $P_{\mathcal{K}_r}$ are continuous mappings, it follows from (4.8) and (4.33) that

$$\begin{aligned} h_1(x_1^*) &= P_{\mathcal{K}_r}(q_1^*) = P_{\mathcal{K}_r}(g_1(x_2^*) - \eta_1 u_1^*), \\ h_2(x_2^*) &= P_{\mathcal{K}_r}(q_2^*) = P_{\mathcal{K}_r}(g_2(x_3^*) - \eta_2 u_2^*), \\ &\vdots \\ h_1(x_{N-1}^*) &= P_{\mathcal{K}_r}(q_{N-1}^*) = P_{\mathcal{K}_r}(g_{N-1}(x_N^*) - \eta_{N-1} u_{N-1}^*), \\ h_N(x_N^*) &= P_{\mathcal{K}_r}(q_N^*) = P_{\mathcal{K}_r}(g_N(x_1^*) - \eta_N u_N^*). \end{aligned} \tag{4.34}$$

Now Lemma 4.1, guarantees that $(x_1^*, \ldots, x_N^*, u_1^*, \ldots, u_N^*)$ is a solution set of system (3.1). □

References

1. M. K. Ahmad and Salahuddin, Perturbed three step approximation process with errors for a general implicit nonlinear variational inequalities, Int. J. Math. Math. Sci., Article ID 43818, (2006), 1-14.
2. M. K. Ahmad and Salahuddin, A stable perturbed algorithms for a new class of generalized nonlinear implicit quasi variational inclusions in Banach spaces, Advances in Pure Mathematics, 2 (2012), no. 2, 139-148.
3. J. Balooee, Y. J. Cho and M. K. Kang, Projection methods and a new system of extended general regularized nonconvex set valued variational inequalities, J. Applied Math., 2012, Article ID 690648, (2012), 18 pages.
4. M. Bounkhel and L. Thibault, Further characterizations of regular sets in Hilbert spaces and their applications to nonconvex sweeping process, Preprint Centro de Modela Minto Matematica (CMM), Universidad de chile 2000.
5. M. Bounkhel, L. Tadj and A. Hamdi, Iterative schemes to solve nonconvex variational problems, J. Inequal. Pure Appl. Math., 4 (2003), 1-14.
6. S. S. Chang, Y. K. Tang and Salahuddin, A system of nonlinear set valued variational Inclusions, SpringerPlus, 3 (2014), 318-328.
7. S. S. Chang, H. W. J. Lee and C. K. Chan, Generalized system for relaxed cocoercive variational inequalities in Hilbert space, Appl. Math. Lett., 20 (2007), no. 3, 329-334.
8. F. H. Clarke, Optimization and nonsmooth analysis, Wiley Int. Science, New York, 1983.
9. F. H. Clarke, Y. S. Ledyaw, R. J. Stern and P. R. Wolenski, Nonsmooth analysis and control theory, Springer-Verlag, New York 1998.
10. X. P. Ding, Existence and algorithm of solutions for a system of generalized mixed implicit equilibrium problem in Banach spaces, Appl. Math. Mech., 31 (2010), no. 9, 1049-1062.
11. X. P. Ding and Salahuddin, On a system of general nonlinear variational inclusions in Banach spaces, Appl. Math. Mech., 36 (2015), no. 12, 1663-1672.
12. M. C. Ferris and J. S. Pang, Engineering and economic applications of complementarity problems, SIAM Review, 39 (1997), no. 4, 669-713.
13. N. J. Huang and Y. P. Fang, Generalized m-accretive mappings in Banach spaces, J. Sichuan University, 38 (2001), no. 4, 591-592.

14. I. Inchan and N. Petrot, System of general variational inequalities involving different nonlinear operators related to fixed point problems and its applications, Fixed point Theory, 2011, Article ID 689478, pages 17, doi 10.1155/2011/689478.
15. M. M. Jin, Iterative algorithms for a new system of nonlinear variational inclusions with (A, η)-accretive mappings in Banach spaces, Comput. Math. Appl., 54 (2007), 579-588.
16. M. F. Khan and Salahuddin, Generalized multivalued quasi variational–like inclusions, Adv. Nonlinear Var. Inequal., 9 (2006), no. 1, 37–50.
17. B. S. Lee, M. F. Khan and Salahuddin, Generalized Nonlinear Quasi-Variational Inclusions in Banach Spaces, Comput. Maths. Appl., 56 (2008), no. 5, 1414-1422.
18. B. S. Lee and Salahuddin, A general system of regularized nonconvex variational inequalities, Appl. Comput. Maths., 3 (2014), no. 4, 14-22.
19. B. S. Lee and Salahuddin, System of generalized nonlinear mixed variational inclusions involving relaxed cocoercive mappings in Hilbert spaces, East Asian Math. J., 31 (2015), no. 3, 383-391.
20. H. Y. Lan, N. J. Huang and Y. J. Cho, New iterative approximation for a system of generalized nonlinear variational inclusions with set valued mappings in Banach spaces, Math. Inequal. Appl., 1 (2006), 175-187.
21. S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math., 30 (1969), 475-488.
22. J. W. Peng, D. L. Zhu and X. P. Zheng, Existence of solutions and convergence of a multistep iterative algorithm for a system of variational inclusions with (H, η)-accretive operators, Fixed Point Theory Appl. 2007, Art. ID 93678, (2007), pp 20.
23. G. Stampacchia, An Formes bilineaires coercitives sur les ensembles convexes, C. R. Acad. Sci. Paris, 258 (1964), 4413-4416.
24. R. A. Poliquin, R. T. Rockafellar and L. Thibault, Local differentiability of distance functions, Trans. Amer. Math. Soc., 352 (2000), 5231-5249.
25. R. U. Verma and Salahuddin, System of nonlinear regularized nonconvex parametric variational inequalities, Pan Amer. Math. J., 26 (2016), no. 1, 54-65.
26. R. U. Verma, Generalized system for relaxed cocoercive variational inequalities and projection methods, J. Optim. Theo. Appl., 121 (2004), 203-210.
27. D. J. Wen, Projection methods for a generalized system of nonconvex variational inequalities with different nonlinear operators, Nonlinear Anal., 73 (2010), 2292-2297.
28. X. Weng, Fixed point iteration for local strictly pseudo contractive mapping, Proc. Amer. Math. Soc., 113 (1991), no. 3, 727-737.