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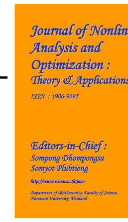

COUPLED FIXED POINTS IN PARTIALLY ORDERED METRIC SPACES BY SAMET's METHOD AND APPLICATION

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In 2006, Bhaskar and Lakshmikantham proved a fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order, using a weak contractivity type of assumption. Recently Luong and Thuan proved some results of coupled fixed point that generalized main results of them. In this paper, By using the samet's method and by using different conditions we prove some coupled fixed point theorems for mapping having mixed monotone property in partially ordered metric space. Also by considering the results of Berinde and Burcut and using the main idea of Samet and Vetro extend the concept of $\alpha-$admissibility for tripled fixed point theorems in metric spaces. As an application, we discuss the existence and solution of a nonlinear integral equation.

KEYWORDS : Coupled fixed point, Tripled fixed point, Mixed monotone mapping, Integral equation.
AMS Subject Classification: 54H25, 47H10

1. INTRODUCTION

In 1987, Guo and Lakshmikantham introduced the notion of coupled fixed points [6]. In the last decade of the previous century other authors obtained important results in this area. In 2006 Bhaskar and Lakshmikantham introduced notions of a mixed monotone mapping and a coupled fixed point [7]. They proved fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order, using a weak contractivity type of assumption.
Recently Luong and Thuan proved some results of coupled fixed point that generalized main results of them [10].
Let us recall some basic definitions of mixed monotone property and α-admissiblity, [11]-[10].

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Definition 1.1. [7], Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$x_1, x_2 \in X,\ x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y)$

and

$y_1, y_2 \in X,\ y_1 \leq y_2 \Longrightarrow F(x, y_1) \geq F(x, y_2)$.

Definition 1.2. [7], An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping $F : X \times X \to X$ if $x = F(x, y)$, and $y = F(y, x)$.

Definition 1.3. [2], An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of $F : X \times X \times X \to X$ if $F(x, y, z) = x,\ F(y, z, x) = y$, and $F(z, x, y) = z$.

Definition 1.4. [11], Let (X, d) be a metric space and $T : X \to X$ be a given mapping. we say that T is an $\alpha - \psi - contractive$ mapping, if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that :

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \qquad forall \quad x, y \in X.$$

Definition 1.5. [11], Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. we say that T is $\alpha - admissible$ if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.6. [9] Let $f : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$. We say that f is a triangular $\alpha-$admissible mapping if $(T1)$ $\alpha(x, y) \geq 1 \ \Rightarrow\ \alpha(fx, fy) \geq 1,\ x, y \in X$
$(T2)$

$$\begin{cases} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{cases}$$

implies $\alpha(x, y) \geq 1, x, y, z \in X$.

Example 1.7. [9] Let $X = \mathbb{R}$, $fx = e^{x^7}$ and $\alpha(x, y) = \sqrt[5]{x - y} + 1$. Hence, f is a triangular $\alpha-$admissible mapping. Again, if $\alpha(x, y) = \sqrt[5]{x - y} + 1 \geq 1$ then $x \geq y$ which implies $fx \geq fy$. That is, $\alpha(fx, fy) \geq 1$. Moreover, if

$$\begin{cases} \alpha(x, z) \geq 1; \\ \alpha(z, y) \geq 1, \end{cases}$$

then $x - y \geq 0$, and hence $\alpha(x, y) \geq 1$.

Lemma 1.8. [11] *(A Coupled Fixed Point is a Fixed Point). Let $F : X \times X \to X$ be a given mapping. Define the mapping $T : X \times X \longrightarrow X \times X$ by*

$$T(x, y) = (F(x, y), F(y, x)),\ forall\ (x, y) \in X \times X.$$

Then, (x, y) is a coupled fixed point of F if and only if (x, y) is a fixed point of T.

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy
(i) φ is continuous and non-decreasing,
(ii) $\varphi(t) = 0$ if and only if $t = 0$,
(iii)$\varphi(t + s) \leq \varphi(t) + \varphi(s)$, $\forall t, s \in [0, \infty)$
and Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy $lim_{t \to r}\psi(t) > 0$ for all $r > 0$ and $lim_{t \to o^+}\psi(t) = 0$.
In [10] the authors gave examples of this functions.

Theorem 1.9. *(see [11]) let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi-$contractive mapping satisfying the following condition:*

(i) *T is $\alpha-$admissible,*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 1.10. [10]*Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with*

$x_0 \leq F(x_o, y_0)$ and $y_0 \geq F(y_0, x_0)$.

Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$\varphi(d(F(x, y), F(u, v))) \leq \frac{1}{2}\varphi(d(x, u) + d(y, v)) - \psi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
    (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*
    (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n,*

then F has a coupled fixed point in X.

2. The main results

Theorem 2.1. *Let (X, d) be a complete metric space and $F : X \times X \to X$ be a mapping having the mixed monotone property on X. suppose that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ and a function $\alpha : X^2 \times X^2 \to [0, \infty)$ such that*

$$\alpha\big((x, y), (u, v)\big)\varphi\big(d\big(F(x, y), F(u, v)\big)\big) \leq \frac{1}{2}\varphi\big(d(x, u) + d(y, v)\big) - \psi\Big(\frac{d(x, u) + d(y, v)}{2}\Big). \tag{2.1}$$

if F is $\alpha - admissible$ and there exists $(x_0, y_0) \in X \times X$ such that

$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1$ and $\alpha((F(y_0, x_0), F(x_0, y_0)), (y_0, x_0)) \geq 1$,

and there exists $x_0, y_0 \in X$ such that

$x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$,

and for all $x, y, u, v \in X$ Suppose either

(a) *F is continuous or*
(b) *X has the following property:*
    (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*
    (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n*

then F has a coupled fixed point in X.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(x_0, y_o)$. we construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \; and \; y_{n+1} = F(y_n, x_n) \; for \; all \; n \geq 0 \tag{2.2}$$

We shall show that

$$x_n \leq x_{n+1} \; and \; y_n \geq y_{n+1} \; for \; all \; n \geq 0. \tag{2.3}$$

We shall use the mathematical induction.

Let $n = 0$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ and $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \leq x_1$ and $y_0 \geq y_1$ thus (2.28) hold for $n = 0$.

Now suppose that (2.28) hold for some fixed $n \geq 0$, then since $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, and by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_{n+1}) \geq F(x_n, y_n) = x_{n+1} \tag{2.4}$$

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}) \leq F(y_n, x_n) = y_{n+1} \tag{2.5}$$

Now from (2.29) and (2.30), we obtain
$x_{n+1} \leq x_{n+2}$ and $y_{n+1} \geq y_{n+2}$.
Thus by the mathematical induction we conclude that (2.28) hold for all $n \geq 0$. Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, so we have

$$\begin{aligned}
&\varphi(d(x_n, x_{n+1})) = \varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1})))\\
&\leq \alpha\Big((x_{n-1}, y_{n-1}), \big(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})\big)\Big)\varphi\Big(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))\Big)\\
&\qquad \leq \frac{1}{2}\varphi\Big(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) - \psi\Big(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2}\Big). \qquad (2.6)
\end{aligned}$$

$$\begin{aligned}
&\varphi(d(y_n, y_{n+1})) = \varphi\Big(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))\Big)\\
&\leq \alpha\Big((F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})), (y_{n-1}, x_{n-1})\Big)\varphi\Big(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))\Big)\\
&\qquad \leq \frac{1}{2}\varphi\Big(d(y_{n-1}, y_n) + d(x_{n-1}, x_n)\Big) - \psi\Big(\frac{d(y_{n-1}, y_n) + d(x_{n-1}, x_n)}{2}\Big). \qquad (2.7)
\end{aligned}$$

Adding (2.33) to (2.36), we get

$$\begin{aligned}
&\beta((\zeta_1, \zeta_2), (\eta_1, \eta_2))\Big(\varphi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) + \varphi(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)))\Big)\\
&\leq \varphi\Big(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\Big) - 2\psi\Big(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\Big). \qquad (2.8)
\end{aligned}$$

with

$$\beta\big((\zeta_1, \zeta_2), (\eta_1, \eta_2)\big) = \min\Big\{\alpha\Big((x_{n-1}, y_{n-1}), \big(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})\big)\Big), \tag{2.9}$$

$$\alpha\Big(\big(F(y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1})\big), (x_{n-1}, y_{n-1})\Big)\Big\}. \tag{2.10}$$

If we consider $Y = X \times X$, so we can define $\beta : Y \times Y \to [0, \infty)$, such that for all $\zeta = (\zeta_1, \zeta_2), \eta = (\eta_1, \eta_2) \in Y$

$$\beta((\zeta_1, \zeta_2), (\eta_1, \eta_2)) = \min\{\alpha((\zeta_1, \zeta_2), (\eta_1, \eta_2)), \alpha((\eta_2, \eta_1), (\zeta_2, \zeta_1)).$$

and $T : Y \to Y$ is given by (1.8). let $\zeta = (\zeta_1, \zeta_2), \eta = (\eta_1, \eta_2) \in Y$ such that $\beta(\zeta, \eta) \geq 1$, we obtain immediately that $\beta(T\zeta, T\eta) \geq 1$. also there exists $(x_0, y_0) \in Y$ such that:

$$\beta\big((x_0, y_0), (F(x_0, y_0), F(y_0, x_0))\big) \geq 1. \tag{2.11}$$

By property (iii) of φ, we have

$$\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big) \leq \varphi(d(x_{n+1}, x_n)) + \varphi(d(y_n, y_{n+1})). \tag{2.12}$$

From (2.37), (2.40) and (2.41), we have

$$\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big) \leq \beta((x_n, y_n), (x_{n-1}, y_{n-1}))\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big)$$

$$\leq \varphi\Big(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\Big) - 2\psi\Big(\frac{d(x_n, x_{n-1}) + d(y_n, y_{n-1})}{2}\Big). \tag{2.13}$$

Which implies

$\varphi\big(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\big) \leq \varphi\big(d(x_n, x_{n-1}) + d(y_n, y_{n-1})\big)$.

Using the fact that φ is non-decreasing, we get

$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq d(x_n, x_{n-1}) + d(y_n, y_{n-1})$.

Set $\delta_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ then sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = \delta. \tag{2.14}$$

We shall show that $\delta = 0$. Suppose to the contrary, that $\delta > 0$, Then taking the limit as $n \to \infty$ of both sides of (2.41) and have in mind that we suppose $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and φ is continuous, we have

$$\begin{aligned}\varphi(\delta) = \lim_{n\to\infty} \varphi(\delta_n) &= \lim_{n\to\infty} \Big[\varphi(\delta_{n-1}) - 2\psi\Big(\frac{\delta_{n-1}}{2}\Big)\Big] \\ &= \varphi(\delta) - 2 \lim_{\delta_{n-1}\to\delta} \psi\big(\frac{\delta_{n-1}}{2}\big) < \varphi(\delta).\end{aligned}$$

a contradiction. Thus $\delta = 0$, that is,

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}\Big[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\Big] = 0. \tag{2.15}$$

In what follows, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least of $\{x_n\}$ or $\{y_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}, \{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \geq \varepsilon. \tag{2.16}$$

Further,corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (2.45). Then

$$d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) < \varepsilon. \tag{2.17}$$

Using (2.45), (2.46) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq r_k := &\ d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.44)

$$lim_{k\to\infty}r_k = lim_{k\to\infty}\Big[d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\Big] = \varepsilon. \tag{2.18}$$

By the triangle inequality

$$\begin{aligned}r_k = d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) \\ &+ d(x_{m(k)+1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) \\ &+ d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) \\ &= \delta_{n(k)} + \delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}).\end{aligned}$$

Using the property of φ, we have

$$\varphi(r_k) = \varphi\big(\delta_{n(k)} + \delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})\big)$$
$$\leq \varphi(\delta_{n(k)} + \delta_{m(k)}) + \varphi(d(x_{n(k)+1}, x_{m(k)+1})) + \varphi(d(y_{n(k)+1}, y_{m(k)+1})). \quad (2.19)$$

Since $n(k) > m(k)$, hence $x_{n(k)} \geq x_{m(k)}$ and $y_{n(k)} \leq y_{m(k)}$. so we have

$$\varphi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) = \varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big) \quad (2.20)$$
$$\leq \alpha\big((x_{n(k)}, y_{n(k)}), (x_{m(k)}, y_{m(k)})\big)\varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big) \quad (2.21)$$
$$\leq \frac{1}{2}\varphi\big(d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})\big) - \psi\Big(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)})}{2}\Big) \quad (2.22)$$
$$= \tfrac{1}{2}\varphi(r_k) - \psi\Big(\tfrac{r_k}{2}\Big), \quad (2.23)$$

and

$$\varphi\big(d(y_{n(k)+1}, y_{m(k)+1})\big) = \varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big)$$
$$\leq \alpha\big((y_{n(k)}, x_{n(k)}), (y_{m(k)}, x_{m(k)})\big)\varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big)$$
$$\leq \frac{1}{2}\varphi\big(d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)})\big) - \psi\Big(\frac{d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)})}{2}\Big)$$
$$= \tfrac{1}{2}\varphi(r_k) - \psi\Big(\tfrac{r_k}{2}\Big). \quad (2.24)$$

From (2.48)-(2.52), we have

$$\varphi(r_k) \leq \varphi\big(\delta_{n(k)} + \delta_{m(k)}\big) + \varphi(r_k) - 2\psi\Big(\tfrac{r_k}{2}\Big).$$

a contradiction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete metric space, there exist $x, y \in X$ such that

$$lim_{n\to\infty} x_n = x \text{ and } lim_{n\to\infty} y_n = y. \quad (2.25)$$

Now, suppose F is continuous. Taking the limit as $n \to \infty$ in (2.27) and by (2.53), we get

$$x = lim_{n\to\infty} x_n = lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F\big(lim_{n\to\infty} x_{n-1}, lim_{n\to\infty} y_{n-1}\big) = F(x, y)$$

and

$$y = lim_{n\to\infty} y_n = lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F\big(lim_{n\to\infty} y_{n-1}, lim_{n\to\infty} x_{n-1}\big) = F(y, x).$$

Therefore F has coupled fixed point.
Finally, suppose that (b) holds. by assumption (b), we have $x_n \geq x$ and $y_n \leq y$ for all n. Since

$$d(x, F(x, y)) \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) = d(x, x_{n+1}) + d(F(x_n, y_n), F(x, y))$$

Therefore

$$\varphi\big(d(x, F(x, y))\big) \leq \varphi\big(d(x, x_{n+1})\big) + \varphi\big(d(F(x_n, y_n), F(x, y))\big)$$
$$\leq \varphi\big(d(x, x_{n+1})\big) + \alpha\big((x_n, y_n), (x, y)\big)\varphi\big(d(F(x_n, y_n), F(x, y))\big)$$
$$\leq \varphi\big(d(x, x_{n+1})\big) + \tfrac{1}{2}\varphi\big(d(x_n, x) + d(y_n, y)\big) - \psi\Big(\tfrac{d(x_n, x) + d(y_n, y)}{2}\Big).$$

Taking the limit of above inequality, using (2.53) and the property of ψ, we get $\varphi\big(d(x, F(x, y))\big) = 0$, thus $d(x, F(x, y)) = 0$. Hence $x = F(x, y)$. Similarly, one can show that $y = F(y, x)$.
Thus we proved that F has a coupled fixed point. □

Theorem 2.1. *In addition to hypothesis of Theorem 2.1, if x_0 and y_0 are comparable then F has a Fixed point.*

Proof. By using a similar proof in Theorem (2.6) of [10] we can deduce the proof. □

Corollary 2.2. *In Theorem 2.1, if 2.1 replaced with:*

$$\alpha\big((x, y), (u, v)\big)\varphi\big(d\big(F(x, y), F(u, v)\big)\big) \leq \frac{1}{4}\varphi\big(d(x, u) + d(y, v) + d(u, F(u, v)) + d(v, F(v, u))\big) - \psi\Big(\frac{d(x, u) + d(y, v) + d(u, F(u, v)) + d(v, F(v, u)}{4}\Big). \tag{2.26}$$

for all $(x, y), (u, v) \in X \times X$, then F has coupled fixed point.

Proof. Let $x_0, y_0 \in X$ be such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(x_0, y_o)$. we constract sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \ and \ y_{n+1} = F(y_n, x_n) \ for \ all \ n \geq 0 \tag{2.27}$$

We shall show that

$$x_n \leq x_{n+1} \ and \ y_n \geq y_{n+1} \ for \ all \ n \geq 0. \tag{2.28}$$

We shall use the mathematical induction. Let $n = 0$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$ and $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$, we have $x_0 \leq x_1$ and $y_0 \geq y_1$ thus (2.28) hold for $n = 0$.
Now suppose that (2.28) hold for some fixed $n \geq 0$, then since $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$, and by mixed monotone property of F, we have

$$x_{n+2} = F(x_{n+1}, y_{n+1}) \geq F(x_n, y_{n+1}) \geq F(x_n, y_n) = x_{n+1} \tag{2.29}$$

$$y_{n+2} = F(y_{n+1}, x_{n+1}) \leq F(y_n, x_{n+1}) \leq F(y_n, x_n) = y_{n+1} \tag{2.30}$$

Now from (2.29) and (2.30), we obtain
$x_{n+1} \leq x_{n+2}$ and $y_{n+1} \geq y_{n+2}$.
Thus by the mathematical induction we conclude that (2.28) hold for all $n \geq 0$.
Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, so we have

$$\varphi(d(x_n, x_{n+1})) = \varphi(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \tag{2.31}$$

$$\leq \alpha\Big((x_{n-1}, y_{n-1}), \big(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})\big)\Big)\varphi\Big(d(F(x_{n-1}, y_{n-1}), F(x_n, y_n))\Big) \tag{2.32}$$

$$\leq \frac{1}{4}\varphi\Big(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(x_{n-1}, F(x_{n-1}, y_{n-1})) + d(y_{n-1}, F(y_{n-1}, x_{n-1}))\Big) - \psi\Big(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(x_{n-1}, F(x_{n-1}, y_{n-1})) + d(y_{n-1}, F(y_{n-1}, x_{n-1}))}{4}\Big)$$

$$\leq \frac{1}{2}\varphi\Big(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\Big) - \psi\Big(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{2}\Big) \tag{2.33}$$

$$\varphi(d(y_n, y_{n+1})) = \varphi(d(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \tag{2.34}$$

$$\leq \alpha\Big(\big(F(y_{n-1},x_{n-1}),F(x_{n-1},y_{n-1})\big),(y_{n-1},x_{n-1})\Big)\varphi\Big(d(F(y_{n-1},x_{n-1}),F(y_n,x_n))\Big) \tag{2.35}$$

$$\begin{aligned}&\leq \frac{1}{4}\varphi\Big(d(y_{n-1},y_n)+d(x_{n-1},x_n)+d(y_{n-1},F(y_{n-1},x_{n-1}))+d(x_{n-1},F(x_{n-1},y_{n-1}))\Big)\\&-\psi\Big(\frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)+d(y_{n-1},F(y_{n-1},x_{n-1}))+d(x_{n-1},F(x_{n-1},y_{n-1}))}{4}\Big)\\&\leq \frac{1}{2}\varphi\Big(d(y_{n-1},y_n)+d(x_{n-1},x_n)\Big)-\psi\Big(\frac{d(y_{n-1},y_n)+d(x_{n-1},x_n)}{2}\Big).\end{aligned} \tag{2.36}$$

Adding (2.33) to (2.36), we get

$$\begin{aligned}&\beta((\zeta_1,\zeta_2),(\eta_1,\eta_2))\Big(\varphi\big(d(F(x_n,y_n),F(x_{n-1},y_{n-1}))\big)+\varphi\big(d(F(y_{n-1},x_{n-1}),F(y_n,x_n))\big)\Big)\\&\leq \varphi\Big(d(x_n,x_{n-1})+d(y_n,y_{n-1})\Big)-2\psi\Big(\frac{d(x_n,x_{n-1})+d(y_n,y_{n-1})}{2}\Big).\end{aligned} \tag{2.37}$$

With

$$\beta\big((\zeta_1,\zeta_2),(\eta_1,\eta_2)\big)=\min\Big\{\alpha\Big((x_{n-1},y_{n-1}),\big(F(x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1})\big)\Big), \tag{2.38}$$

$$\alpha\Big(\big(F(y_{n-1},x_{n-1}),F(x_{n-1},y_{n-1})\big),(x_{n-1},y_{n-1})\Big)\Big\}. \tag{2.39}$$

If we consider $Y=X\times X$, so we can define $\beta: Y\times Y\to[0,\infty)$, such that for all $\zeta=(\zeta_1,\zeta_2),\eta=(\eta_1,\eta_2)\in Y$

$$\beta((\zeta_1,\zeta_2),(\eta_1,\eta_2))=\min\{\alpha((\zeta_1,\zeta_2),(\eta_1,\eta_2)),\alpha((\eta_2,\eta_1),(\zeta_2,\zeta_1)).$$

and $T: Y\to Y$ is given by (1.8). let $\zeta=(\zeta_1,\zeta_2),\eta=(\eta_1,\eta_2)\in Y$ such that $\beta(\zeta,\eta)\geq 1$, we obtain immediately that $\beta(T\zeta,T\eta)\geq 1$. also there exists $(x_0,y_0)\in Y$ such that:

$$\beta\big((x_0,y_0),(F(x_0,y_0),F(y_0,x_0))\big)\geq 1. \tag{2.40}$$

By property (iii) of φ, we have

$$\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)\leq\varphi(d(x_{n+1},x_n))+\varphi(d(y_n,y_{n+1})). \tag{2.41}$$

From (2.37), (2.40) and (2.41), we have

$$\begin{aligned}\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)&\leq\beta((x_n,y_n),(x_{n-1},y_{n-1}))\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)\\&\leq\varphi\Big(d(x_n,x_{n-1})+d(y_n,y_{n-1})\Big)-2\psi\Big(\frac{d(x_n,x_{n-1})+d(y_n,y_{n-1})}{2}\Big).\end{aligned} \tag{2.42}$$

Which implies

$$\varphi\big(d(x_{n+1},x_n)+d(y_{n+1},y_n)\big)\leq\varphi\big(d(x_n,x_{n-1})+d(y_n,y_{n-1})\big).$$

Using the fact that φ is non-decreasing, we get

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)\leq d(x_n,x_{n-1})+d(y_n,y_{n-1}).$$

Set $\delta_n = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ then sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)] = \delta. \tag{2.43}$$

We shall show that $\delta = 0$. Suppose to the contrary, that $\delta > 0$, Then taking the limit as $n \to \infty$ of both sides of (2.41) and have in mind that we suppose $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and φ is continuous, we have

$$\begin{aligned}\varphi(\delta) &= \lim_{n\to\infty} \varphi(\delta_n) = \lim_{n\to\infty} \Big[\varphi(\delta_{n-1}) - 2\psi\Big(\frac{\delta_{n-1}}{2}\Big)\Big] \\ &= \varphi(\delta) - 2 \lim_{\delta_{n-1}\to\delta} \psi\big(\frac{\delta_{n-1}}{2}\big) < \varphi(\delta).\end{aligned}$$

a contradiction. Thus $\delta = 0$, that is,

$$lim_{n\to\infty}\delta_n = lim_{n\to\infty}\Big[d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\Big] = 0. \tag{2.44}$$

In what follows, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least of $\{x_n\}$ or $\{y_n\}$ is not Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}, \{x_{m(k)}\}$ of $\{x_n\}$ and $\{y_{n(k)}\}, \{y_{m(k)}\}$ of $\{y_n\}$ with $n(k) > m(k) \geq k$ such that

$$d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \geq \varepsilon. \tag{2.45}$$

Further,corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (2.45). Then

$$d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) < \varepsilon. \tag{2.46}$$

Using (2.45), (2.46) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \leq r_k &:= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &+ d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) \\ &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\ &+ d(x_{m(k)}, x_{m(k+1)}) + d(y_{m(k)}, y_{m(k+1)}) \\ &\leq \delta_{n(k-1)} + \varepsilon + \delta_{m(k)} \leq 2\delta_{n(k-1)} + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.44)

$$\begin{aligned}lim_{k\to\infty}r_k = lim_{k\to\infty}\Big[&d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &+d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)}))\Big] = \varepsilon.\end{aligned} \tag{2.47}$$

By the triangle inequality

$$\begin{aligned}r_k &= d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \\ &+ d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) \\ &\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) \\ &+ d(y_{n(k)+1}, y_{m(k)+1}) + d(y_{m(k)+1}, y_{m(k)}) + d(x_{m(k)}, x_{m(k+1)}) + d(y_{m(k)}, y_{m(k+1)}) \\ &= \delta_{n(k)} + 2\delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}).\end{aligned}$$

Using the property of φ, we have

$$\begin{aligned}\varphi(r_k) &= \varphi\big(\delta_{n(k)} + 2\delta_{m(k)} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1})\big) \\ &\leq \varphi(\delta_{n(k)} + 2\delta_{m(k)}) + \varphi(d(x_{n(k)+1}, x_{m(k)+1})) + \varphi(d(y_{n(k)+1}, y_{m(k)+1})).\end{aligned} \tag{2.48}$$

Since $n(k) > m(k)$, hence $x_{n(k)} \geq x_{m(k)}$ and $y_{n(k)} \leq y_{m(k)}$. so we have

$$\varphi\big(d(x_{n(k)+1}, x_{m(k)+1})\big) = \varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big) \quad (2.49)$$

$$\leq \alpha\big((x_{n(k)}, y_{n(k)}), (x_{m(k)}, y_{m(k)})\big)\varphi\big(d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}))\big)$$
$$\leq \frac{1}{4}\varphi\big(d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)}))\big)$$
$$- \psi\Big(\frac{d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)}))}{4}\Big)$$
$$= \frac{1}{4}\varphi(r_k) - \psi\Big(\frac{r_k}{4}\Big) \quad (2.50)$$

$$\varphi\big(d(y_{n(k)+1}, y_{m(k)+1})\big) = \varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big) \quad (2.51)$$

$$\leq \alpha\big((y_{n(k)}, x_{n(k)}), (y_{m(k)}, x_{m(k)})\big)\varphi\big(d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))\big)$$
$$\leq \frac{1}{4}\varphi\big(d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)}))\big)$$
$$-\psi\Big(\frac{d(y_{n(k)}, y_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})) + d(x_{m(k)}, F(x_{m(k)}, y_{m(k)}))}{4}\Big)$$
$$= \frac{1}{4}\varphi(r_k) - \psi\Big(\frac{r_k}{4}\Big). \quad (2.52)$$

From (2.48)-(2.52), we have

$$\varphi(r_k) \leq \varphi\big(\delta_{n(k)} + 2\delta_{m(k)}\big) + \tfrac{1}{2}\varphi(r_k) - 2\psi\Big(\tfrac{r_k}{4}\Big).$$

Letting $k \to \infty$ and using (2.44) and (2.47), we have

$$\varphi(\varepsilon) \leq \varphi(0) + \tfrac{1}{2}\varphi(\varepsilon) - 2lim_{k\to\infty}\psi\Big(\tfrac{r_k}{4}\Big) = \tfrac{1}{2}\varphi(\varepsilon) - 2lim_{r_k\to\varepsilon}\psi\Big(\tfrac{r_k}{4}\Big) < \varphi(\varepsilon),$$

a contradiction. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete metric space, there exist $x, y \in X$ such that

$$lim_{n\to\infty}x_n = x \ and \ lim_{n\to\infty}y_n = y. \quad (2.53)$$

Now, suppose F is continuous. Taking the limit as $n \to \infty$ in (2.27) and by (2.53), we get

$x = lim_{n\to\infty}x_n = lim_{n\to\infty}F(x_{n-1}, y_{n-1}) = F\big(lim_{n\to\infty}x_{n-1}, lim_{n\to\infty}y_{n-1}\big) = F(x, y)$

and

$y = lim_{n\to\infty}y_n = lim_{n\to\infty}F(y_{n-1}, x_{n-1}) = F\big(lim_{n\to\infty}y_{n-1}, lim_{n\to\infty}x_{n-1}\big) = F(y, x).$

Therefore F has coupled fixed point.
Finally, suppose that (b) holds. by assumption (b), we have $x_n \geq x$ and $y_n \leq y$ for all n. Since

$$d(x, F(x, y)) \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y)) = d(x, x_{n+1}) + d(F(x_n, y_n), F(x, y)).$$

Therefore

$$\begin{aligned}\varphi\big(d(x,F(x,y))\big) &\leq \varphi\big(d(x,x_{n+1})\big)+\varphi\big(d(F(x_n,y_n),F(x,y))\big)\\ &\leq \varphi\big(d(x,x_{n+1})\big)+\alpha\big((x_n,y_n),(x,y)\big)\varphi\big(d(F(x_n,y_n),F(x,y))\big)\\ &\leq \varphi\big(d(x,x_{n+1})\big)+\frac{1}{4}\varphi\big(d(x_n,x)+d(y_n,y)+d(x_n,F(x_n,y_n))+d(y_n,F(y_n,x_n))\big)\\ &-\psi\Big(\frac{d(x_n,x)+d(y_n,y)+d(y_n,F(y_n,x_n))+d(x_n,F(x_n,y_n))}{4}\Big)\\ &=\varphi\big(d(x,x_{n+1})\big)+\frac{1}{4}\varphi\big(d(x_n,x)+d(y_n,y)+d(x_n,x_{n+1})+d(y_n,y_{n+1})\big)\\ &-\psi\Big(\frac{d(x_n,x)+d(y_n,y)+d(y_n,y_{n+1})+d(x_n,x_{n+1})}{4}\Big)\\ &=\varphi\big(d(x,x_{n+1})\big)+\frac{1}{4}\varphi\big(d(x_n,x)+d(y_n,y)+\delta_n\big)-\psi\Big(\frac{d(x_n,x)+d(y_n,y)+\delta_n}{4}\Big).\end{aligned} \tag{2.54}$$

Taking the limit of above inequality, using (2.53) and the property of ψ, we get $\varphi\big(d(x,F(x,y))\big)=0$, thus $d(x,F(x,y))=0$. Hence $x=F(x,y)$. Similarly, one can show that $y=F(y,x)$.
Thus we proved that F has a coupled fixed point. □

Theorem 2.2. *Let (X,d) be a complete metric space $\alpha:X\times X\to[0,\infty)$ a function, $\psi\in\Psi$, and $T:X\to X$ be a continuous, non-decreasing triangular $\alpha-$admissible mapping such that*

$$\alpha(x,y)\psi\big(d(Tx,Ty)\big)\leq\psi\Big(\frac{1}{2}\big(d(x,Ty)+d(y,Tx)\big)\Big)-\varphi\Big(d(x,Ty),d(y,Tx)\Big). \tag{2.55}$$

For all $x,y\in X$,
where $\varphi:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(x,y)=0$ if and only if $x=y$. and there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$. Then T has a fixed point.

Proof. Take $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq1$ and define sequence $\{x_n\}$ in X with $x_{n+1}=Tx_n$.
$\alpha(x_0,Tx_0)=\alpha(x_0,x_1)\geq1\Rightarrow\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\geq1$.
By continuity of this process we have $\alpha(x_n,x_{n+1})\geq1$, Thus

$$\begin{aligned}\psi\big(d(x_n,x_{n+1})\big)&=\psi\big(d(Tx_{n-1},Tx_n)\big)\leq\alpha(x_{n-1},x_n)\psi\big(d(Tx_{n-1},Tx_n)\big)\\ &\leq\psi\Big(\frac{1}{2}\big(d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})\big)\Big)-\varphi\Big(d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\Big)\\ &\leq\psi\Big(\frac{1}{2}\big(d(x_{n-1},x_{n+1})+d(x_n,x_n)\big)\Big)-\varphi\Big(d(x_{n-1},x_{n+1}),d(x_n,x_n)\Big)\\ &\leq\psi\Big(\frac{1}{2}\big(d(x_{n-1},x_{n+1})\big)\Big)-\varphi\Big(d(x_{n-1},x_{n+1}),0\Big)\leq\psi\Big(\frac{1}{2}d(x_{n-1},x_{n+1})\Big).\end{aligned}$$

Since ψ is non-decreasing function, we get

$$\psi\big(d(x_n,x_{n+1})\big)\leq\psi\big(\tfrac{1}{2}d(x_{n_1},x_{n+1})\big)\Rightarrow d(x_n,x_{n+1}\leq\tfrac{1}{2}d(x_{n-1},x_{n+1}).$$

Hence $d(x_n,x_{n+1})_{n\geqslant1}$ is non-decreasing sequence and there is $r\geqslant0$ such that

$$r=\lim_{n\to\infty}d(x_n,x_{n+1}). \tag{2.56}$$

Also we have

$$d(x_n, x_{n+1}) \leq \tfrac{1}{2}d(x_{n-1}, x_{n+1}) \leq \tfrac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$$

$lim_{n\to\infty}d(x_{n-1}, x_{n+1}) = 2r$ Letting $n \to \infty$ and using (2.56), we get

$$r \leq lim_{n\to\infty}\tfrac{1}{2}d(x_{n-1}, x_{n+1}) \leq \tfrac{1}{2}(r + r).$$

Hence by using of continuity ψ, φ, we have

$$\psi(r) \leq \psi\big(\tfrac{1}{2}(2r)\big) - \varphi(2r, 0).$$

Which implies that $\varphi(2r, 0) = 0 \Rightarrow r = 0$ and $lim_{n\to\infty}d(x_n, x_{n+1}) = r = 0$. Now we show that $\{x_n\}$ is a cauchy sequence in x. Suppose to the contrary that $\{x_n\}$ is not cauchy sequence, then there exists $\varepsilon > 0$ for which we can find two subsequence $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which $n_i > m_i > 0$, $d(x_{n_i}, x_{m_i}) \geq \varepsilon$, Thus $d(x_{m_i}, x_{n_{i-1}}) < \varepsilon$, and we have

$$\begin{aligned}\varepsilon \leq d(x_{m_i}, x_{n_i}) &\leq d(x_{m_i}, x_{m_{i+1}}) + d(x_{m_{i+1}}, x_{n_{i-1}}) + d(x_{n_{i-1}}, x_{n_i})\\ &\leq 2d(x_{m_i}, x_{m_{i+1}}) + d(x_{m_i}, x_{n_i}) + 2d(x_{m_{i-1}}, x_{n_i})\\ &\leq 2d(x_{m_i}, x_{m_{i+1}}) + \varepsilon + 3d(x_{m_{i-1}}, x_{n_i}).\end{aligned}$$

Letting $i \to \infty$ we get

$$lim_{i\to\infty}d(x_{m_i}, x_{n_i}) = lim_{i\to\infty}d(x_{m_{i+1}}, x_{n_{i+1}}) = lim_{i\to\infty}d(x_{m_{i+1}}, x_{n_i}) = \varepsilon. \quad (2.57)$$

Also we have

$$\begin{aligned}\psi\big(d(x_{m_{i+1}}, x_{n_i})\big) &= \psi\big(d(Tx_{m_i}, Tx_{n_{i-1}})\big) \leq \alpha(x_{m_i}, x_{n_i})\psi\big(d(Tx_{m_i}, Tx_{n_{i-1}})\big)\\ &\leq \psi\Big(\frac{1}{2}\big(d(x_{m_i}, Tx_{n_{i-1}}) + d(x_{n_{i-1}}, Tx_{m_i})\big)\Big) - \varphi\Big(d(x_{m_i}, Tx_{n_{i-1}}), d(x_{n_{i-1}}, Tx_{m_i})\Big)\\ &= \psi\Big(\frac{1}{2}\big(d(x_{m_i}, x_{n_i}) + d(x_{n_{i-1}}, x_{m_{i+1}})\big)\Big) - \varphi\Big(d(x_{m_i}, x_{n_i}), d(x_{n_{i-1}}, x_{m_{i+1}})\Big).\end{aligned}$$

By letting $i \to \infty$ and use of continuity of φ and ψ, we get

$$\psi(\varepsilon) \leqslant \psi(\varepsilon) - \varphi(\varepsilon, \varepsilon).$$

Hence we get $\varphi(\varepsilon, \varepsilon) = 0$, and $\varepsilon = 0$, a contradiction, thus $\{x_n\}$ is cauchy sequence in X and there exists $x \in X$ such that $lim_{n\to\infty}x_n = x$.
Since T is continuous and $x_n \to x$, we obtain $x_{n+1} = Tx_n \to Tx$ and $Tx = x$. Thus T has fixed point. □

Theorem 2.3. *let $F : X \times X \times X \to X$ be a given mapping in complete metric space (X, d) and suppose that there exist $\psi \in \Psi$ and a function $\alpha : X^3 \times X^3 \to [0, \infty)$ such that*

$$\alpha\big((x, y, z), (u, v, w)\big)d\big(F(x, y, z), F(u, v, w)\big) \leqslant \frac{1}{3}\psi\big(d(x, u) + d(y, v) + d(z, w)\big) \quad (2.58)$$

for all $(x, y, z), (u, v, z) \in X \times X \times X$. suppose also that

(i) *For all $(x, y, z), (u, v, w) \in X \times X \times X$, we have $\alpha\big((x, y, z), (u, v, w)\big) \geqslant 1 \Rightarrow$*

$$\alpha\Big(\big(F(x, y, z), F(y, z, x), F(z, x, y)\big), \big(F(u, v, w), F(v, w, u), F(w, u, v)\big)\Big) \geqslant 1,$$

(ii) *there exists $(x_0, y_0, z_0) \in X \times X \times X$ such that*

$$\alpha\Big((x_0, y_0, z_0), \big(F(x_0, y_0, z_0), F(y_0, z_0, x_0), F(z_0, x_0, y_0)\big)\Big) \geqslant 1$$

$$\alpha\Big(\big(F(y_0, z_0, x_0), F(z_0, x_0, y_0), F(x_0, y_0, z_0)\big), (y_0, z_0, x_0)\Big) \geqslant 1$$

$$\alpha\Big((z_0, x_0, y_0), \big(F(z_0, x_0, y_0,), F(x_0, y_0, z_0), F(y_0, z_0, x_0)\big)\Big) \geqslant 1$$

(iii) *and F is continuous.*
Then, F has a tripled fixed point, that is, there exists $(x^*, y^*, z^*) \in X \times X \times X$ such that $x^* = F(x^*, y^*, z^*)$ and $y^* = F(y^*, z^*, x^*)$ and $z^* = F(z^*, x^*, y^*)$.

Proof. The idea consists in transporting the problem to the complete metric space (Y, δ) where $Y = X \times X \times X$ and $\delta\big((x, y, z), (u, v, w)\big) = d(x, u) + d(y, v) + d(z, w)$ for all $(x, y, z), (u, v, w) \in X \times X \times X$. also we have

$$\alpha\big((x, y, z), (u, v, w)\big)d\big(F(x, y, z), F(u, v, w)\big) \leqslant \frac{1}{3}\psi\Big(\delta\big((x, y, z), (u, v, w)\big)\Big) \quad (2.59)$$

and

$$\alpha\big((v, w, u), (y, z, x)\big)d\big(F(v, w, u), F(y, z, x)\big) \leqslant \frac{1}{3}\psi\Big(\delta\big((v, w, u), (y, z, x)\big)\Big) \quad (2.60)$$

and

$$\alpha\big((z, x, y), (w, u, v)\big)d\big(F(z, x, y), F(w, u, v)\big) \leqslant \frac{1}{3}\psi\Big(\delta\big((z, x, y), (w, u, v)\big)\Big). \quad (2.61)$$

Now if $T : Y \to Y$ is defined by

$$T(\tau_1, \tau_2, \tau_3) = \big(F(\tau_1, \tau_2, \tau_3), F(\tau_2, \tau_3, \tau_1), F(\tau_3, \tau_1, \tau_2)\big). \quad (2.62)$$

for all $(\tau_1, \tau_2, \tau_3) \in Y$, and $\beta : Y \times Y \to [0, \infty)$ is the function defined by

$$\beta\big((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3)\big) = \\ \min\big\{\alpha\big((x, y, z), (u, v, w)\big), \alpha\big((v, w, u), (y, z, x)\big), \alpha\big((z, x, y), (w, u, v)\big)\big\},$$

Then by summing up the inequalities (2.59)-(2.61), and using of (2.62) we get

$$\beta(\xi, \eta)\delta(T(x, y, z), T(u, v, w)) \leqslant \psi\big(\delta((x, y, z), (u, v, w))\big). \quad (2.63)$$

for all $\xi = (\xi_1, \xi_2, \xi_3), \eta = (\eta_1, \eta_2, \eta_3) \in Y$. Then T is continuous and $\beta-\psi-$contractive mapping and $\beta(\xi, \eta) \geqslant 1$. It is easy to check that T is $\beta-$admissible and we know that there exists $(x_0, y_0, z_0) \in Y$ such that

$$\beta\big((x_0, y_0, z_0), T(x_0, y_0, z_0)\big) \geqslant 1.$$

All the hypotheses of 1.9 are satisfied, and so we deduce the existence of a fixed point of T. □

Example 2.3. Let $X = [0, +\infty)$ equipped with the standard metric $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X,d) is complete metric space. Define the mapping $F : X \times X \times X \to X$ by

$$F(x, y, z) = \begin{cases} \frac{x-y-z}{6} & x \geq y \geq z \\ 0 & otherwise \end{cases}$$

Clearly F is continuous mapping. Define $\alpha : X^3 \times X^3 \to [0, +\infty)$ by

$$\alpha\big((x, y, z), (u, v, w)\big) = \begin{cases} 1 & u \geq v \geq w \\ 0 & otherwise \end{cases}$$

Then, (2.58) is satisfied with $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Also it is easy to check that $\alpha\big((x, y, z), (u, v, w)\big) \geq 1$ implies

$$\alpha\Big(\big(F(x, y, z), F(y, z, x), F(z, x, y)\big), \big(F(u, v, w), F(v, w, u), F(w, u, v)\big)\Big) \geq 1,$$

for all $(x, y, z), (u, v, w)$
$\in X \times X \times X$. On the other hand, the condition (ii) of Theorem (2.3) is satisfied with $(x_0, y_0, z_0) = (0, 0, 0)$. All the required hypotheses of same Theorem are true

and so we deduce the existence of a tripled fixed point of F. Hence $(0,0,0)$ is a tripled fixed point of F.

Corollary 2.4. *Let (X,d) be a complete metric space and $F: X\times X\times X\to X$ be a mapping having the mixed monotone property on X. Suppose that there exist $\psi\in\Psi$ and $\varphi\in\phi$ and a function $\alpha: X^3\times X^3\to[0,\infty)$ such that*

$$\alpha\big((x,y,z),(u,v,w)\big)\varphi\Big(d\big(F(x,y,z),F(u,v,w)\big)\Big)$$

$$\leqslant \frac{1}{3}\varphi\Big(d(x,u)+d(y,v)+d(z,w)\Big)-\psi\Big(\frac{d(x,u)+d(y,v)+d(z,w)}{3}\Big). \tag{2.64}$$

If F is $\alpha-$admissible mapping satisfying the following conditions

(i) *For all $(x,y,z),(u,v,w)\in X\times X\times X$, we have*
$\alpha\big((x,y,z),(u,v,w)\big)\geqslant 1\Rightarrow$

$$\alpha\Big(\big(F(x,y,z),F(y,z,x),F(z,x,y)\big),\big(F(u,v,w),F(v,w,u),F(w,u,v)\big)\Big)\geqslant 1.$$

(ii) *there exists $(x_0,y_0,z_0)\in X\times X\times X$ such that*

$$\alpha\Big((x_0,y_0,z_0),\big(F(x_0,y_0,z_0),F(y_0,z_0,x_0),F(z_0,x_0,y_0)\big)\Big)\geqslant 1,$$
$$\alpha\Big(\big(F(y_0,z_0,x_0),F(z_0,x_0,y_0),F(x_0,y_0,z_0)\big),(y_0,z_0,x_0)\Big)\geqslant 1,$$
$$\alpha\Big((z_0,x_0,y_0),\big(F(z_0,x_0,y_0,),F(x_0,y_0,z_0),F(y_0,z_0,x_0)\big)\Big)\geqslant 1.$$

(iii) *There exists $x_0,y_0,z_0\in X$ such that*
$x_0\leqslant F(x_0,y_0,z_0)$ $y_0\geqslant F(y_0,z_0,x_0)$ *and* $z_0\leqslant F(z,x_0,y_0)$,
(iv) *for $x,y,z,u,v,w\in X$ with $x\geqslant u, y\leqslant v, z\geqslant w$,*
(v) *F is continuous or*
(a) *If a non-decreasing sequence $\{x_n\}\to x$, then $x_n\leqslant x$ for all n,*
(b) *If a non-increasing sequence $\{y_n\}\to y$, then $y_n\geqslant y$ for all n,*
(c) *If a non-decreasing sequence $\{z_n\}\to z$, then $z_n\leqslant z$ for all n. Then F has tripled fixed point in X.*

3. Application

In this section, we study the existence of a solution to a nonlinear integral equation, as an application to the fixed point theorem.
let Θ denote the class of those functions $\theta:[0,\infty)\to[0,\infty)$ which satisfies the following conditions:
(i) θ is increasing.
(ii) There exists $\psi\in\Psi$ such that $\theta(x)=\frac{x}{2}-\psi(\frac{x}{2})$ for all $x\in[0,\infty)$.
For example, $\theta(x)=kx$, where $0\le k\le\frac{1}{2}, \theta(x)=\frac{x^2}{2(x+1)}, \theta(x)=\frac{x}{2}-\frac{ln(x+1)}{2}$ are in Θ.
Consider the following integral equation

$$x(t)=\int_a^b (K_1(t,s)+K_2(t,s))(f(s,x(s))+g(s,x(s))ds+h(t) \tag{3.1}$$
$$t\in I=[a,b].$$

We assume that K_1,K_2,f,g,e satisfy the following conditions (i) $K_1(t,s)\ge 0$ and $K_2(t,s)\le 0$ for all $t,s\in[a,b]$.
(ii) There exist $\lambda,\mu>0$ and $\theta\in\Theta$ such that for all $x,y\in\mathbb{R}$, $x\ge y$.
$0\le f(t,x)-f(t,y)\le\lambda\theta(x-y)$

and
$-\mu\theta(x-y) \leq g(t,x) - g(t,y) \leq 0$
and Also
$\max\{\lambda,\mu\} \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))ds \leq \frac{1}{2}$.
(iii) Also there exists $(x_0, y_0) \in C(I) \times C(I)$such that for all $t \in I$, we have
$e((x_0(t), y_0(t)), (\int_a^b K_1(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds$
$+ \int_a^b K_2(t,s)(f(s,y_0(s)) + g(s,x_0(s)))ds + h(t),$
$(\int_a^b K_1(t,s)(f(s,y_0(s)) + g(s,x_0(s)))ds + \int_a^b K_2(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds + h(t)))) \geq 0.$
and
$((\int_a^b (_1(t,s)(f(s,y_0(s))+g(s,x_0(s)))ds+\int_a^b K_2(t,s)(f(s,x_0(s))+g(s,y_0(s)))ds+h(t),$
$(\int_a^b K_1(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds + \int_a^b K_2(t,s)(f(s,y_0(s))$
$+g(s,x_0(s)))ds + h(t), (y_0(t), x_0(t))))) \geq 0.$
(v) For all $t \in I, x, y \in C(I)$, $e((x(t), y(t)), (u(t), v(t))) \geq 0$ implies that
$e(\int_a^b K_1(t,s)(f(s,x(s)) + g(s,y(s)))ds + \int_a^b K_2(t,s)(f(s,y(s)) + g(s,x(s)))ds + h(t),$
$\int_a^b K_1(t,s)(f(s,y(s)) + g(s,x(s)))ds + \int_a^b K_2(t,s)(f(s,x(s)) + g(s,y(s)))ds + h(t)),$
$(\int_a^b K_1(t,s)(f(s,u(s)) + g(s,v(s)))ds + \int_a^b K_2(t,s)(f(s,v(s)) + g(s,u(s)))ds + h(t),$
$\int_a^b K_1(t,s)(f(s,v(s)) + g(s,u(s)))ds + \int_a^b K_2(t,s)(f(s,u(s)) + g(s,v(s)))ds + h(t)) \geq 0.$

Definition 3.1. An element $(\beta,\gamma) \in C(I,\mathbb{R}) \times C(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equation (3.1) if $\beta(t) \leq \gamma(t)$ and

$\beta(t) \leq \int_a^b K_1(t,s)(f(s,\beta(s)) + g(s,\gamma(s)))ds + \int_a^b K_2(t,s)(f(s,\gamma(s)) + g(s,\beta(s)))ds + h(t),$
and

$\gamma(t) \geq \int_a^b K_1(t,s)(f(s,\gamma(s)) + g(s,\beta(s)))ds + \int_a^b K_2(t,s)(f(s,\beta(s)) + g(s,\gamma(s)))ds + h(t),$
for all $t \in [a,b]$.

Theorem 3.2. *Consider the integral equation (3.1) with $K_1, K_2 \in C(I \times I, \mathbb{R})$, $f, g \in C(I \times \mathbb{R}, \mathbb{R})$ and $h \in C(I \times \mathbb{R}, \mathbb{R})$ and suppose that conditions of (i), (ii), (iii) and (iv) are satisfied. Then the existence of a coupled lower and upper solution for (3.1) provides the existence of a solution of (3.1) in $C(I,\mathbb{R})$.*

Proof. Let $X = C(I,\mathbb{R})$. X is a partially ordered set if we define the following order relation in X
$x, y \in C(I,\mathbb{R}) \Leftrightarrow x(t) \leq y(t)$, for all $t \in [a,b]$.
And (X,d) is a complete metric space with metric
$d(x,y) = \sup_{t\in I} |x(t) - y(t)|,\ x, y \in C(I,\mathbb{R})$
Suppose $\{u_n\}$ is a monotone non-decreasing sequence in X that converges to $u \in X$. Then for every $t \in I$, the sequence of real numbers
$u_1(t) \leq u_2(t) \leq \ldots \leq u_n(t) \leq \ldots$
converges to $u(t)$. Therefore, for all $t \in I, n \in \mathbb{N}, u_n(t) \leq u(t)$. Hence $u_n \leq u$, for all n.
Similarly, we can verify that $\lim v(t)$ of a monotone non-increasing sequence $v_n(t)$ in X is a lower bound for all the elements in the sequence. That is, $v \leq v_n$ for all n. Therefore, condition (b) of Theorem (2.1) holds. Also, $X \times X = C(I,\mathbb{R}) \times C(I,\mathbb{R})$

is a partially ordered set if we define the following order relation in $X \times X$
$(x, y), (u, v) \in X \times X, (x, y) \leq (u, v) \Leftrightarrow x(t) \leq u(t)$ and $y(t) \geq v(t),\ \forall t \in I$.
For any $x, y \in X, \max\{x(t), y(t)\}$ and $\min\{x(t), y(t)\}$, for each $t \in I$, are in X and are the upper and lower bounds of x, y, respectively. Therefore, for every $(x, y), (u, v) \in X \times X$, there exists a $(\max\{x, u\}, \min\{y, v\}) \in X \times X$ that is comparable to (x, y) and(u, v). Define $F : X \times X \to X$ by

$$F(x,y)(t) = \int_a^b K_1(t,s)(f(s,x(s))+g(s,y(s)))ds+\int_a^b K_2(t,s)(f(s,y(s))+g(s,x(s)))ds+h(t)$$

for all $t \in [a, b]$.
Now we shall show that F has the mixed monotone property. Indeed, for $x_1 \leq x_2$, that is, $x_1(t) \leq x_2(t)$, for all $t \in [a, b]$, we have
$F(x_1, y)(t) - F(x_2, y)(t) = \int_a^b K_1(t, s)(f(s, x_1(s)) + g(s, y(s)))ds$
$+ \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_1(s)))ds + h(t)$
$- \int_a^b K_1(t, s)(f(s, x_2(s)) + g(s, y(s)))ds$
$- \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x_2(s)))ds - h(t)$
$= \int_a^b K_1(t, s)(f(s, x_1(s)) + f(s, x_2(s)))ds$
$+ \int_a^b K_2(t, s)(g(s, x_1(s)) + g(s, x_2(s)))ds,$
by Assumption (i) and (ii). Hence $F(x_1, y)(t) \leq F(x_2, y)(t), \forall t \in I$, that is,

$$F(x_1, y) \leq F(x_2, y).$$

Similarly, if $y_1 \geq y_2$, that is, $y_1(t) \geq y_2(t)$, for all $t \in [a, b]$, we have
$F(x, y_1)(t) - F(x, y_2)(t) = \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y_1(s)))ds$
$+ \int_a^b K_2(t, s)(f(s, y_1(s)) + g(s, x(s)))ds + h(t)$
$- \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y_2(s)))ds$
$- \int_a^b K_2(t, s)(f(s, y_2(s)) + g(s, x(s)))ds - h(t)$
$= \int_a^b K_1(t, s)(g(s, y_1(s)) - g(s, y_2(s)))ds$
$+ \int_a^b K_2(t, s)(f(s, y_1(s)) - f(s, y_2(s)))ds \leq 0,$
by Assumption (i) and (ii). Hence $F(x, y_1)(t) \leq F(x, y_2)(t), \forall t \in I$, that is,

$$F(x, y_1) \leq F(x, y_2).$$

Thus, $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y.
Define the function $\alpha : C(I)^2 \times C(I)^2 \to [0, \infty)$ by

$$\alpha((x,y),(u,v)) = \begin{cases} 1 & e((x(t),y(t)),(u(t),v(t))) \geq 0,\ t \in I \\ 0 & otherwise. \end{cases}$$

Now, for $x \geq u$, $y \leq v$, that is, $x(t) \geq u(t)$, $y(t) \leq v(t)$ for all $t \in I$, we have
$\alpha((x, y), (u, v))d(F(x, y), F(u, v)) \leq d(F(x, y), F(u, v))$
$= \sup_{t\in I} |F(x, y)(t) - F(u, v)(t)|$
$= \sup_{t\in I} | \int_a^b K_1(t, s)(f(s, x(s)) + g(s, y(s)))ds$
$+ \int_a^b K_2(t, s)(f(s, y(s)) + g(s, x(s)))ds + h(t)$
$-(\int_a^b K_1(t, s)(f(s, u(s)) + g(s, v(s)))ds$
$+ \int_a^b K_2(t, s)(f(s, v(s)) + g(s, u(s)))ds + h(t))|$
$= \sup_{t\in I} | \int_a^b K_1(t, s)[(f(s, x(s)) - f(s, u(s))) + (g(s, y(s)) - g(s, v(s)))]ds$
$+ \int_a^b K_2(t, s)[(f(s, y(s)) - f(s, v(s))) + (g(s, x(s)) - g(s, u(s)))]ds|$
$= \sup_{t\in I} | \int_a^b K_1(t, s)[(f(s, x(s)) - f(s, u(s))) - (g(s, v(s)) - g(s, y(s)))]ds$

$- \int_a^b K_2(t,s)[(f(s,v(s)) - f(s,y(s))) - (g(s,x(s)) - g(s,u(s)))]ds|$
$\leq \sup_{t\in I} | \int_a^b K_1(t,s)[\lambda\theta(x(s) - u(s)) + \mu\theta(v(s) - y(s))]ds$
$- \int_a^b K_2(t,s)[\lambda\theta(v(s) - y(s)) + \mu\theta(x(s) - u(s))]ds|$
$\leq \max\{\lambda,\mu\} \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))[\theta(x(s) - u(s)) + \theta(v(s) - y(s))]ds.$
As the function θ is increasing and $x \geq u, y \leq v$, then
$\theta(x(s) - u(s)) \leq \theta(d(x,u)), \theta(v(s) - y(s)) \leq \theta(d(v,y)),$
for all $s \in [a,b]$, we obtain
$d(F(x,y), F(u,v)) \leq \max\{\lambda,\mu\} \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))$
$\times[\theta(x(s) - u(s)) + \theta(v(s) - y(s))]ds.$
$\leq \max\{\lambda,\mu\}.[\theta(d(x,u)) + \theta(d(v,y))]. \sup_{t\in I} \int_a^b (K_1(t,s) - K_2(t,s))ds$
$\leq \frac{1}{2}[\theta(d(x,u)) + \theta(d(v,y))]$
$\leq \theta(d(x,u)) + \theta(d(v,y))$
$= \frac{d(x,u)+d(v,y)}{2} - \psi(\frac{d(x,u)+d(v,y)}{2}).$
Therefore, for $x \geq u, y \leq v$, we have
$d(F(x,y), F(u,v)) \leq \frac{d(x,u)+d(v,y)}{2} - \psi(\frac{d(x,u)+d(v,y)}{2}).$

Now, let (β,γ) be a coupled lower and upper solution of the integral equation (3.1) then we have $\beta(t) \leq F(\beta,\gamma)(t)$ and $\gamma(t) \geq F(\gamma,\beta)(t)$ for all $t \in [a,b]$, that is, $\beta \leq F(\beta,\gamma)$ and $\gamma \geq F(\gamma,\beta)$.
From condition (iv), for all $(x,y),(u,v) \in C(I) \times C(I)$,
$\alpha((x,y),(u,v)) \geq 1 \Longrightarrow e((x(t),y(t)),(u(t),v(t))) \geq 0$
$\Longrightarrow e((F(x(t),y(t)), F(y(t),x(t))),(F(u(t),v(t)), F(v(t),u(t)))) \geq 0$
$\Longrightarrow \alpha((F(x,y), F(y,x)),(F(u,v), F(v,u))) \geq 1.$
Then F is α-admissible.
From (iii), there exists $(x_0,y_0) \in C(I) \times C(I)$ such that
$\alpha((x_0,y_0),(F((x_0,y_0)), F(y_0,x_0))) \geq 1, \alpha((F(y_0,x_0), F((x_0,y_0))),(x_0,y_0)) \geq 1.$
Finally, Theorem (3.2) give that F has a coupled fixed point (x,y). Since $\beta \leq \gamma$, then the hypothesis of Theorem (3.2) is satisfied. Therefore $x = y$, that is $x(t) = y(t)$, for all $t \in [a,b]$ implying $x = F(x,x)$ and $x(t)$ is the solution of equation (3.1). □

References

1. V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (2011). 48894897.
2. M. Borcut, V. Berinde Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Applied Mathematics and Computation. 218 (2012). 5929-5936.
3. S. S. Chang and Y. H. Ma, Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solutions for a class of functional equations arising in dynamic programming, Journal of Mathematical Analysis and Applications, vol. 160, no. 2, pp. 468479, 1991.
4. Y. Z. Chen, Existence theorems of coupled fixed points, Journal of Mathematical Analysis and Applications, vol. 154, no. 1, pp. 142150, 1991.
5. Z. Drici, F. McRae, J.V. Devi, Fixed point theorems for mixed monotone operators with PPF dependence, Nonlinear Anal. 69 (2) (2008) 632-636.
6. D. Guo, V. Lakshmikantham, Coupled fixed points of
7. T. Gnana Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis.TMA 65 (2006) 1379-1393. nonlinear operators with applications, Nonlinear Analysis, vol. 11, no. 5, pp. 623632, 1987.
8. S. Jingxian and L. Lishan, Iterative method for coupled quasisolutions of mixed monotone operator equations, Applied Mathematics and Computation, vol. 52, no. 2-3, pp. 301308, 1992.
9. W. Long, B. E. Rhoades, and M. Rajovic, Coupled coincidence points for two mappings in metric spaces and cone metric spaces, Fixed Point theory and Applications, vol. 2012, article 66, 9 pages, 2012.

10. N. V. Luong, N. X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Analysis. 74 (2011) 983-992.
11. B. Samet, C. Vetro, P. Vetro, *Fixed point theorems for* $\alpha - \psi - contractive$ *type mappings*, J. Nonlinear Analysis :Theory, Methods & Applications, Vol. 75, (2012) 2154-2165.
12. W. Shatanawi, Fixed point theorems for nonlinear weakly C-contractive mappings in metric spaces, Mathematical and Computer Modelling 54 (2011) 2816-2826.
13. Y. Sun, A fixed point theorem for mixed monotone operators with applications, Journal of Mathematical Analysis and Applications, vol. 156, no. 1, pp. 240252, 1991.